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Digital Signal Processing

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Dedicated to .....

My Son

Sanidhya

R. A. Barapate

Preface

Dear students,

We are extremely happy to come out with the Second revised edition of**"Digital Signal Processing"** for you. We have divided the subject into small chapters so that the topics can be arranged and understood properly. The topics within the chapters have been arranged in a proper sequence to ensure smooth flow of the subject.

A large number of solved examples have been included. So, we are sure that this book will cater all your needs for this subject.

We are thankful to Shri. Pradeep Lunawat and Shri. Sachin Shah for the encouragement and support that they have extended. We are also thankful to Pallavi Dabir and all staff members of Tech-Max Publications and others for their efforts to make this book as good as it is. We have jointly made every possible efforts to eliminate all the errors in this book. However if you find any, please let us know, because that will help us to improve further.

We are also thankful to our family members and friends for their patience and encouragement.

- Authors

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Syllabus

1. **Discrete Time Signals and System :**

   Discrete–time signals, Discrete–time systems, Analysis of discrete-time LTI systems, Discrete-time systems described by differential equations, Implementation of discrete-time systems, Correlation of discrete-time systems. **(Refer chapter DTSS)**

2. **Z-Transform :**

   Definition and Properties of Z-transform, Rational Z-transforms, Inverse Z-transform, one-sided Z-transform, Analysis of LTI systems in Z-domain

   (Refer chapter ZT and AOLTIZT)

3. **Frequency Analysis of Signals and Systems :**

   Frequency analysis: Continuous time signals and Discrete-time signals, Properties of the Fourier transform for discrete-time signals. Frequency domain characteristics of LTI systems, LTI system as a frequency selective filter, Inverse systems and deconvolution. **(Refer chapter FACTS, FFT and FADTS)**

4. **Discrete Fourier Transform:**

   Frequency domain sampling, Properties of DFT, Linear filtering method based on DFT, Frequency analysis of signals using DFT, FFT algorithm, Applications of FFT, Geortzel algorithm, Quantisation effects in the computation of DFT **(Refer chapter DFT)**

5. **Implementation of Discrete Time Systems :**

   Structure of FIR systems, Structure of IIR systems, quantization of filter coefficients, round-off effects in digital filters **(Refer chapter FIRFS and IIRFT)**

6. **Design of Digital Filters :**

   Design of FIR filters, Design of IIR filters from analog filters, frequency transformations, Design of digital filters based on least-squares method digital filters from analog filters, Properties of FIR digital filters, Design of FIR filters using windows, Comparison of IIR and FIR filters, and Linear phase filters. **(Refer chapter FIRFD and IIRFD)**

7. **Introduction to DSP :**

   Co-processors : TMS 320C40/50, Analog Devices. **(Refer chapter DPA)**

8. **Applications :**

   Image processing, Control, Speech, Audio, Telecommunication.

   (Refer chapter APP-DSP)

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Introduction to DSP

Section No.	Name of the Topic
1.1	Introduction
1.2	Basic Elements of DSP and its Requirements
1.3	Advantages of Digital over Analog Signal Processing
1.4	Limitations of Digital Signal Processing
1.5	Comparison between Digital and Analog Signal Processing
1.6	Applications of DSP

• Code : DSP19

1.1 Introduction :

The world of science and engineering is filled with signals such as images from remote space probes, voltages generated by the heart and brain and countless other applications.

1.1.1 What is DSP ?

Digital signal processing is used in a wide variety of applications. It is hard to get exact definition of DSP.

Let us first look at the dictionary meanings of these words :

Digital : Operating by the use of discrete signals to represent data in the form of numbers.

Signal : A variable parameter by which information is conveyed through an electronic circuit.

Processing : To perform operations on data according to programmed instructions. This leads to a simple definition of DSP.

Definition of DSP : DSP is defined as changing or analysing information which is measured as discrete sequences of numbers.

1.1.2 System :

System is associated with signals.

System is nothing but physical device that performs an operation on a signal.

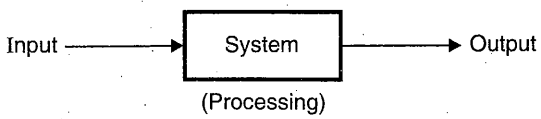


Fig. S-1 : Signal processing

As shown in Fig. S-1, when we apply an input signal to the system, the system will process the signal and we get the expected response. This phenomenon is called as signal processing.

1.2 Basic Elements of DSP and its Requirements :

Fig. S-2 shows that the basic elements of digital signal processing system

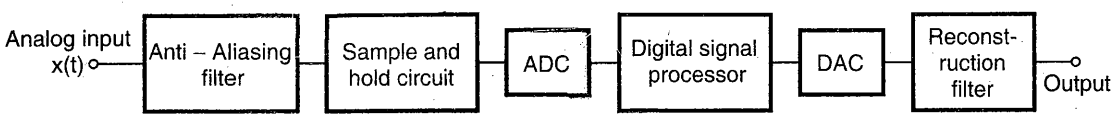


Fig. S-2

The different blocks of this system are as follows :

1. Input signal :

It is the signal generated from some transducer or from some communication system. It may be biomedical signal like ECG or EEG. Generally input signal is analog in nature. It is denoted by x(t).

2. Anti-aliasing filter :

Anti aliasing filter is basically a low pass filter. It is used for the following purposes.

(a) It removes the high frequency noise contain in input signal.

(b) As the name indicates; it avoids aliasing effect. That means it is used to band limit the signal.

3. Sample and hold circuit :

As the name indicates; this block takes the samples of input signal. It keeps the voltage level of input signal relatively constant which is the requirement of ADC.

Some times amplifiers are used to bring the voltage level of input signal upto the required voltage level of ADC.

4. Analog to digital converter (ADC) :

As the name indicates; this block is used to convert analog signal into digital form. This is required because digital signal processor accepts the signal which is digital in nature.

5. Digital signal processor :

It processes input signal digitally. In a simple languages processing of input signal making modifying the signal as per requirement. For this purpose DSP processors like ADSP 2100 or TMS 320 can be used.

6. Digital to analog converter (DAC) :

The output of digital signal processor is digital in nature. But the required final output is analog in nature. So to convert digital signal into analog signal DAC is used.

7. Reconstruction filter :

Output signal of DAC is analog, that means it is a continuous signal. But it may contain high frequency components. Such high frequency components are unwanted. To remove these components; reconstruction filter is used.

1.3 Advantages of Digital over Analog Signal Processing :

1. Versatility :

Digital systems can be reprogrammed for other applications (where programmable DSP chips are used). Moreover, digital systems can be ported to different hardware.

2. Repeatability :

Digital systems can be easily duplicated. These systems do not depend upon component tolerances and temperature.

3. Simplicity :

It is easy to built any digital system as compared to an analog one.

4. Accuracy :

To design analog system; analog components like resistors, capacitors and inductors are used. The tolerance of these components reduce accuracy of analog system. While in case of DSP ; much better accuracy is obtained.

5. Remote processing :

Analog signals are difficult to store because of problems like noise and distortion. While digital signal can be easily stored on storage media like magnetic tapes, disks etc. Thus compared to analog signals; digital signals can be easily transposed. So remote processing of digital signal can be done easily.

6. Implementation of algorithms :

The mathematical processing algorithms can be easily implemented in case of digital signal processing. But such algorithms are difficult to implement in case of analog signals.

7. Easy upgradations :

Because of the use of software; digital signal processing systems can be easily upgraded compared to analog system.

8. Compatibility :

In case of digital systems; generally all applications needs standard hardware. Thus operation of dsp system is mainly dependent on software. Hence universal compatibility is possible compared to analog systems.

9. Cheaper :

In many applications ; the digital systems are comparatively cheaper than analog systems.

1.4 Limitations of Digital Signal Processing :

The digital signal processing systems have many advantages. Even though there are certain disadvantages as follows :

1. System complexity :

The digital signal processing system, makes use of converters like ADC and DAC. This increases the system complexity compared to analog systems. Similarly in many applications; the time required for this conversion is more.

2. Bandwidth limitation :

In case of DSP system; if input signal is having wide bandwidth then it demands for high speed ADC. This is because, to avoid alising effect, the sampling rate should be atleast twice the bandwidth. Thus such signals require fast digital signal processors. But always there is a practical limitation in the speed of processors and ADC.

3. Power consumption :

A typical digital signal processing chip contains more than 4 lakh transistors. Thus power dissipation is more in dsp systems compared to analog systems.

4. For small applications digital signal processing systems are expensive compared to analog systems.

1.5 Comparison between Digital and Analog Signal Processing :

Table S-1 shows comparison between digital and analog signal processing

Table S-1 : Comparison between digital and analog signal processing

Digital Signal Processing	Analog Signal Processing
1. More versatile	1. Less versatile
2. More repeatability	2. Less repeatability
3. More accuracy	3. Less accuracy
4. Such signals can be easily stored	4. It is difficult to store analog signals
5. Mathematical processing algorithm can be easily implemented.	5. It is difficult to implement mathematical processing algorithms.
6. Upgradation can be done easily	6. Upgradation is difficult
7. Universal compatibility is possible	7. Universal compatibility is not possible
8. More system complexity	8. Less system complexity
9. Requires more power consumption.	9. Requires less power consumption.

1.6 Applications of DSP :

Sr. No.	Application of DSP	Fields where it is used
(1)	Image processing	(a) Pattern recognition (b) Animation (c) Robotic vision (d) Image enhancement
(2)	Instrumentation and control	(a) Spectral analysis (b) Noise reduction (c) Data compression
(3)	Speech/Audio	(a) Speech recognition (b) Speech synthesis (c) Equalisation
(4)	Biomedical	(a) Scanners (b) ECG analysis (c) Patient monitoring
(5)	Telecommunication	(a) Echo cancellation (b) Spread spectrum (c) Data communication
(6)	Military	(a) Sonar processing (b) Radar processing (c) Secure communication
(7)	Consumer applications	(a) Digital audio and video (b) Television and music systems etc
(8)	Industrial applications	(a) Robotics and CNC (b) Power line monitors
(9)	Automotive applications	(a) Vibration analysis (b) Voice commands (c) Cellular telephones

Theory Questions

Q. 1 What is DSP ?

Q. 2 Name basic elements of DSP.

Q. 3 Give limitations of DSP.

Q. 4 List applications of DSP.

❑❑❑

Note

Discrete Time Signals and Systems

Section No.	Name of the Topic
1.1	Introduction
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• Code : DSP1

1.1 Introduction :

In a communication system, the word 'signal' is very commonly used. Therefore we must know its exact meaning.

- Mathematically, signal is described as a function of one or more independent variables.
- Basically it is a physical quantity. It varies with some dependent or independent variables.
- So the term signal is defined as "A physical quantity which contains some information and which is function of one or more independent variables."

1.2 Classification of Signals :

There are various types of signals. Every signal is having its own characteristic. The processing of signal mainly depends on the characteristics of that particular signal. So classification of signal is necessary. Broadly the signals are classified as follows :

- Continuous and discrete time signals
- Continuous valued and discrete valued signals
- Periodic and non-periodic signals
- Even and odd signals
- Energy and power signals
- Deterministic and random signals
- Multichannel and multidimensional signals.

1.2.1 Continuous and Discrete Time Signals :

Continuous signal :

A signal of continuous amplitude or time is known as continuous signal or analog signal. This signal is having some value at every instant of time.

Examples :

Sinewave, cosinewave, triangular wave etc. Similarly certain electrical signals derived in proportion with physical quantity such as temperature, pressure, sound etc. are also examples of continuous signal. Some of the continuous signals are as shown in Fig. A-1.

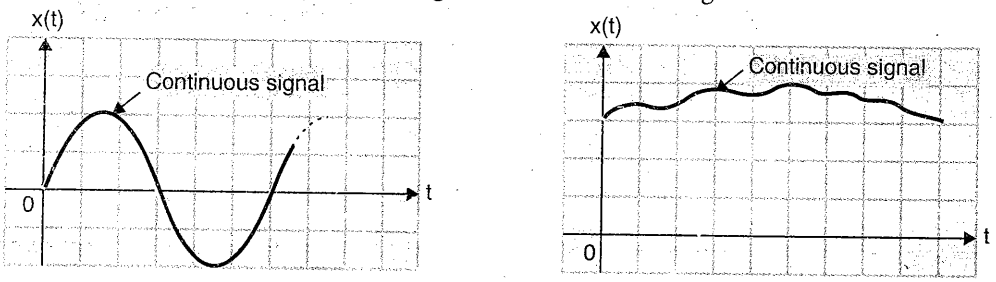


Fig. A-1 : Examples of continuous signals

Mathematical expression :

Mathematically a continuous signal (eg. sinewave) can be expressed as,

$$x(t) = A \sin(\omega t + \theta)$$

Here A = Amplitude of signal

ω = Angular frequency = $2\pi f$

θ = Phase shift

Characteristics :

- For every fix value of t, x (t) is periodic in nature.
- If the frequency $\left(\frac{1}{t}\right)$ is increased then the rate of oscillation also changes.

Discrete time signal :

In this case the value of signal is specified only at specific time. So the signal represented at "discrete interval of time" is called as discrete time signal.

The discrete time signal is generated from continuous time signal by using the sampling operation. We will study the sampling operation in detail, later in this chapter. This process is shown in Fig. A-2.

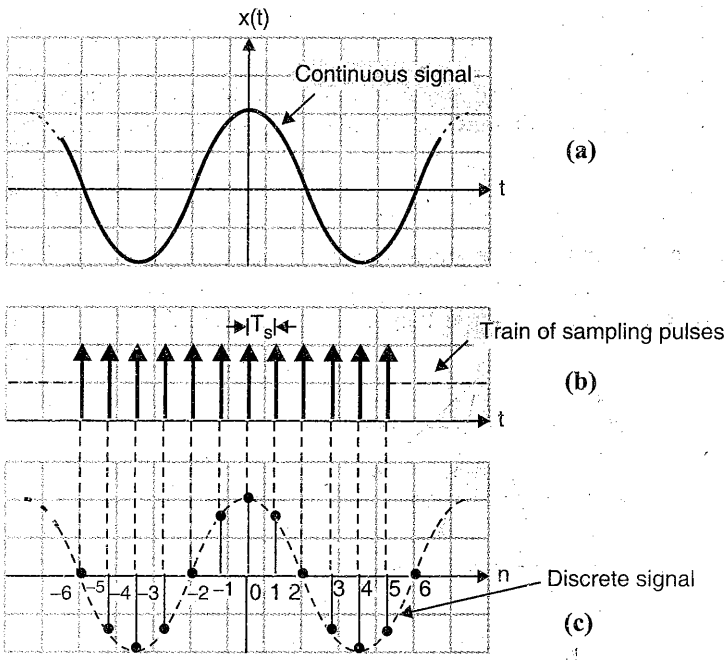


Fig. A-2 : Discrete time signal

- Consider a continuous analog signal as shown in Fig. A-2(a). This signal is continuous in nature from $-\infty$ to $+\infty$.
- The sampling pulses are shown in Fig. A-2(b). These are train of pulses. Here the samples are taken at $\pm T_s, \pm 2T_s, \pm 3T_s$ ... and T_s is the sampling time.

- Fig. A-2(c) shows discrete time signal. Observe that this signal takes the value, only where the sampling pulse is present. In between the two sampling pulses the signal is absent. So this is called as discrete time signal.
- In Fig. A-2(a), on X-axis time (t) is plotted. On Y-axis the amplitude is plotted. So continuous time signal is represented by $x(t)$. Observe Fig. A-2(c). On X-axis index n is plotted. Here n is the number of corresponding sample. So discrete time signal is denoted by $x(n)$.
- For signal in Fig. A-2(a), the expression is,

$$x(t) = A \cos \omega t$$

and for signal shown in Fig. A-2(c), the expression is,

$$x(n) = A \cos \omega n$$

Characteristics :

- Discrete time sinusoidal signals are identical when their frequencies are separated by integer multiple of 2π.
- If the frequency of discrete time sinusoidal is a rational number, then such signal is periodic in nature.
- For the discrete time sinusoidal, the highest oscillation is obtained when angular frequency $\omega = \pm\pi$.

1.2.2 Continuous Valued or Discrete Valued Signals :

Continuous valued signal :

If the variation in the amplitude of signal is continuous then, it is called as continuous valued signal. Such signal may be continuous or discrete in nature. Such signals are as shown in Figs. A-2(a) and A-2(c).

Discrete valued signal :

If the variation in the amplitude of signal is not continuous; but the signal has certain discrete amplitude levels then such signal is called as discrete valued signal. Such signal may be again continuous or discrete in nature as shown in Figs. A-3(a) and A-3(b).

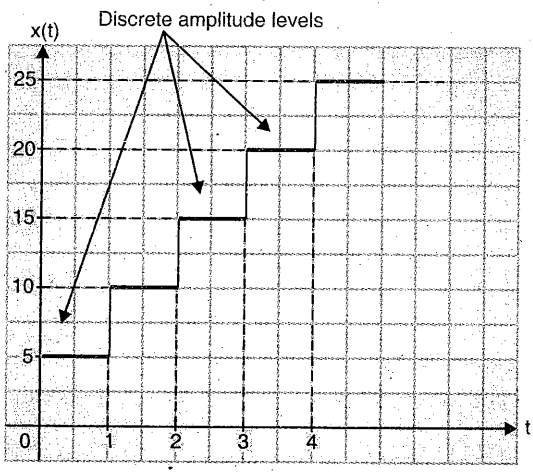


Fig. A-3(a) : Discrete amplitude signal continuous in nature

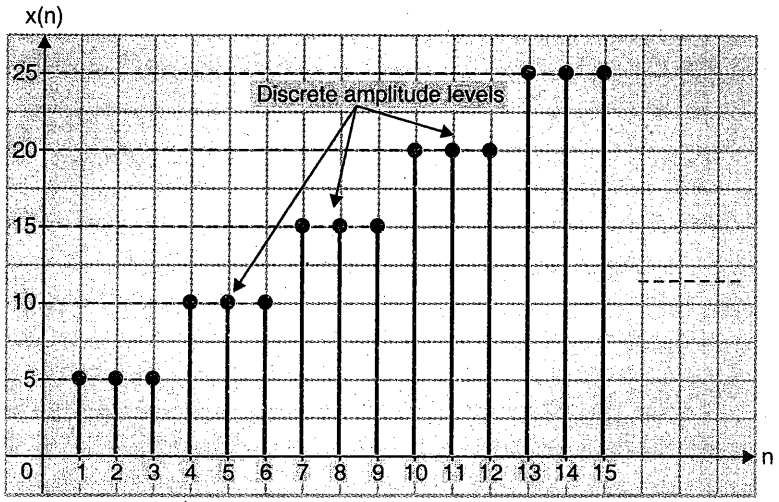


Fig. A-3(b) : Discrete amplitude signal discrete in nature

As shown in Fig. A-3(a), the signal is defined at all instants of time. So it is continuous signal. But it takes only certain discrete amplitude levels. The amplitude is not continuously changing with time. So it is discrete amplitude signal continuous in nature.

As shown in Fig. A-3(b), the signal is defined only at discrete intervals of time. So it is discrete signal. And this signal takes only certain discrete amplitude levels. So it is discrete amplitude signal discrete in nature.

1.2.3 Periodic and Non-periodic Signals :

Periodic signal :

A signal which repeats itself after a fixed time period or interval is called as periodic signal. The periodicity of continuous time signal can be defined mathematically as,

$$x(t) = x(t + T_0) \quad \ldots(1)$$

This is called as condition of periodicity. Here T_0 is called as fundamental period. That means after this period the signal repeats itself.

For the discrete time signal, the condition of periodicity is,

$$x(n) = x(n + N) \quad \ldots(2)$$

Here number 'N' is the period of signal. The smallest value of N for which the condition of periodicity exists is called as fundamental period.

Periodic signals are shown in Figs. A-4(a) and (b).

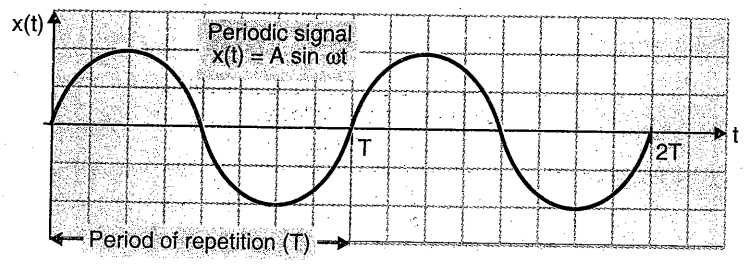


Fig. A-4(a) : Continuous time periodic signal

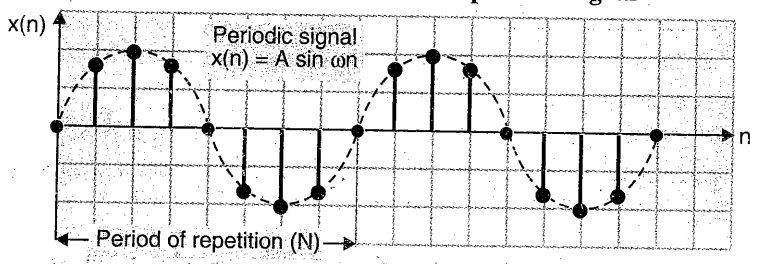


Fig. A-4(b) : Discrete time periodic signal

Non-periodic signal :

A signal which does not repeat itself after a fixed time period or does not repeat at all is called as non-periodic or aperiodic signal. Thus mathematical expression for non-periodic signal is,

$$x(t) \neq x(t + T_0) \quad \ldots(3)$$

and

$$x(n) \neq x(n + N) \quad \ldots(4)$$

Sometimes it is said that non-periodic signal has a period $T = \infty$ as shown in Fig. A-4(c). This is exponential signal having period, $T = \infty$.

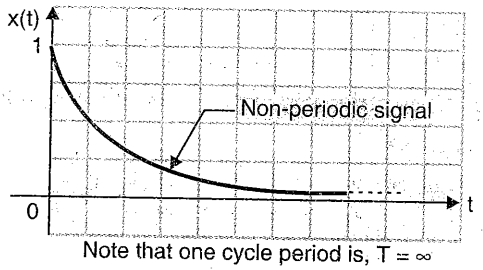


Fig. A-4(c) : Aperiodic signal having period, $T = \infty$

Important condition for periodicity :

- A discrete time sinusoidal signal is periodic only if its frequency (f_0) is rational. That means frequency f_0 should be in the form of ratio of two integers.

Proof :

For the discrete signal, the condition of periodicity is,

$$x(n+N) = x(n) \quad ...(1)$$

Let $x(n)$ be the cosine wave. So it can be expressed as,

$$x(n) = A\cos(2\pi f_0 n + \theta) \quad ...(2)$$

Here A = Amplitude

and θ = Phase shift

Now the equation of $x(n+N)$ can be obtained by replacing 'n' by 'n + N' in Equation (2).

$$\therefore \quad x(n+N) = A\cos[2\pi f_0 (n+N) + \theta] \quad ...(3)$$

According to condition of periodicity (Equation (1)); we can equate Equations (2) and (3).

$$\therefore \quad A\cos[2\pi f_0 (n+N) + \theta] = A\cos(2\pi f_0 n + \theta)$$

$$\therefore \quad A\cos(2\pi f_0 n + 2\pi f_0 N + \theta) = A\cos(2\pi f_0 n + \theta) \quad ...(4)$$

To satisfy this equation,

$$2\pi f_0 N = 2\pi k \quad ...(5)$$

where k is an integer

$$\therefore f_0 = \frac{k}{N} \quad ...(6)$$

Here k and N both are integers. Thus discrete time (DT) signal is periodic if its frequency f_0 is rational.

Periodicity condition for $x(n) = x_1(n) + x_2(n)$:

Here input sequence is expressed as summation of two discrete time sequences. We can calculate the values of f_1 and f_2 corresponding to $x_1(n)$ and $x_2(n)$. Then we can express f_1 and f_2 as,

$$f_1 = \frac{k_1}{N_1} \text{ and } f_2 = \frac{k_2}{N_2}$$

The resultant signal $x(n)$ is periodic if $\frac{N_1}{N_2}$ is ratio of two integers. The period of $x(n)$ will be least common multiple of N_1 and N_2.

Similarly if continuous time signals is,

$$x(t) = x_1(t) + x_2(t)$$

We can calculate the values of T_1 and T_2 corresponding to $x_1(t)$ and $x_2(t)$. Then the

resultant x (t) is periodic if $\frac{T_1}{T_2}$ is ratio of two integers. The fundamental period of x (t) will be least common multiple of T_1 and T_2.

Solved Examples :

Prob. 1 : Few discrete time sequence are given below

(i) cos (0.01 π n)

(ii) cos (3 π n)

(iii) sin (3 n)

(iv) $\cos\left(\frac{n}{8}\right)\cos\left(\frac{\pi n}{8}\right)$

Determine whether they are periodic or non periodic. If a sequence is periodic, determine its fundamental period.

Soln. :

(1) Given sequence is

$$x(n) = \cos(0.01\pi n) \quad \ldots(1)$$

We have the standard equation,

$$x(n) = \cos \omega n \quad \ldots(2)$$

Comparing Equations (1) and (2) we get,

$$\omega = 0.01\pi$$

But $\omega = 2\pi f$

$\therefore 2\pi f = 0.01\pi$

$$\therefore f = \frac{0.01\pi}{2\pi} = \frac{0.01}{2}$$

$$\therefore f = \frac{1}{200} \text{ cycles per sample} \quad \ldots(3)$$

Since frequency 'f' is expressed as the ratio of two integers; this sequence is periodic. Now we have the condition of periodicity,

$$f = \frac{k}{N} \quad \ldots(4)$$

Here 'N' indicates, the fundamental period.

Comparing Equation (3) and (4)

Fundamental period = N = 200 samples

(ii) Given equation is

$$x(n) = \cos(3\pi n) \quad \ldots(5)$$

Comparing with Equation (2) we get,

$$\omega = 3\pi$$

$$\therefore \quad 2\pi f = 3\pi$$

$$\therefore \quad f = \frac{3}{2} \text{ cycles/sample} \qquad ...(6)$$

Since 'f' is ratio of two integers; the given sequence is periodic. Comparing Equation (4) and (6) we get,

Fundamental period = N = 2 samples

(iii) Given sequence is,

$$x(n) = \sin 3n \qquad ...(7)$$

Comparing with Equation (2) we get,

$$\omega = 3$$

$$\therefore \quad 2\pi f = 3$$

$$\therefore \quad f = \frac{3}{2\pi}$$

Here 2π is not an integer. That means 'f' cannot be expressed as the ratio of two integers. Thus the given sequence is non-periodic.

(iv) Given sequence is,

$$x(n) = \cos\left(\frac{n}{8}\right)\cos\left(\frac{\pi n}{8}\right) \qquad ...(8)$$

The standard equation can be expressed as,

$$x(n) = \cos \omega_1 n \cos \omega_2 n \qquad ...(9)$$

Comparing Equations (8) and (9) we get,

$$\omega_1 = \frac{1}{8} \text{ and } \omega_2 = \frac{\pi}{8}$$

But $\omega = 2\pi f$

$$\therefore \quad 2\pi f_1 = \frac{1}{8} \text{ and } 2\pi f_2 = \frac{\pi}{8}$$

$$\therefore \quad f_1 = \frac{1}{16\pi} \text{ and } f_2 = \frac{1}{16}$$

Hence f_1 is ratio of non-integer values. So it is non-periodic. While f_2 is ratio of two integer values. So it is periodic. But the total signal is multiplication of non-periodic and periodic signals. So the resultant signal $x(n)$ is non-periodic.

Prob. 2 : State whether the following signals $x(t)$ and $y(n)$ are periodic or not, giving reasons. If they are periodic, find the corresponding period.

(i) $x(t) = 2\cos 100\pi t + 5\sin 50t$

(ii) $y(n) = \sin\frac{\pi}{5}n + 2\cos\frac{\pi}{\sqrt{2}}n$

Soln. :

(i) The given signal is,

$$x(t) = 2\cos 100\pi t + 5\sin 50t \quad ...(1)$$

$$\text{Let } x(t) = x_1(t) + x_2(t) \quad ...(2)$$

$$\text{Here } x_1(t) = 2\cos 100\pi t \quad ...(3)$$

$$\text{and } x_2(t) = 5\sin 50t \quad ...(4)$$

The standard equation can be expressed as,

$$x_1(t) = A\cos \omega_1 t \quad ...(5)$$

Comparing Equations (3) and (5) we get,

$$\omega_1 = 100\pi$$

Now we have,

$$\omega_1 = 2\pi f_1 = \frac{2\pi}{T_1}$$

$$\therefore \frac{2\pi}{T_1} = 100\pi$$

$$\therefore T_1 = \frac{2\pi}{100\pi} = \frac{1}{50} \quad ...(6)$$

Similarly standard equation for $x_2(t)$ can be written as,

$$x_2(t) = A\sin \omega_2 t \quad ...(7)$$

Comparing Equations (4) and (7) we get,

$$\omega_2 = 50$$

$$\therefore 2\pi f_2 = \frac{2\pi}{T_2} = 50$$

$$\therefore T_2 = \frac{2\pi}{50} \quad ...(8)$$

Here $x(t)$ is expressed as summation of two signals. We know that the resultant signal $x(t)$ is periodic if $\frac{T_1}{T_2}$ is the ratio of two integers. From Equations (6) and (7) we get,

$$\frac{T_1}{T_2} = \frac{1}{50} \cdot \frac{50}{2\pi} = \frac{1}{2\pi}$$

It is not the ratio of two integers. Thus $x(t)$ is non-periodic.

(ii) Given :

$$y(n) = \sin\frac{\pi}{5}n + 2\cos\frac{\pi}{\sqrt{2}}n.$$

$$\text{Let } y(n) = y_1(n) + y_2(n) \quad ...(9)$$

$$\text{Here } y_1(n) = \sin\frac{\pi}{5}n \quad ...(10)$$

$$\text{and } y_2(n) = 2\cos\frac{\pi}{\sqrt{2}}n \quad ...(11)$$

The standard equation for $y_1(n)$ can be written as,

$$y_1(n) = \sin\omega_1 n \quad ...(12)$$

Comparing Equations (10) and (12) we get,

$$\omega_1 = \frac{\pi}{5} \qquad \therefore \quad 2\pi f_1 = \frac{\pi}{5}$$

$$\therefore \quad f_1 = \frac{1}{10}$$

We know that $f_1 = \frac{k_1}{N_1}$

Thus $N_1 = 10$. That means $y_1(n)$ is periodic with period 10 samples.

Now the standard equation for $y_2(n)$ can be written as,

$$y_2(n) = A\cos\omega_2 n \quad ...(13)$$

Comparing Equations (11) and (13) we get,

$$\omega_2 = \frac{\pi}{\sqrt{2}} \qquad \therefore \quad 2\pi f_2 = \frac{\pi}{\sqrt{2}}$$

$$\therefore \quad f_2 = \frac{1}{2\sqrt{2}}$$

But we have,

$$f_2 = \frac{k_2}{N_2} \qquad \therefore \quad N_2 = 2\sqrt{2}$$

Now the resultant signal $y(n)$ is periodic if the ratio $\frac{N_1}{N_2}$ is the ratio of two integers.

$$\text{Here } \frac{N_1}{N_2} = \frac{10}{2\sqrt{2}} = \frac{5}{\sqrt{2}}$$

Thus $y(n)$ is non-periodic in nature.

Prob. 3 : Determine which of the following signals are periodic. If periodic determine the fundamental period.

(i) $10 \sin (12 \pi t) + 4 \sin (18 \pi t)$

(ii) $3 \sin 4t$

(iii) $3 + t^2$

Soln. :

(i) Given signal is,

$$x(t) = 10 \sin (12 \pi t) + 4 \sin (18 \pi t) \quad ...(1)$$

$$\text{Let } x(t) = x_1(t) + x_2(t)$$

$$\text{Here } x_1(t) = 10 \sin (12 \pi t) \quad ...(2)$$

$$\text{and } x_2(t) = 4 \sin (18 \pi t) \quad ...(3)$$

The standard equation can be written as,

$$x(t) = A \sin \omega t \quad ...(4)$$

Comparing Equations (2) and (4) we get,

$$\omega_1 = 12 \pi \qquad \therefore 2 \pi f_1 = 12 \pi$$

$$\therefore f_1 = \frac{12 \pi}{2 \pi} = \frac{6}{1}$$

$$\therefore T_1 = \frac{1}{f_1} = \frac{1}{6} \quad ...(5)$$

Similarly comparing Equations (3) and (4) we get,

$$\omega_2 = 18 \pi$$

$$\therefore 2 \pi f_2 = 18 \pi \qquad \therefore f_2 = \frac{18 \pi}{2 \pi} \therefore f_2 = 9$$

$$\therefore T_2 = \frac{1}{f_2} = \frac{1}{9} \quad ...(6)$$

The resultant signal $x(t)$ is periodic if $\frac{T_1}{T_2}$ is the ratio of two integers. From Equations (5) and (6) we get,

$$\frac{T_1}{T_2} = \frac{1/6}{1/9} = \frac{1}{6} \cdot \frac{9}{1} = \frac{9}{6}$$

It is the ratio of two integers. Thus $x(t)$ is periodic. Now the fundamental period of $x(t)$ is least common multiple of T_1 and T_2. Thus fundamental period is $\frac{1}{3}$ sec.

(ii) Given $x(t) = 3 \sin 4t \quad ...(7)$

We have the standard equation,

$$x(t) = A \sin \omega t \quad ...(8)$$

Comparing Equations (7) and (8) we get,

$$\omega = 4 \qquad \therefore 2\pi f = 4$$

$$\therefore f = \frac{4}{2\pi}$$

It is not the ratio of integer values. Thus this signal is non- periodic in nature.

(iii) Given $x(t) = 3 + t^2$ ...(9)

We know that a continuous time signal is periodic in nature if it satisfies the equation,

$$x(t) = x(t + T_0) \qquad ...(10)$$

Where T_0 is the fundamental period of repetation.

From Equation (9) we can write,

$$x(t + T_0) = 3 + (t + T_0)^2$$

$$\therefore x(t + T_0) = 3 + t^2 + 2tT_0 + T_0^2 \qquad ...(11)$$

For any value of 'T_0' Equations (9) and (11) cannot be made equal. Thus given signal is non-periodic.

Prob. 4 : Find if the following sequence are periodic or not. If yes find its fundamental time period.

(i) $x_1(n) = e^{j\left(\frac{\pi}{4}\right)n}$ (ii) $x_2(t) = \cos(50\pi t) + \cos(100\pi t)$

(iii) $x_3(n) = 3\sin\left(\frac{1}{8}n\right)$

Soln. :

(i) Given, $x_1(n) = e^{j\left(\frac{\pi}{4}\right)n}$ ...(1)

A discrete time complex exponential is periodic if its relative frequency is a rational number. We have the standard equation of a discrete time complex exponential.

$$x(n) = e^{j\frac{2\pi kn}{N}} \qquad ...(2)$$

Comparing Equations (1) and (2) we get,

$$\frac{2\pi kn}{N} = \frac{\pi}{4}n \qquad \therefore \frac{2k}{N} = \frac{1}{4}$$

$$\therefore \frac{k}{N} = \frac{1}{8}$$

Since it is ratio of two integers; the given sequence is periodic. Its fundamental period is N = 8 samples.

(ii) Given $x_2(t) = \cos(50\pi t) + \cos(100\pi t)$ ...(3)

Let $x_2(t) = x_a(t) + x_p(t)$ ...(4)

Here $x_a(t) = \cos(50\pi t)$ ...(5)

and $x_b(t) = \cos(100\pi t)$ ...(6)

The standard equation can be written as,

$$x(t) = \cos \omega t \quad ...(7)$$

Comparing Equations (5) and (7) we get,

$$\omega_a = 50\pi \qquad \therefore 2\pi f_a = 50\pi \qquad \therefore f_a = \frac{50\pi}{2\pi} = 25$$

$$\therefore T_a = \frac{1}{25} \quad ...(8)$$

Similarly comparing Equations (6) and (7) we get,

$$\omega_b = 100\pi$$

$$\therefore 2\pi f_b = 100\pi \qquad \therefore f_b = 50$$

$$\therefore T_b = \frac{1}{50} \quad ...(9)$$

We know that $x(t)$ is periodic if $\frac{T_a}{T_b}$ is the ratio of two integers.

Here $\frac{T_a}{T_b} = \frac{\frac{1}{25}}{\frac{1}{50}} = \frac{1}{25}\,50 = \frac{2}{1}$

Since it is ratio of two integers; the given signal is periodic. Now the fundamental period is least common multiple of T_a and T_b. Thus fundamental period is $\frac{1}{5}$ sec.

(iii) Given $x_3(n) = 3\sin\left(\frac{1}{8}n\right)$ ...(10)

We have the standard equation,

$$x_3(n) = A \sin \omega n \quad ...(11)$$

Comparing Equations (10) and (11) we get,

$$\omega = \frac{1}{8} \qquad \therefore 2\pi f = \frac{1}{8}$$

$$\therefore f = \frac{1}{16\pi}$$

It is not the ratio of two integers. Thus given sequence is non- periodic.

1.2.4 Even and Odd Signals :

Even signals :

An even signal is also called as symmetrical signal. A continuous time (C.T.) signal $x(t)$ is said to be symmetrical or even if it satisfies the following condition :

Condition for symmetry : $x(t) = x(-t)$ ... for C.T. signal.

Here $x(-t)$ indicates that the signal is present for negative time period. That means $x(-t)$ is the signal which is reflected about vertical (Y) axis. So even signals are symmetric about vertical axis or at $t = 0$.

Examples : Figs. A-5(a) and (b) are examples of C.T. even signals.

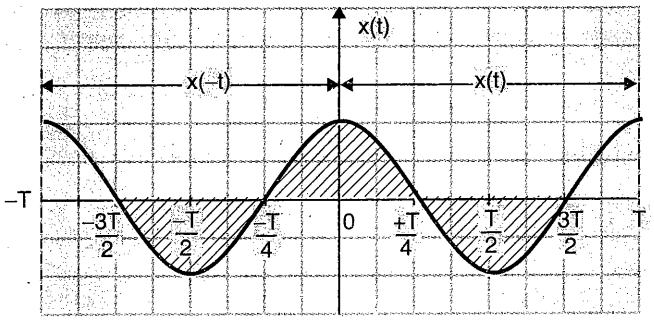


Fig. A-5(a) : Cosinewave $x(t) = x(-t)$ symmetrical or even signal

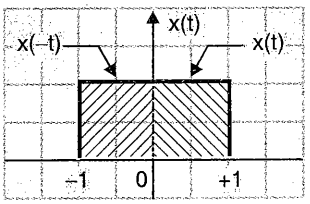
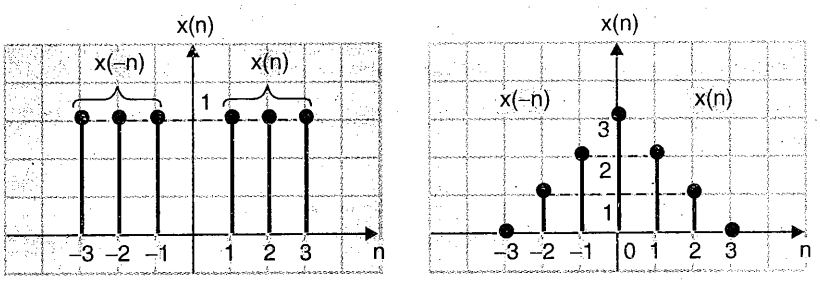


Fig. A-5(b) : Pulse, centered at origin $x(t) = x(-t)$

A discrete time real valued signal is said to be symmetric (even) if,

Condition of symmetry : $x(n) = x(-n)$ ... for D.T. signal

Examples : Fig. A-6(a) and (b) are examples of D.T. even signals.



(a) D.T. even signal **(b) D.T. odd signal**

Fig. A-6

Odd (Antisymmetric) signal :

A continuous time (C.T.) signal $x(t)$ is said to be antisymmetric or odd if it satisfies the following condition

$$\text{Condition of antisymmetric : } x(-t) = -x(t) \quad \text{... for C.T. signal}$$

Here $x(-t)$ indicates that the signal is present for negative time period. While $-x(t)$ indicates that the amplitude of signal negative. Thus antisymmetric signal is not symmetric about vertical axis.

Example : Figs. A-7(a) and (b) are examples of C.T. odd signals.

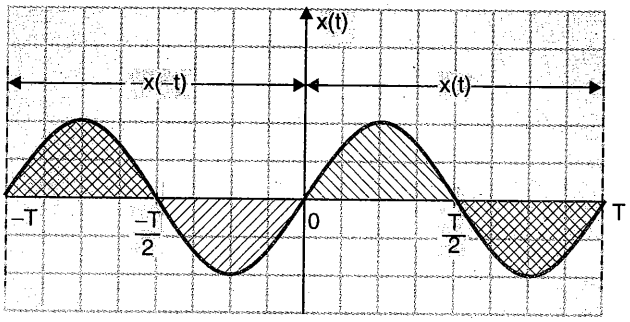


Fig. A-7(a) : Sinewave $x(-t) = -x(t)$ odd signal

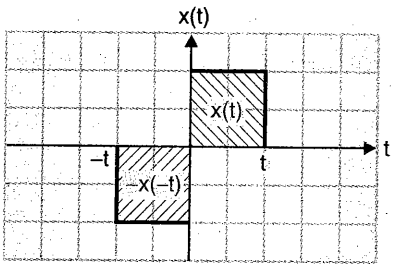


Fig. A-7(b) : Pulse $x(t) = -x(t)$ odd signal

Note that amplitude of antisymmetric signal is always zero at origin.

Similarly discrete time (D.T.) signal $x(n)$ is said to be antisymmetric or odd if it satisfies the following condition.

$$\text{Condition of antisymmetric : } x(-n) = -x(n) \quad \text{... For D.T. signal}$$

Example : Fig. A-8 is example of D.T. odd signal.

Even and odd parts :

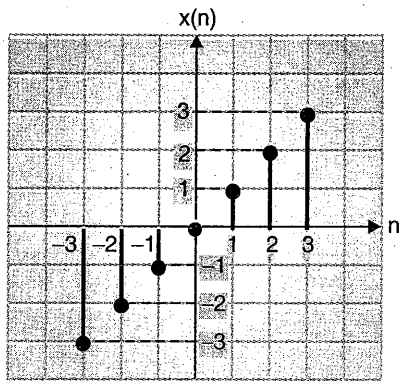


Fig. A-8 : D.T. odd signal

Any C.T. signal can be expressed as the summation of even part and odd part.

$$\therefore \quad x(t) = x_e(t) + x_o(t) \qquad ...(1)$$

Here $x_e(t)$ = Even part of $x(t)$

and $x_o(t)$ = Odd part of $x(t)$

Now even parts and odd parts can be expressed as,

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)] \qquad ...(2)$$

$$\text{and} \quad x_o(t) = \frac{1}{2}[x(t) - x(-t)] \qquad ...(3)$$

Similarly for D.T. signal we can write,

$$x(n) = x_e(n) + x_o(n) \qquad ...(4)$$

$$x_e(n) = \frac{1}{2}[x(n) + x(-n)] \qquad ...(5)$$

$$\text{and} \quad x_o(n) = \frac{1}{2}[x(n) - x(-n)] \qquad ...(6)$$

Solved Problems :

Prob. 1 : Prove that any arbitrary signal can be expressed as the sum of two signals, one of which is EVEN and other is ODD.

Soln. :

Let the given D.T. signal be $x(n)$. It can be expressed as,

$$x(n) = \frac{x(n)}{2} + \frac{x(n)}{2} \qquad ...(1)$$

Adding and subtracting $\frac{x(-n)}{2}$ we get,

$$x(n) = \frac{x(n)}{2} + \frac{x(n)}{2} + \frac{x(-n)}{2} - \frac{x(-n)}{2}$$

Rearranging the terms,

$$x(n) = \left[\frac{x(n) + x(-n)}{2}\right] + \left[\frac{x(n) - x(-n)}{2}\right] \qquad ...(2)$$

The terms in the bracket satisfies the definition of even and odd signals.

$$\therefore \quad x(n) = x_e(n) + x_o(n) \qquad \ldots(3)$$

Put $n = -n$

$$\therefore \quad x(-n) = x_e(-n) + x_o(-n) \qquad \ldots(4)$$

But from the definition of even and odd signals we can write,

$$x_e(-n) = x_e(n) \qquad \ldots(5)$$

and $$x_o(-n) = -x_o(n) \qquad \ldots(6)$$

Putting Equations (5) and (6) in Equation (4) we get,

$$x(-n) = x_e(n) - x_o(n) \qquad \ldots(7)$$

Adding Equations (3) and (7) we get,

$$x(n) + x(-n) = 2x_e(n)$$

$$\therefore \quad x_e(n) = \frac{1}{2}[x(n) + x(-n)] \qquad \ldots(8)$$

Now subtracting Equation (7) from Equation (3) we get,

$$x(n) - x(-n) = 2x_o(n)$$

$$\therefore \quad x_o(n) = \frac{1}{2}\left[\frac{x(n) - x(-n)}{2}\right] \qquad \ldots(9)$$

Prob. 2 : Function x (t) is as shown in Fig. A-9. Draw even and odd parts of x (t).

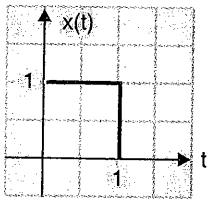


Fig. A-9 : Given function x (t)

Soln. :

Even and odd parts are calculated by using equations $x_e(t) = \frac{1}{2}[x(t) + x(-t)]$ and $x_o(t) = \frac{1}{2}[x(t) - x(-t)]$

This calculation is as shown in Fig. A-10.

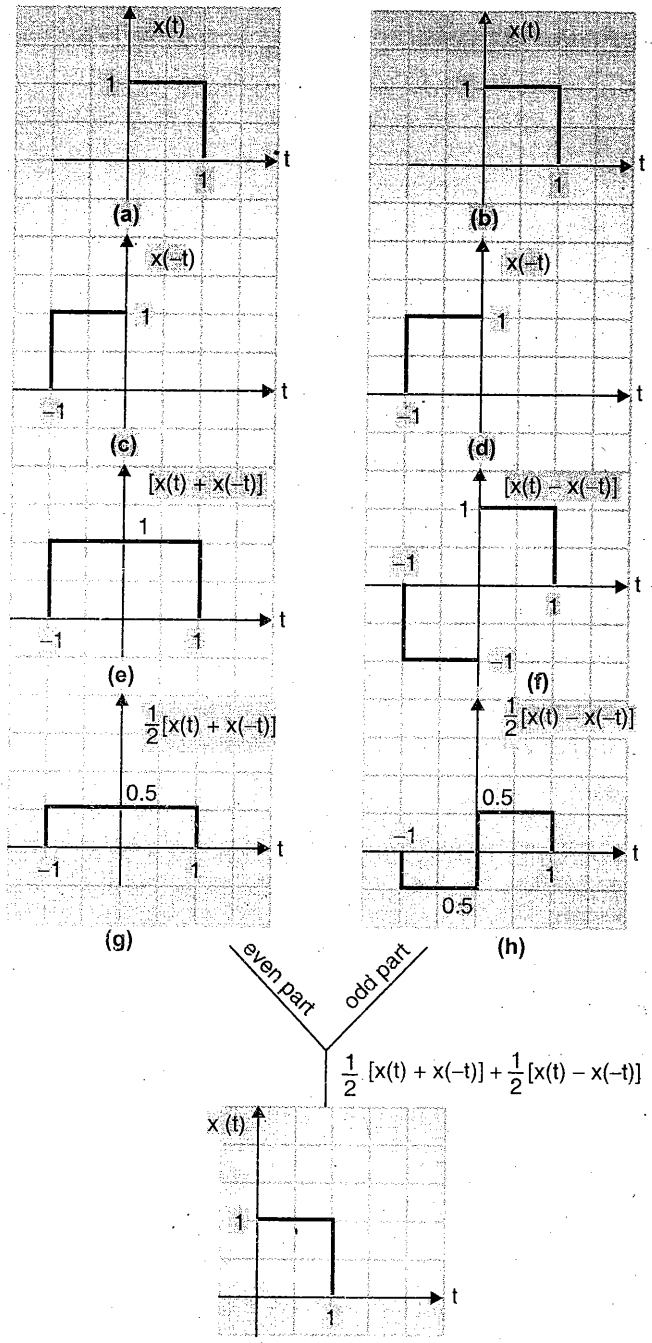

x(t)
1
t
1
(a)
x(–t)
1
t
–1
(c)
[x(t) + x(–t)]
1
t
–1
1
(e)
$\frac{1}{2}$[x(t) + x(–t)]
0.5
t
–1
1
(g)
x(t)
1
t
1
(b)
x(–t)
1
t
–1
(d)
[x(t) – x(–t)]
1
–1
t
1
–1
(f)
$\frac{1}{2}$[x(t) – x(–t)]
0.5
–1
t
1
0.5
(h)
even part
odd part
$\frac{1}{2}$ [x(t) + x(–t)] + $\frac{1}{2}$[x(t) – x(–t)]
x (t)
1
t
1

Fig. A-10

1.2.5 Energy and Power Signals :

Power signals :

There are three power signals :

(A) Instantaneous power

(B) Normalized power

(C) Average normalized power.

(A) Instantaneous power :

An instantaneous power across resistor R is given by,

$$\text{Instantaneous power} = \frac{V^2}{R} \qquad ...(1)$$

Here 'V' is the voltage across resistor R. Let this voltage be denoted by $x(t)$. Then Equation (1) becomes,

$$\text{Instantaneous power} = \frac{x^2(t)}{R} \qquad ...(2)$$

Similarly in terms of current we have,

$$\text{Instantaneous power} = I^2 R \qquad ...(3)$$

If we-say current is denoted by $x(t)$ then we get,

$$\text{Instantaneous power} = x^2(t)\,R \qquad ...(4)$$

(B) Normalized power :

Everytime we may not know whether $x(t)$ is a voltage signal or current signal. So to make the equation of power independent of nature of $x(t)$; we will normalize it by putting $R = 1$. Thus Equations (2) and (4) becomes,

$$\text{Normalized instantaneous power} = x^2(t) \qquad ...(5)$$

(C) Average normalized power :

Average normalized power is given by,

$$P = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |x^2(t)|\,dt \qquad ...(6)$$

- Here magnitude of $x(t)$ is written so this equation is also applicable, if $x(t)$ is complex.
- Integration is taken from $-\frac{T_0}{2}$ to $+\frac{T_0}{2}$ that means it is for entire time period T_0.
- T_0 take the average value; integration term is multiplied by $\frac{1}{T_0}$.

Based on above discussion we can define the power signal.

Definition of Power Signal :

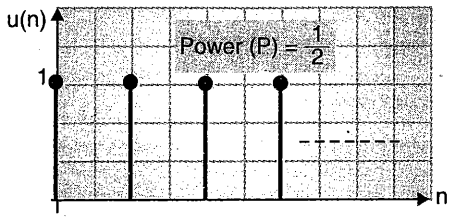


Fig. A-11(a)

A signal x (t) is said to be power signal if its normalized average power is non-zero and finite. Thus for the power signals the normalized average power, P is finite and non-zero.

For the discrete time signals, average power P is given by,

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x^2(n) \qquad ...(7)$$

In Equation (7), it is expected that N >> 1.

The power signal is as shown in Fig. A-11(a).

Energy signal :

The total normalized energy for a "real" signal x (t) is given by,

$$E = \int_{-\infty}^{\infty} x^2(t)\, dt \qquad ...(8)$$

But if the signal x (t) is complex then Equation (8) is modified as,

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt \qquad ...(9)$$

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt \qquad ...(10)$$

Definition of energy signal :

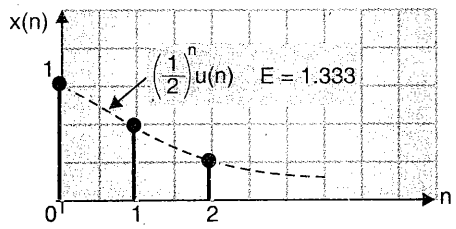


Fig. A-11(b)

A signal x (t) is said to be an energy signal if its normalized energy is non-zero and finite. Hence for the energy signals the total normalized energy (E) is non-zero and finite.

The energy of discrete time signal is denoted by E. It is given by,

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 \qquad ...(11)$$

The energy signal is as shown in Fig. A-11(b).

Note :

Signal	Condition
Energy signal	$0 < E < \infty$
Power signal	$0 < P < \infty$

Table A-1 : Comparison between energy and power signal

Sr. No.	Power signals	Energy signals
1.	The signal having finite non-zero power is called as power signal.	The signal having finite non-zero energy is called as energy signal.
2.	Almost all periodic signals in practice are power signals.	Almost all non-periodic signals are energy signals.
3.	Power signals can exist over an infinite time. They are not time limited.	Energy signals exist over a short period of time. They are time limited.
4.	Energy of a power signal is infinite.	Power of an energy signal is zero.

Prob. 1 : Obtain energy for the signal $x(n) = a^n u(n)$ where $|a| < 1$.

Soln. : For D.T. signal energy is given by,

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 \quad ...(1)$$

The given equation of $x(n)$ is,

$$x(n) = a^n u(n) \quad ...(2)$$

Here $u(n)$ represents unit step. Its value is unity (1) for the range 0 to ∞; that means for the positive values of 'n' only. Actually multiplication by $u(n)$ does not change the amplitude of a^n; but it indicates that a^n is present only from $n = 0$ to $n = \infty$.

Thus Equation (1) becomes,

$$E = \sum_{n=0}^{\infty} |a^n|^2 \quad ...(3)$$

Rearranging the term we get,

$$E = \sum_{n=0}^{\infty} [a^2]^n \quad ...(4)$$

Expanding Equation (4) we get,

$$E = [a^2]^0 + [a^2]^1 + [a^2]^2 + [a^2]^3 + ...$$

$$\therefore \quad E = 1 + [a^2]^1 + [a^2]^2 + [a^2]^3 + \ldots \quad ...(5)$$

Now we have the geometric series,

$$\sum_{n=0}^{\infty} A^n = 1 + A + A^2 + A^3 + \ldots = \frac{1}{1-A} \text{ if } |A| < 1.$$

Let $A = a^2$. Thus Equation (5) becomes,

$$E = \frac{1}{1-a^2} \text{ if } |a^2| < 1.$$

...Ans.

Prob. 2 : $x(n) = (0.5)^n u(n)$. State whether it is an energy or power signal. Justify.

Soln. :

First we will calculate the energy of signal $x(n)$. It is given by,

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 \quad \ldots(1)$$

The given signal is,

$$x(n) = (0.5)^n u(n) \quad \ldots(2)$$

Since it is multiplied by unit step; this signal is present from $n = 0$ to $n = \infty$. Thus Equation (1) becomes,

$$E = \sum_{n=0}^{\infty} \left[(0.5)^n \right]^2$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^{2n}$$

$$E = \sum_{n=0}^{\infty} \left(\frac{1}{4} \right)^n \quad \ldots(3)$$

We have the standard geometric series formula,

$$\sum_{n=0}^{\infty} A^n = 1 + A + A^2 + \ldots = \frac{1}{1-A}$$

$$\therefore \quad E = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

That means this is finite energy.

Justification :

If $0 < E < \infty$ then the signal is energy signal. Since the calculated value of energy is finite; the given signal is energy signal.

Prob. 3 : Determine the energy and power of signal given by,

$$x(n) = \left(\frac{1}{2}\right)^n \qquad n \geq 0$$
$$= 3^n \qquad n < 0$$

Soln. :

The energy of signal is given by,

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 \qquad ..(1)$$

The given signal is $x(n) = \left(\frac{1}{2}\right)^n$ for the range $n \geq 0$. That means for the range $n = 0$ to $n = \infty$. And $x(n) = 3^n$ for $n < 0$. That means for the range $n = -1$ to $n = -\infty$. Thus Equation (1) becomes,

$$E = \sum_{n=0}^{\infty} \left[\left(\frac{1}{2}\right)^n\right]^2 + \sum_{n=-1}^{-\infty} \left[3^n\right]^2$$

$$\therefore \quad E = \sum_{n=0}^{\infty} \left[\left(\frac{1}{2}\right)^2\right]^n + \sum_{n=-1}^{-\infty} \left[3^2\right]^n$$

$$\therefore \quad E = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n + \sum_{n=-1}^{-\infty} (9)^n \qquad ...(2)$$

Consider the first summation term. Using geometric series formula we can express it as follows :

$$\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{1-\frac{1}{4}} = \frac{4}{3} \qquad ...(3)$$

Now consider the second summation term. To make the limits of summation positive; put $n = -m$. Then the limits will change as follows.

When $n = -1 \Rightarrow -m = -1 \quad \therefore \; m = 1$

When $n = -\infty \Rightarrow -m = -\infty \quad \therefore \; m = \infty$

$$\therefore \sum_{n=-1}^{-\infty} (9)^n = \sum_{m=+1}^{\infty} (9)^{-m} = \sum_{m=1}^{\infty} \left(\frac{1}{9}\right)^m$$

Now use the standard summation formula,

$$\sum_{n=1}^{\infty} A^n = \frac{A}{1-A}$$

$$\therefore \sum_{m=1}^{\infty} \left(\frac{1}{9}\right)^m = \frac{1/9}{1-\frac{1}{9}} = \frac{\frac{1}{9}}{8/9}$$

$$\therefore \sum_{m=1}^{\infty} \left(\frac{1}{9}\right)^m = \frac{1}{8} \qquad ...(4)$$

Putting Equations (3) and (4) in Equation (2) we get,

$$E = \frac{4}{3} + \frac{1}{8}$$

$$\therefore E = \frac{35}{24}$$

This is the finite energy. That means the given signal is energy signal. We know that if the energy of signal is finite then its power is zero. Thus the power of given signal is zero.

$$\therefore P = 0$$

Prob. 4 : State whether unit step sequence is energy signal or power signal. Calculate corresponding energy or average power as the case may be

Soln. : The unit step sequence is power signal. We will prove it as follows :

Step I : Calculation of energy :

Energy of input sequence x (n) is given by,

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 \qquad ...(1)$$

Here x (n) = Unit step, u (n). Unit step is present in the range n = 0 to n = ∞

$$\therefore E = \sum_{n=0}^{\infty} |u(n)|^2 \qquad ..(2)$$

The value of unit step is 1 for energy value of n.

$$\therefore E = 1 + 1 + 1 + 1 \ldots\ldots \infty$$

This is infinite value.

Thus unit step is not an energy signal.

Step II : Calculation of power :

The average power of input signal is given by,

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 \quad ...(3)$$

But here $x(n)$ = Unit step, $u(n)$. Since the range of unit step is from $n = 0$ to $n = \infty$, we will change the limits of summation as follows :

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=0}^{N} |u(n)|^2$$

We know that $|u(n)| = 1$ and $|u(n)|^2 = 1^2 = 1$. Thus summation can be expanded as :

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \underbrace{[1+1+1.....+1]}_{n=0 \text{ to } N}$$

Thus we can write,

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=0}^{N} (1)^n \quad ...(4)$$

Now use standard equation of summation

$$\sum_{n=N_1}^{N_2} a^n = \begin{cases} \dfrac{a^{N_1} - a^{N_2+1}}{1-a} & \text{for } a \neq 1 \\ N_2 - N_1 + 1 & \text{for } a = 1 \end{cases} \quad ...(5)$$

Comparing summation term of Equation (4) with Equation (5),

$$a = 1, N_1 = 0 \text{ and } N_2 = N.$$

Thus Equation (4) becomes,

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \left[N_2 - N_1 + 1 \right] = \lim_{N \to \infty} \frac{1}{2N+1} [N - 0 + 1]$$

$$\therefore \quad P = \lim_{N \to \infty} \frac{N+1}{2N+1}$$

Dividing numerator and denominator by N we get,

$$P = \lim_{N \to \infty} \frac{1 + \frac{1}{N}}{2 + \frac{1}{N}} \qquad \therefore \quad P = \frac{1}{2}$$

Since the power is finite; unit step is power signal.

1.2.6 Deterministic and Random Signals :

Deterministic signal :

A signal which can be described by a mathematical expression, loop-up table or some well defined rule is called as the deterministic signal.

Examples :

Sine wave, cosine wave, square wave etc.

Fig. A-12(a) shows C.T. sine wave signal, which is deterministic signal. Because it can be represented mathematically as,

$$x(t) = A \sin(2\pi ft)$$

Here A = Amplitude of signal

f = Frequency of signal.

Similarly for D.T. wave we have,

$$x(n) = A \sin(2\pi fn)$$

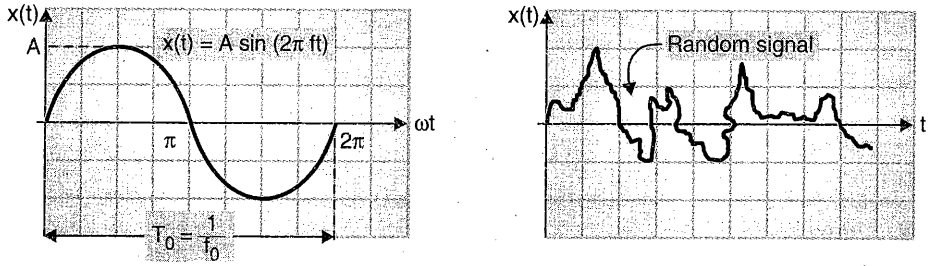
Note : The deterministic signals such as sine wave, cosine wave etc. are periodic. But some deterministic signals may not be periodic. Exponential signal is an example of non-periodic signal.

Random signal :

A signal which cannot be described by any mathematical expression is called as random signal. Due to this it is not possible to predict about the amplitude of such signals at a given instant of time.

Example :

A good example of random signal is "noise" in the communication system. Such a random signal is as shown in Fig. A-12(b).



(a) Deterministic signal (b) Random signal

Fig. A-12

1.2.7 Multichannel and Multidimensional Signals :

Multichannel signals :

- As the name indicates, multichannel signals are generated by multiple sources or multiple sensors.

- The resultant signal is the vector sum of signals from all channels.

Example :

A common example of multichannel signal is ECG waveform. To generate ECG waveform; different leads are connected to the body of a patient. Each lead is acting as individual channel. Since there are 'n' number of leads; the final ECG waveform is a result of multichannel signal. Mathematically final wave is expressed as,

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} ; \text{ if three leads are used.}$$

Multidimensional signals :

- If a signal is a function of single independent variable, the signal is called as one-dimensional signal. On the other hand, if the signal is a function of multi (many) independent variables then it is called as multidimensional signal.
- A good example of multidimensional signal is the picture displayed on the TV screen. To locate a pixel (a point) on the TV screen two co-ordinates namely X and Y are required. Similarly this point is a function of time also. So to display a pixel, minimum three dimensions are required; namely x, y and t. Thus this is multidimensional signal. Mathematically it can be written as, $P(x, y, t)$.

Comparison of multichannel and multidimensional signal :

Multichannel signal	Multidimensional signal
1. Such signals are generated by multiple sources or multiple sensors.	Such signals are function of many independent variables.
2. Example : ECG signal. Mathematically it is represented by, $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$	Example : Picture displayed on TV screen. Mathematically it is represented by $P(x, y, t)$.

Comparison of continuous valued and discrete valued signals :

Continuous valued signal	Discrete valued signal
1. Such signals are defined at each and every instant of time.	Such signals are defined at discrete instants of time.
2. Example : Sinewave.	Example : Signal represented by $x(n) = \{1, 1, 1, 2, 2, 2, 3, 3, 3, ...\}$

1.2.8 Summary of Classification of Signals :

Table A-2 gives quick recap of different signals.

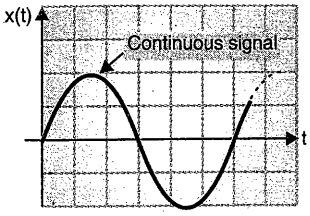
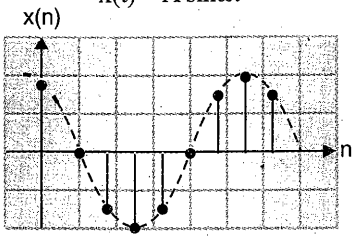
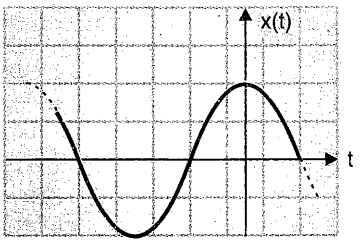
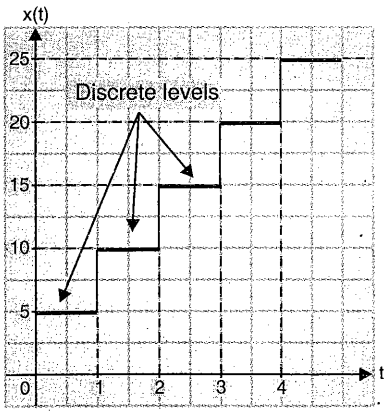
Sr. No.	Type	D. T. Waveform	C. T. Waveform
1.	Cotinuous or Discrete	• A signal of continuous amplitude or time is called as continuous signal. • A signal represented at discrete interval of time is called as discrete signal.	 $x(t) = A \sin\omega t$  $x(n) = A \cos\omega n$
2.	Continuous valued or Discrete valued	• If variation in the amplitude of signal is continuous then it is continuous valued signal. • If variation in amplitude of signal is not continuous but the signal has certain discrete amplitude levels then it is discrete amplitude signal.	 $x(t) = A \cos\omega t$  x(t) = 5 for t = 0 to t = 1, x(t) = 10 for t = 1 to t = 2 .....

Table A-2 contd...

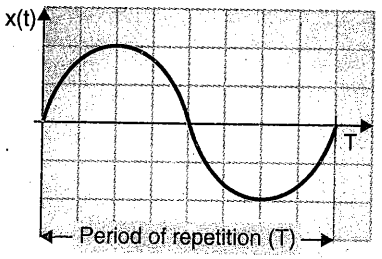
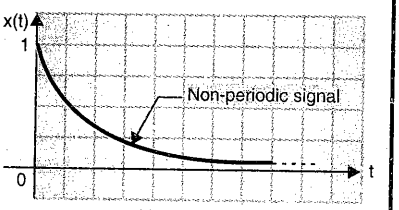
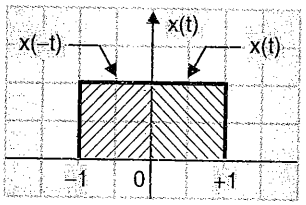
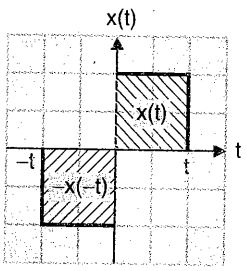
Sr. No.	Type	D. T. Waveform	C. T. Waveform
3.	Periodic or Non-periodic	• A signal which repeats itself after a fixed time interval is called as periodic signal.	 $x(t) = x(t + T_0)$
		• A signal which does not repeat itself after a fixed time period is called as non-periodic signal.	 $x(t) \neq x(t + T_0)$
4.	Even or Odd	• A signal is said to be even if it is symmetric about vertical axis or at t = 0.	 $x(t) = x(-t)$
		• A signal is said to be odd if it is not symmetric about vertical axis.	 $x(-t) = -x(t)$

Table A-2 contd...

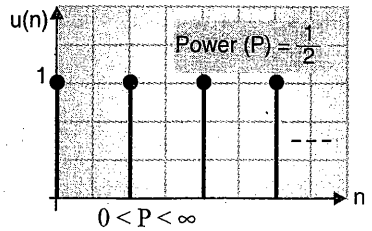
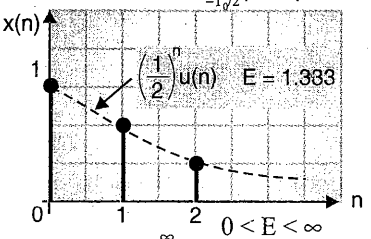
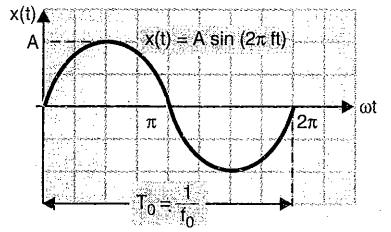
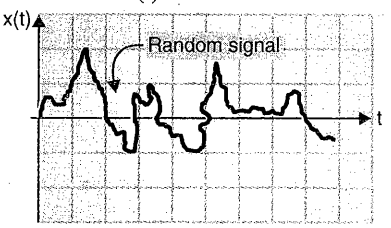
Sr. No.	Type	D. T. Waveform	C. T. Waveform
5.	Power or Energy	• A signal is said to be power signal if its normalized average power is non-zero and finite. • A signal is said to be an energy signal if its normalized energy is non-zero and finite.	 $P = 1/T_0 \int_{-T_0/2}^{T_0/2} \lvert x^2(t) \rvert dt$  $E = \int_{-\infty}^{\infty} \lvert x(t) \rvert^2 dt$
6.	Deterministic or Random	• A signal which can be described by a mathematical expression is called as deterministic signal. • A signal which cannot be described by any mathematical expression is called random signal.	 $x(t) = A\sin\omega t$ 
7.	Multichannel or Multidimensional	• A signal generated by multiple sources or multiple sensors is called as multichannel signal. • If a signal is function of many independent variables then it is called as multidimensional signal.	• ECG waveform generated by 'n' ECG electrodes. $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$ • Pixel displayed on TV screen P(x,y,t)

Table A-2

1.3 Representation of Discrete Time Sequences :

The discrete time sequence is denoted by x (n). Consider such a discrete time signal as shown in Fig. A-13.

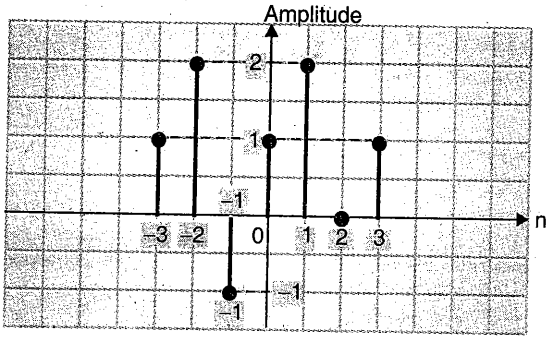


Fig. A-13 : Discrete time signal x (n)

On the X-axis index 'n' is plotted. Here 'n' is corresponding number of the sample. In the given diagram value of n varies from – 3 to + 3. On the Y-axis, amplitude of signal is plotted. The signal is having some amplitude at each value of n. Now the different methods used to represent the signal x (n) are as follows :

1. Functional representation 2. Tabular representation 3. Sequence representation

1. Functional representation : For the signal shown in Fig. A-13, the functional representation of signal is as follows :

$$x(n) = \begin{cases} 2 & \text{for } n = -2, 1 \\ 1 & \text{for } n = -3, 0, 3 \\ 0 & \text{for } n = 2 \\ -1 & \text{for } n = -1 \end{cases}$$

2. Tabular representation : The tabular representation of the given signal is as follows :

n	.....	– 3	– 2	– 1	0	1	2	3
x (n)	.....	1	2	– 1	1	2	0	1

Here the amplitude of signal is written below the corresponding value of n.

3. Sequence representation : The sequence representation of given signal is as follows :

$$x(n) = \{1, 2, -1, 1, 2, 0, 1\}$$
$$\uparrow$$

Here all the amplitudes of signal are written sequentially starting from the leftmost amplitude. *ARROW ALWAYS INDICATES THE AMPLITUDE OF SAMPLE AT n = 0.* If arrow is not shown in the sequence then by default it is at first position.

Example : $x(n) = \{1, 2, 3, 4, 5\}$

Here arrow is not shown; so by default it is at first sample.

That means we can write, $x(n) = \{1, 2, 3, 4, 5\}$
$$\uparrow$$

- Number of samples contained in the given sequence is called as the length of sample.
- To adjust the length of sequence we can add any number of zeros at the beginning or at the end sequence. *This is called as ZERO PADDING.*

Example : If $x(n) = \{1, 2, 0, 1, 2\}$ (↑ on 1)

Then we can write,

$x(n) = \{0, 0, 1, 2, 0, 1, 2\}$ or $x(n) = \{1, 2, 0, 1, 2, 0, 0\}$ (↑ on 1 in both)

- Remember that the position of pointer (arrow) does not change.

Prob. 1 : Represent the following signals graphically :

(i) $x(n) = \{1, 2, 0, -1, 1\}$ (↑ on 1)

(ii) $x(n) = \{0, 0, -1, 2, 3\}$ (↑ on 2)

(iii) $x(n) = \{0, 1, -1, 1, -1\}$ (↑ on 1)

Soln. : These signals are as shown in Fig. A-14(a), (b) and (c) respectively.

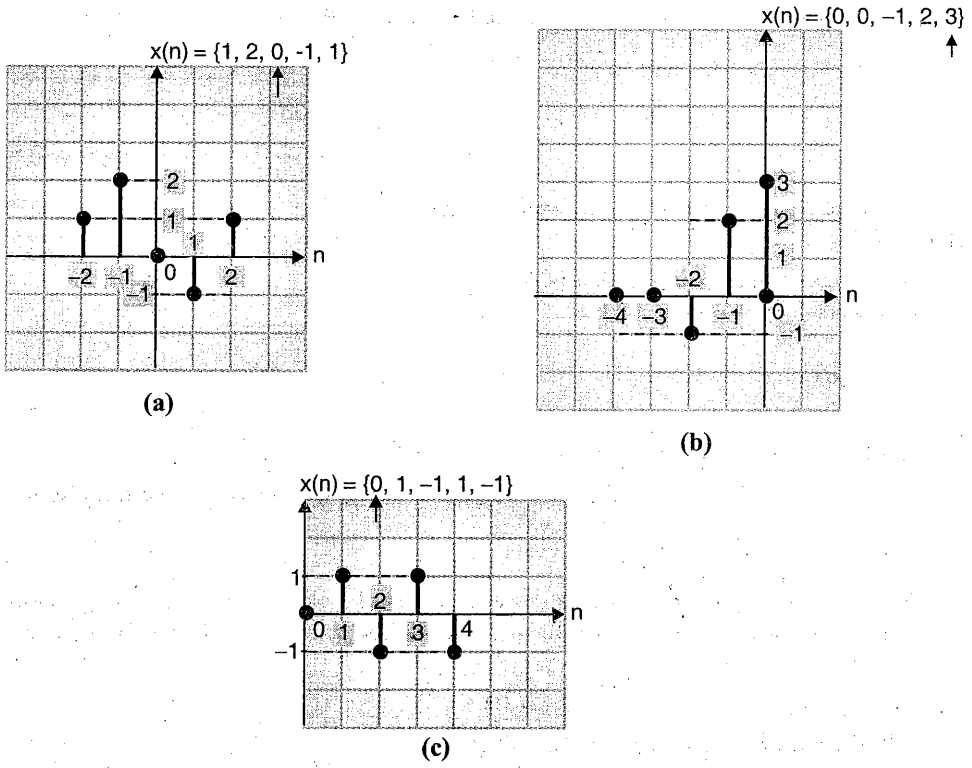


Fig. A-14 : Graphical representation of given sequences

1.4 Basic Sequences :

In the analysis of communication systems, standard test signals play a vital role. Such signals are used to check the performance of a system. Applying such signals at the system; the output is checked. Now depending on the input-output characteristics of that particular system; study of different properties of a system can be done. Some standard test signals are as follows :

- Delta or unit impulse function.
- Unit step signal
- Unit ramp signal
- Exponential signal
- Sinusoidal signal

1.4.1 Delta or Unit Impulse Function :

- A discrete time unit impulse function is denoted by $\delta(n)$. Its amplitude is 1 at n = 0 and for all other values of n; its amplitude is zero.

$$\therefore \quad \delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

In the sequence form it can be represented as,

$$\delta(n) = \{\ldots, 0, 0, 0, 1, 0, 0, 0\}$$
$$\uparrow$$

or $\delta(n) = \{1\}$

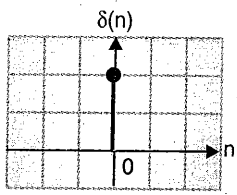


Fig. A-15(a) : Unit impulse function $\delta(n)$

The graphical representation of delta function is as shown in Fig. A-15(a).

- A continuous time delta function is denoted by $\delta(t)$. Mathematically it is expressed as,

$$\delta(t) = 0; \quad \text{For } t \neq 0.$$

It is as shown in Fig. A-15(b).

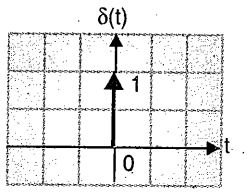


Fig. A-15(b) : Unit impulse function $\delta(t)$

1.4.2 Unit Step Signal :

- A discrete time unit step signal is denoted by u(n). Its value is unity (1) for all positive values of n. That means its value is one for $n \geq 0$. While for other values of n; its value is zero.

$$\therefore \quad u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

In the form of sequence it can be written as,

$$u(n) = \{1, 1, 1, 1, \ldots.\}$$

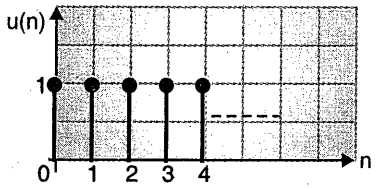
Here arrow is absent; so by default it is at first position.

Graphically, it is represented as shown in Fig. A-16(a).

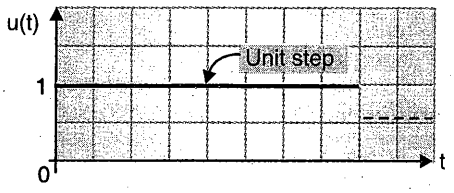
- A continuous time unit step is denoted by u(t). Mathematically it can be expressed as,

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

It is as shown in Fig. A-16(b).



(a) Unit step u(n)



(b) Unit step u(t)

Fig. A-16

1.4.3 Unit Ramp Signal :

- A discrete time unit ramp signal is denoted by $u_r(n)$. Its value increases linearly with sample number n. Mathematically it is defined as,

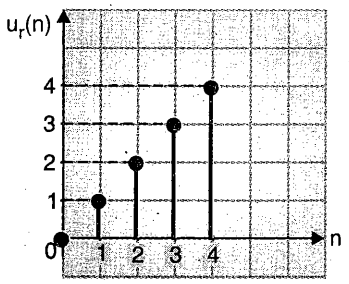
$$u_r(n) = \begin{cases} n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

Graphically it is represented as shown in Fig. A-17(a).

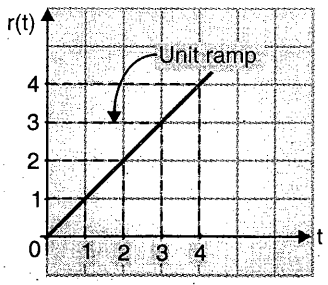
- A continuous time ramp signal is denoted by r(t). Mathematically it is expressed as,

$$r(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

It is as shown in Fig. A-17(b).



(a) Unit ramp signal $u_r(n)$



(b) Ramp signal r(t)

Fig. A-17

1.4.4 Exponential Signal :

A discrete time exponential signal is expressed as,

$$x(n) = a^n \quad \ldots(1)$$

Here 'a' is some real constant.

If 'a' is the complex number then x (n) is written as,

$$x(n) = re^{j\theta} \quad \ldots(2)$$

Here θ denotes the phase.

Now depending upon value of θ we have different cases.

Case (i) : When $a > 1$:

Let a = 3. Thus we have,

$$x(n) = a^n = 3^n.$$

Graphically such signal is represented as shown in Fig. A-18(a).

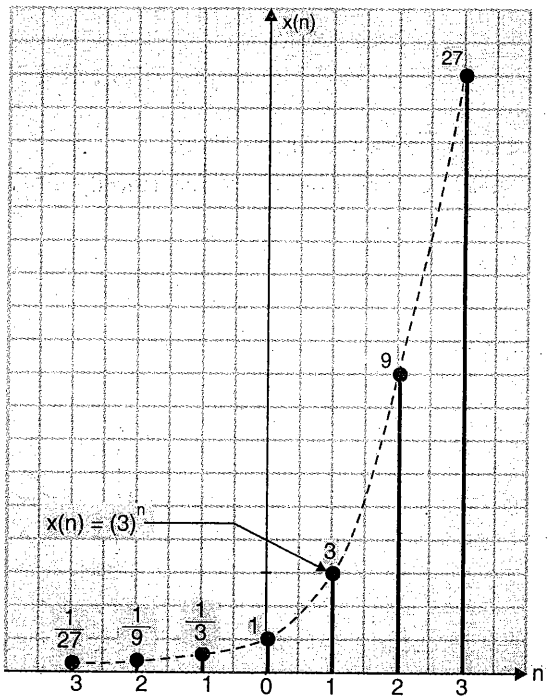


Fig. A-18(a) : Rising exponential signal

Since the signal is exponentially growing; it is called as rising exponential signal.

Case (ii) : When $0 < a < 1$

In this case we will get decaying exponential sequence. Let $a = \frac{1}{3}$

$$\therefore \quad x(n) = a^n = \left(\frac{1}{3}\right)^n.$$

Graphically such signal is represented as shown in Fig. A-18(b).

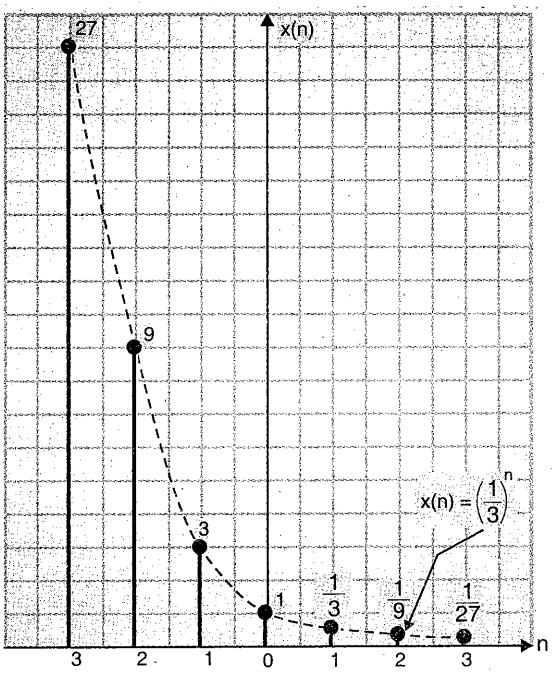


Fig. A-18(b) : Decaying exponential signal

Case (iii) : When $a < -1$

In this case we will get double sided rising exponential signal.

Let $a = -3$

$$\therefore \quad x(n) = (-3)^n$$

Graphically such signal is represented as shown in Fig. A-18(c).

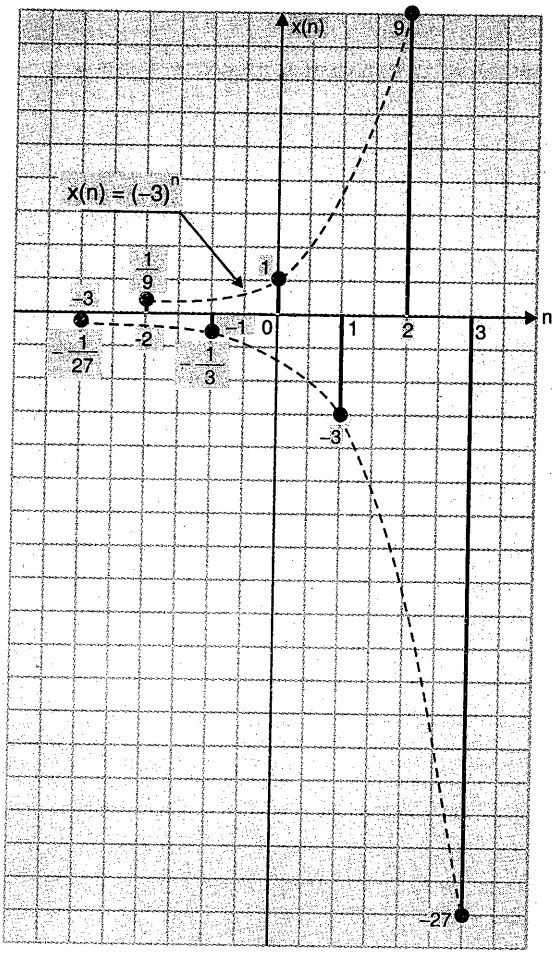


Fig. A-18(c) : Double sided growing exponential signal

Case (iv) : When $-1 < a < 0$

In this case we will get double sided decaying exponential signal.

Let $a = -\frac{1}{3}$

$$\therefore \quad x(n) = a^n = \left(-\frac{1}{3}\right)^n.$$

Graphically such signal is as shown in Fig. A-18(d).

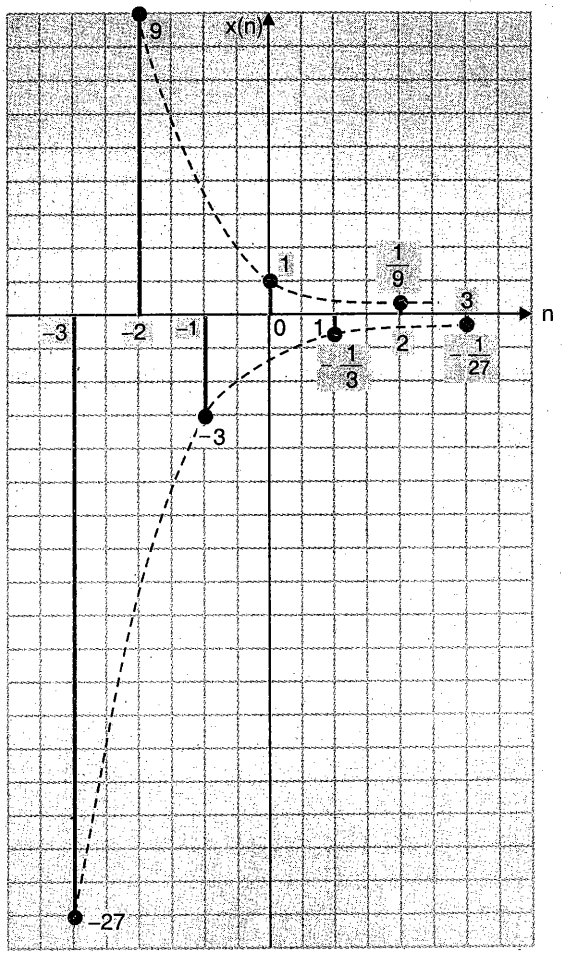


Fig. A-18(d) : Double sided decaying exponential signal

- A continuous time exponential signals for various values of a are as shown in Fig. A-19.

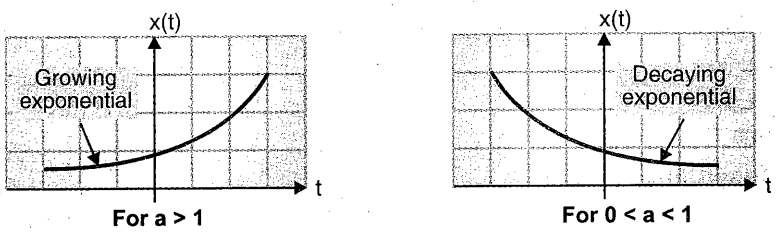
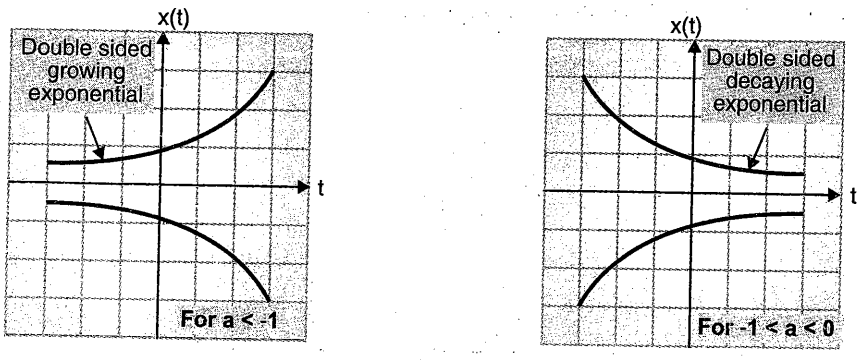


Fig. A-19 : Continuous time exponential signals



(c) (d)

Fig. A-19 : Continuous time exponential signals

1.4.5 Sinusoidal Waveform :

A discrete time sinusoidal waveform is denoted by,

$$x(n) = A \sin \omega n$$

Here A = Amplitude

ω = Angular Frequency = $2\pi f$

This waveform is as shown in Fig. A-20.

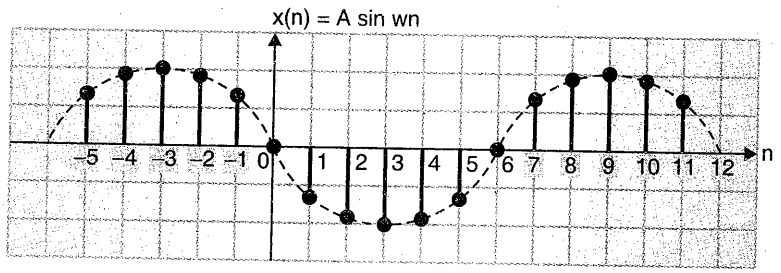


Fig. A-20 : Discrete time sinusoidal waveform

1.4.6 Some Other Important Signals :

Some other important signals are as follows :

1. Signum function
2. Rectangular pulse
3. Sinc function

1. Signum function :

Signum function is denoted by sgn (t). Mathematically it is represented as,

$$\text{sgn}(t) = \begin{cases} 1 & \text{for } t > 0 \\ -1 & \text{for } t < 0 \end{cases}$$

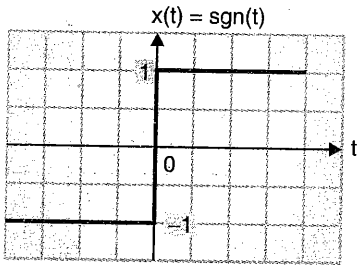


Fig. A-21(a) : Signum function

This function is as shown in Fig. A-21(a). This is an odd or antisymmetric function.

2. Rectangular pulse :

A unit rectangular pulse is centered about Y axis at t = 0. It is denoted by rect (t). Mathematically it is represented as,

$$\text{rect}(t) = \begin{cases} 1 & \text{for} -1/2 \le t \le \frac{1}{2} \\ 0 & \text{elsewhere} \end{cases}$$

A unit rectangular pulse is as shown in Fig. A-21(b).

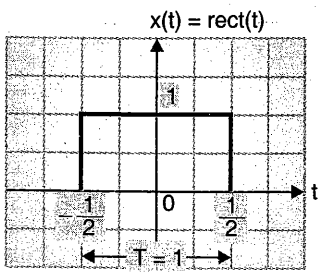
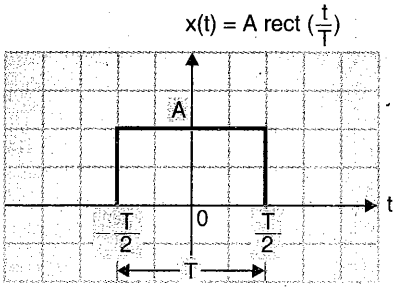


Fig. A-21(b) : Unit rectangular pulse



A rectangular pulse having amplitude 'A' and duration 'T' is as shown in Fig. A-21(c). It is denoted by $A\ \text{rect}\left(\frac{t}{T}\right)$. Mathematically it is expressed as,

$$A\ \text{rect}\left(\frac{t}{T}\right) = \begin{cases} A & \text{for} -\frac{T}{2} \le t \le \frac{T}{2} \\ 0 & \text{elsewhere} \end{cases}$$

Fig. A-21(c) : Rectangular pulse

3. Sinc function :

The sinc function is mathematically represented as,

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

Here x is an independent variable.

Sinc function passes through zero at multiple values of x that means at $x = \pm 1, \pm 2, \pm 3$ ..... This is because $\sin(\pi x) = 0$. For multiple values of x and the amplitude of "sinc" function decreases with increase in the magnitude of x.

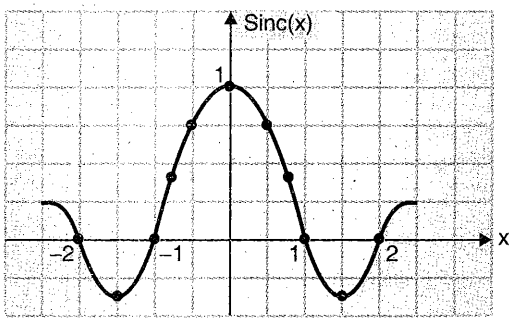


Fig. A-21(d)

1.4.7 Summary of Standard Signals :

Table A-3 gives the summary of standard signals.

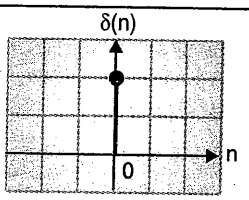
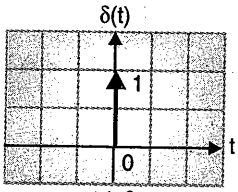
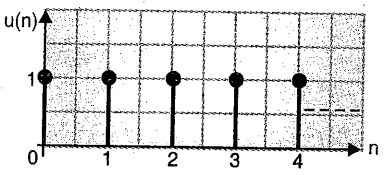
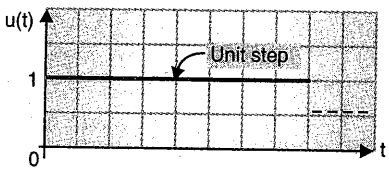
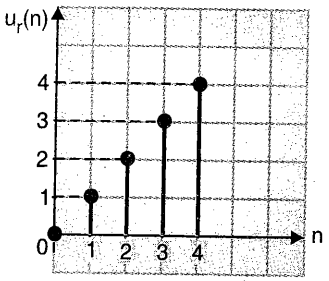
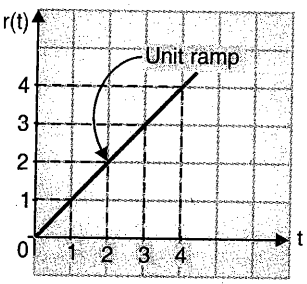
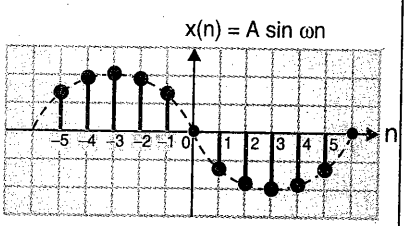
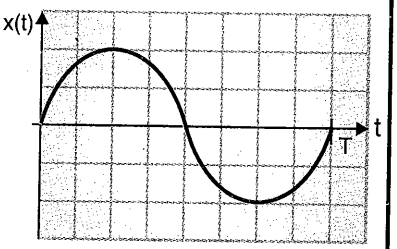
Sr. No.	Type	D. T. Waveform	C. T. Waveform
1.	Delta or unit impulse	 $d(n) = \begin{cases} 1 \text{ for } n = 0 \\ 0 \text{ for } n \neq 0 \end{cases}$	 $d(n) = \begin{cases} 1 \text{ for } t = 0 \\ 0 \text{ for } t \neq 0 \end{cases}$
2.	Unit step	 $u(n) = \begin{cases} 1 \text{ for } n \geq 0 \\ 0 \text{ for } n < 0 \end{cases}$	 $u(t) = \begin{cases} 1 \text{ for } t \geq 0 \\ 0 \text{ for } t < 0 \end{cases}$
3.	Unit ramp	 $u_r(n) = \begin{cases} n \text{ for } n \geq 0 \\ 0 \text{ for } n < 0 \end{cases}$	 $r(t) = \begin{cases} t \text{ for } t \geq 0 \\ 0 \text{ for } t < 0 \end{cases}$
4.	Sinusoidal	 x(n) = A sinwn	 x(t) = A sinwt

Table A-3 contd..

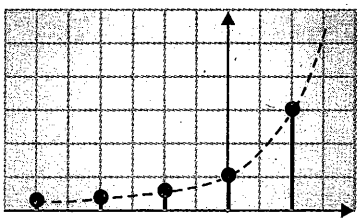
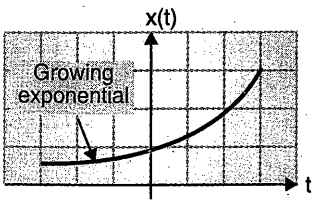
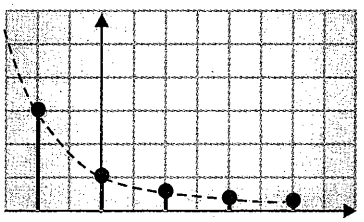
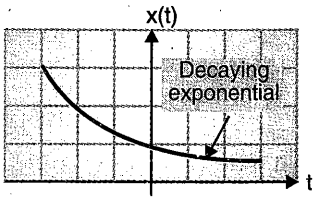
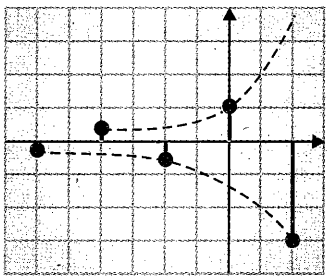
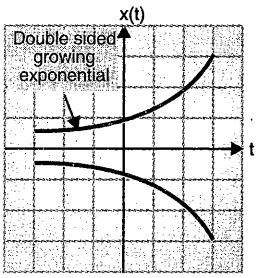
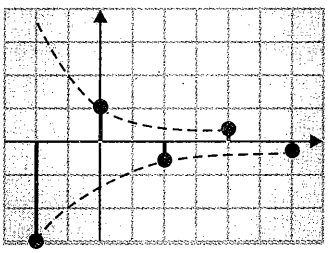
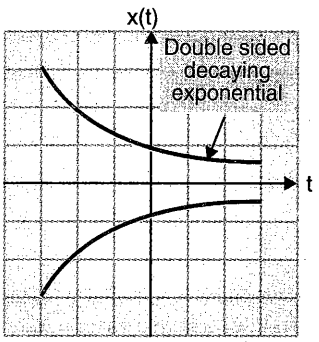
Sr. No.	Type	D. T. Waveform	C. T. Waveform
5.	Exponential		
(i)	When $a > 1$	 $x(n) = a^n$	 $x(t) = a^t$
(ii)	When $0 < a < 1$	 $x(n) = a^n$	 $x(t) = a^t$
(iii)	When $a < -1$	 $x(n) = a^n$	 $x(t) = a^t$
(iv)	When $-1 < a < 0$	 $x(n) = a^n$	 $x(t) = a^t$

Table A-3

1.5 Sequence Operations :

Many times it is necessary to modify the original signal. This modification is achieved by performing different operations on given discrete time signal. These operations are divided into following types :

- Time shifting operations
- Time scaling operations
- Amplitude scaling operations

1.5.1 Time Shifting Operations :

The different time shifting operations are as follows :

- Time delay
- Folding
- Folding and advance
- Time advance
- Folding and delay

1. Time Delay (T.D.) :

As the name indicates, delay means signals are not appearing instantly at the receiver. Almost every signal used in communication provides time delay at the output. Some common examples are TV signals, telephone signals, radar signals etc.

In case of discrete time signals, the given sequence can be delayed by few samples. We know that discrete time signal is denoted by $x(n)$. Suppose we want to delay this sequence by 'k' samples. It will be denoted by $x(n-k)$

$\therefore$ $x(n) \rightarrow$ Original sequence

and $x(n-k) \rightarrow$ Original sequence delayed by k samples.

Here k is an integer.

Example : Let the given signal be,

$$x(n) = \{1, 2, 3, 4, 5\}$$
$$\uparrow$$

This sequence is as shown in Fig. A-22(a).

Suppose we want to delay this sequence by 2 samples. Then here $k = 2$. Thus delayed sequence will be denoted by $x(n-2)$. This sequence will be shifted towards right by 2 samples. This delayed sequence is shown in Fig. A-22(b).

From Fig. A-22(b), we can write the delayed sequence as,

$$x(n-k) = x(n-2) = \{0, 0, 1, 2, 3, 4, 5\}$$
$$\uparrow$$

We know that, arrow always indicate the $n = 0$ sample. Since delayed sequence is shifted towards right; so the first sample starts at $n = 2$. In the original sequence it is starting at $n = 0$. Thus to delay the sequence,

- Shift the diagram towards right by 'k' samples.
- Shift the arrow towards left by 'k'.
- To adjust the length of sequence, add zeros.

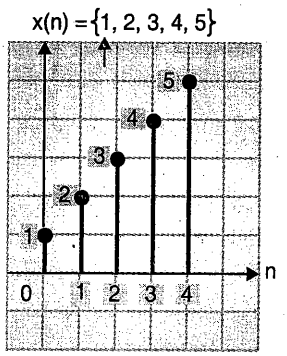


Fig. A-22(a) : $x(n) = \{1, 2, 3, 4, 5\}$ ↑

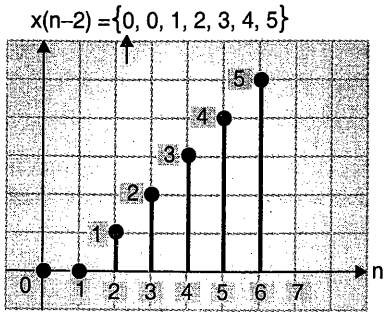


Fig. A-22(b) : Delayed version $x(n-k)$

2. Time Advance (T.A.) :

Time advance operation means the signals are present before they are generated. In a real time operation, this is practically impossible. But if the signals are stored in the memory of computer before starting a particular operation then it is possible to have time advanced signal.

Time advance operation is opposite to the time delay operation. Consider the same sequence $x(n)$ shown in Fig. A-22(a).

$$\therefore \quad x(n) = \{1, 2, 3, 4, 5\}$$
$$\uparrow$$

If we want to advance this sequence by two samples then it is denoted by $x(n+2)$. In this case the diagram is shifted towards left by two samples. This sequence is shown in Fig. A-22(c).

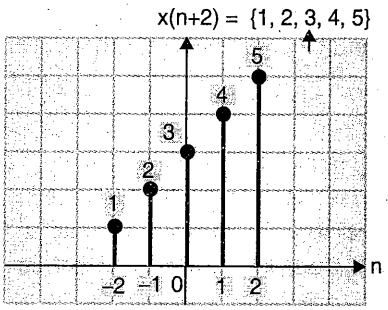


Fig. A-22(c) : Advanced version, $x(n+k)$

From Fig. A-22(c), we can write advanced sequence as,

$$x(n+k) = x(n+2) = \{1, 2, 3, 4, 5\}$$
$$\uparrow$$

3. Folding (FD) :

Folding is also called as reflection. So folding means taking the mirror image of signal. That means the signal is folded about time origin n = 0. Here independent variable 'n' is replaced by – n.

Thus if $x(n)$ represents input signal then $x(-n)$ represents folded input signal.

Consider the same input signal $x(n)$ as shown in Fig. A-22(a)

$$\therefore \quad x(n) = \{1, 2, 3, 4, 5\}$$

Then folded signal $x(-n)$ is as shown in Fig. A-22(d).

So the sequence $x(-n) = \{5, 4, 3, 2, 1\}$
$$\uparrow$$

Thus to obtain folded version of sequence,

- Take the mirror image of sequence about n = 0.
- Write the sequence in reverse order.
- Keep the position of arrow as it is. In the sequence $x(n)$ arrow is at 1 and in the sequence $x(-n)$ arrow is still at 1.

4. Folding and delay : This operation is a combination of folding and delaying operation. Now let us take a quick view at the modifications used for different operations.

(A) For drawing diagrams :

- To obtain delayed sequence, shift the original diagram towards right by 'k' samples.
- To obtain advanced sequence, shift the original diagram towards left by 'k' samples.
- To obtain the folded version, take the mirror image of the diagram at n = 0.

(B) For writing the sequence :

- The delayed version of $x(n)$ is denoted by $x(n-k)$
  Negative sign always indicates delay operation of $x(n)$.
- The advanced version of $x(n)$ is denoted by $x(n+k)$
  Positive sign always indicates advanced operation of $x(n)$.
- To obtain the folded version of $x(n)$; replace n by $-n$

Now folding and delay operation means :

- First fold the sequence $x(n)$; that means obtain $x(-n)$
- Then delay the folded sequence by k samples.

One major difference between $x(n)$ and $x(-n)$ $x(-n)$ denotes the mirror image of $x(n)$. We know that in case of mirror image; left and right sides are reversed. Now if $x(n)$ is original sequence then its delayed version is denoted by $x(n-k)$. The folded sequence is denoted by $x(-n)$. Since it is a mirror image the delayed version of folded signal is denoted by $x(-n+k)$.

$$\therefore \quad x(n) \xrightarrow{\text{Delay}} x(n-k)$$

$$\text{Folding} \downarrow$$

$$x(-n) \xrightarrow{\text{Delay}} x[-(n-k)] = x(-n+k)$$

But stick to the basic concepts. Delay means shift the diagram towards right by k samples.

Consider the same original sequence $x(n) = \{1, 2, 3, 4, 5\}$ as shown in Fig. A-22(a). The
$\uparrow$

folded sequence $x(-n)$ is shown in Fig. A-22(d). Suppose we want to delay this folded sequence $x(-n)$ by '2' samples then it will be denoted by $x(-n+2)$. This sequence is as shown in Fig. A-22(e).

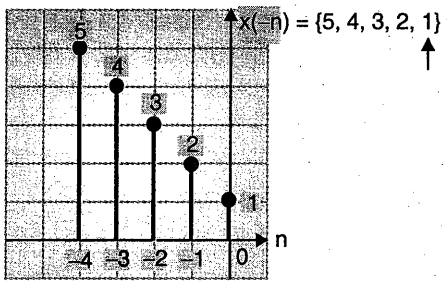


Fig. A-22(d) : Folded version x (− n)

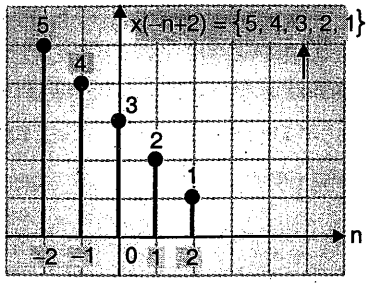


Fig. A-22(e) : Delay of folded sequence x (− n + 2)

From Fig. A-22(e), the sequence x (− n + 2) can be written as,

$$x(-n+2) = \{5, 4, 3, 2, 1\}$$
$$\uparrow$$

5. Folding and advance : The advanced version of original sequence x (n) is denoted by x (n + k). The folded version of x (n) is denoted by x (− n). Since folding means mirror image of sequence; the advanced version of folded sequence is denoted by x (− n − k).

$$\therefore \quad x(n) \xrightarrow{\text{Advance}} x(n+k)$$

$$\text{Folding} \downarrow$$

$$x(-n) \xrightarrow{\text{Advance}} x[-(n+k)] = x(-n-k)$$

Consider the same folded sequence x (− n) shown in Fig. A-22(d). Suppose we want to advance this sequence by '2' samples, then the advanced version is denoted by x (− n − 2). Such a sequence is as shown in Fig. A-22(f).

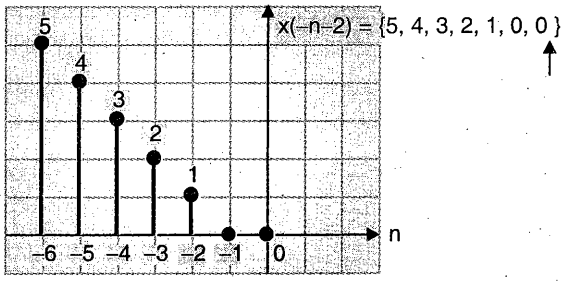


Fig. A-22(f) : Advance of folded sequence, x (− n − 2)

Remember the basic rule. Advancing the sequence means shifting the diagram towards left by 'k' samples. From Fig. A-22(f), we can write the sequence x (− n − 2) as,

$$x(-n-2) = \{5, 4, 3, 2, 1, 0, 0\}$$
$$\uparrow$$

Prob. 1 : Prove that folding and time delaying or advancing of a signal are not commutative operations.

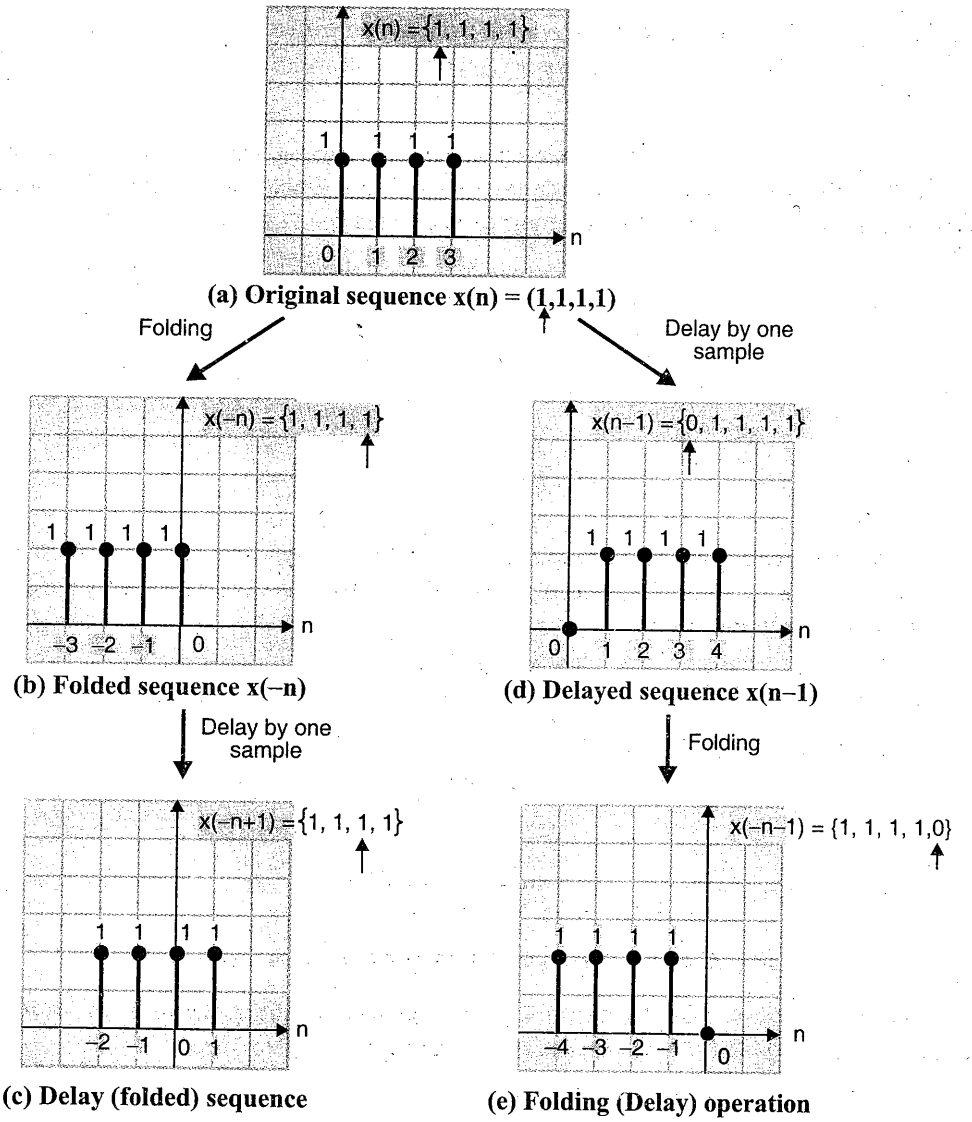
Soln. : In this case we have to prove,

$$\text{Delaying (Folding)} \neq \text{Folding (Delaying)} \quad \ldots(1)$$

Let the input sequence be,

$$x(n) = \{1, 1, 1, 1\}$$
$$\uparrow$$

This sequence is represented in Fig. A-23(a)



(a) Original sequence x(n) = (1,1,1,1)

(b) Folded sequence x(–n)

(d) Delayed sequence x(n–1)

(c) Delay (folded) sequence

(e) Folding (Delay) operation

Fig. A-23

Now consider L.H.S.

Step I : First fold the sequence x (n). The folded sequence x (– n) is shown in Fig. A-23(b).

Step II : Delay this sequence by 1 sample. This gives delaying of folding operation. This sequence is shown in Fig. A-23(c).

Step III : Now consider the R.H.S.
First delay the sequence x (n) by one sample. This sequence is as shown in Fig. A-23(d).

Step IV : Now take the folded version of signal x (n – 1). This gives folding of delaying operation. Such sequence is as shown in Fig. A-23(e).

Compare Fig. A-23(c) and Fig. A-23(e). Since these two diagrams are not same so;

Delaying (Folding) ≠ Folding (Delaying)

Hence proved.

1.5.2 Time Scaling Operations :

As the name indicates, time scaling operations are related to the change in time scale. There are two types of time scaling operations.

- Down scaling (Compression)
- Up scaling (Expansion)

1. Down Scaling or Compression :

Consider the same sequence x (n) = {1, 2, 3, 4, 5}. It is shown in Fig. A-22(a). Let this
↑

x (n) be input to some device which produces output y (n). Also say,

$$y(n) = x(2n)$$

Now from given sequence x (n) we can write,

x (0) = 1 x (1) = 2
x (2) = 3 x (3) = 4
x (4) = 5 x (5) = 0 ...

This gives the value of x (n) for different values of n.

Now we have, $y(n) = x(2n)$

$\therefore \quad y(0) = x(0) = 1$

$y(1) = x(2) = 3$

$y(2) = x(4) = 5$

$y(3) = x(6) = 0$

$\therefore \quad y(n) = x(2n) = \{1, 3, 5, 0, ...\}$
↑

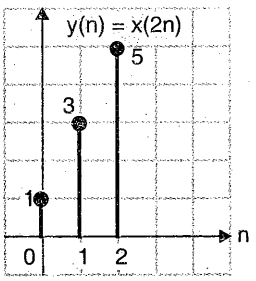


Fig. A-24(a) : Downsampling

This sequence y (n) = x (2n) is represented as shown in Fig. A-24(a).

Compare this diagram with original signal x (n) shown in Fig. A-22(a). Since the scale is compressed this is called as down scaling operation.

2. Up scaling or expansion :

Consider same input sequence $x(n) = \{1, 2, 3, 4, 5\}$ is applied to certain device which produces output $y(n) = x\left(\frac{n}{2}\right)$. Thus in this case,

$$y(0) = x\left(\frac{0}{2}\right) = x(0) = 1$$

$$y(1) = x\left(\frac{1}{2}\right) \rightarrow \text{No sample}$$

$$y(2) = x\left(\frac{2}{2}\right) = x(1) = 2$$

$$y(3) = x\left(\frac{3}{2}\right) = x(1.5) \rightarrow \text{No sample}$$

$$y(4) = x\left(\frac{4}{2}\right) = x(2) = 3$$

$$y(5) = x\left(\frac{5}{2}\right) = x(2.5) \rightarrow \text{No sample}$$

$$y(6) = x\left(\frac{6}{2}\right) = x(3) = 4$$

$$y(7) = x\left(\frac{7}{2}\right) = x(3.5) \rightarrow \text{No sample}$$

$$y(8) = x\left(\frac{8}{2}\right) = x(4) = 5$$

Thus $y(n) = x\left(\frac{n}{2}\right) = \{1, 0, 2, 0, 3, 0, 4, 0, 5\}$

This sequence is shown in Fig. A-24(b).

Compare this diagram with Fig. A-22(a). Since the scale is expanded, this is called as upscaling operation.

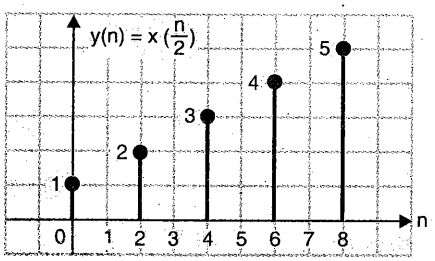


Fig. A-24(b) : Upsampling

1.5.3 Amplitude Scaling Operation :

As the name indicates, in case of amplitude scaling operations, amplitude of signal is changed. Different amplitude scaling operations are as follows :

- Upscaling (Amplification)
- Downscaling (Attenuation)
- Addition
- Multiplication

1. Upscaling or amplification : Consider same input signal x (n) shown in Fig. A-25(a).

$$\therefore \quad x(n) = \{1, 2, 3, 4, 5\}$$
$$\uparrow$$

Here $x(0) = 1, \quad x(1) = 2, \quad x(2) = 3,$
$x(3) = 4, \quad x(4) = 5.$

Now let, $y(n) = 2x(n)$

This indicates multiplication of every amplitude by 2.

$$\therefore \quad y(n) = 2x(n) = \{2, 4, 6, 8, 10\}$$
$$\uparrow$$

This sequence is shown in Fig. A-25(a)

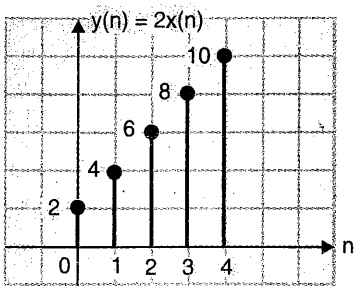


Fig. A-25(a) : Upscaling

2. Downscaling or attenuation : Downscaling or attenuation means reducing the amplitude of signal. Consider same sequence $x(n) = \{1, 2, 3, 4, 5\}$
$\uparrow$

$$\text{Let} \quad y(n) = \frac{x(n)}{2}$$

This operation indicates the division of every amplitude by 2.

$$\therefore \quad y(n) = \frac{x(n)}{2} = \{0.5, 1, 1.5, 2, 2.5\}$$
$$\uparrow$$

This sequence is shown in Fig. A-25(b).

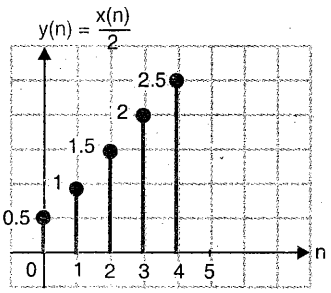


Fig. A-25(b) : Downscaling

3. Addition : Consider two sequences,

$$x_1(n) = \{1, 1, 0, 1, 1\} \quad \text{and} \quad x_2(n) = \{2, 2, 0, 2, 2\}$$
$$\uparrow \qquad\qquad\qquad\qquad \uparrow$$

$$\text{Let} \quad y(n) = x_1(n) + x_2(n)$$

$$\therefore \quad y(n) = \{3, 3, 0, 3, 3\}$$
$$\uparrow$$

Addition operation is shown in Fig. A-26(a).

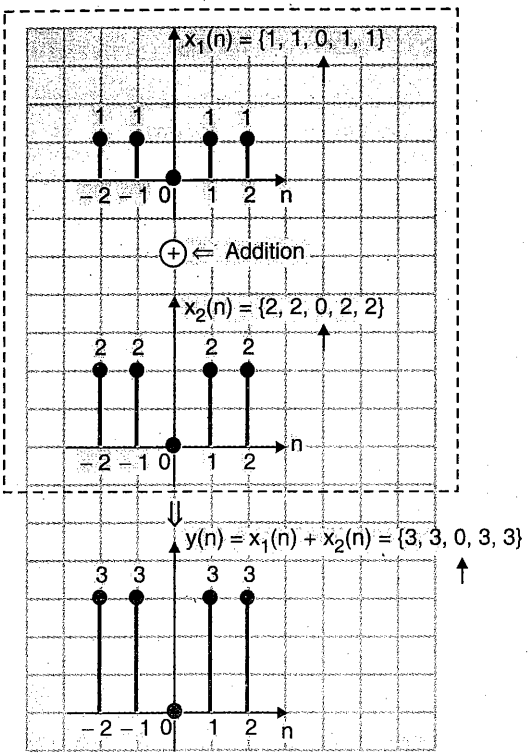


Fig. A-26(a) : Addition of two sequences

4. Multiplication :

Consider $x_1(n) = \{1, 1, 0, 1, 1\}$ ↑

and $x_2(n) = \{2, 2, 0, 2, 2\}$ ↑

Let $y(n) = x_1(n) \times x_2(n)$

$\therefore \quad y(n) = \{2, 2, 0, 2, 2\}$ ↑

Multiplication operation is as shown in Fig. A-26(b).

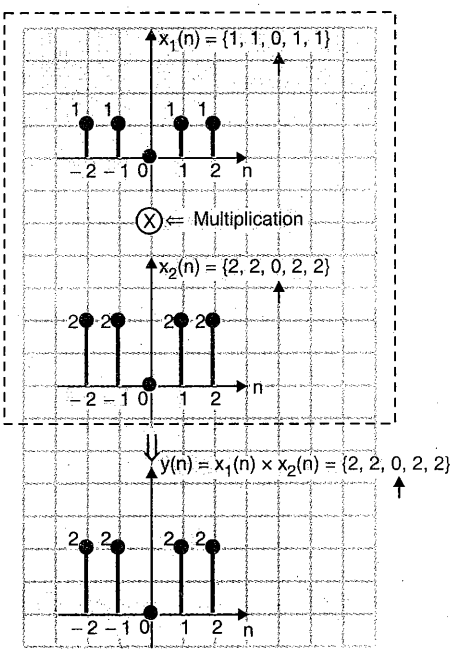


Fig. A-26(b) : Multiplication of two sequences

Prob. 1 : A discrete time signal is given by,

$$x(n) = \{1, 1, \underset{\uparrow}{1}, 1, 2\}$$

Sketch the following signals :

(a) $x(n-2)$ (b) $x(n+1)$ (c) $x(3-n)$

(d) $x(n)\,u(n-1)$ (e) $x(n-1)\cdot\delta(n-1)$

(f) Even samples of $x(n)$ (g) Odd samples of $x(n)$

Soln. : The given sequence is,

$$x(n) = \{1, 1, \underset{\uparrow}{1}, 1, 2\}$$

We know that arrow indicates the position of sample at $n = 0$. Thus the value of $x(n)$ for different values of 'n' are as follows :

$x(-1) = 1$, $x(0) = 1$, $x(1) = 1$,

$x(2) = 1$ and $x(3) = 2$

This sequence is shown in Fig. A-27(a).

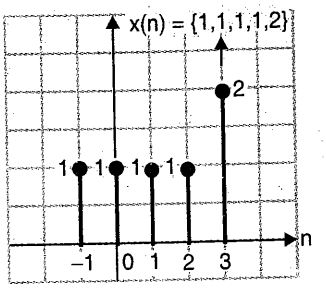


Fig. A-27(a) : Given sequence $x(n)$

(a) $x(n-2)$:

Here negative sign indicates delay operation. Delay operation means shift the diagram towards right. So we have to shift given sequence $x(n)$ towards right by 2 samples. This sequence is shown in Fig. A-27(b).

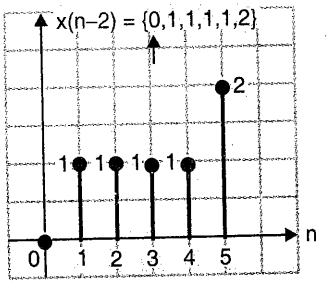


Fig. A-27(b) : Sequence $x(n-2)$

(b) $x(n+1)$:

Here positive sign indicates advance operation. Advance operation means shift the diagram towards left. So we have to shift given sequence $x(n)$ towards left by 1 sample. This sequence is shown in Fig. A-27(c).

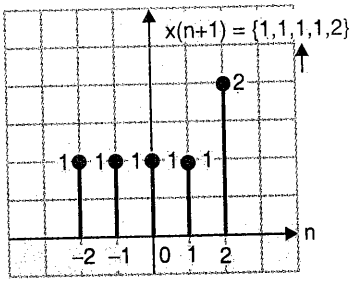


Fig. A-27(c) : Sequence $x(n+1)$

(c) $x(3-n)$:

This sequence can be written as $x(-n+3)$.

- Here '$-n$' $\Rightarrow$ folding operation.
- Remember that in case of folded signal, positive sign $\Rightarrow$ delay operation. Delay means shift the diagram towards right.

- So fold the sequence x (n) to obtain x (– n) and then shift this diagram towards right by 3 samples.

  This sequence is shown in Fig. A-27(d).

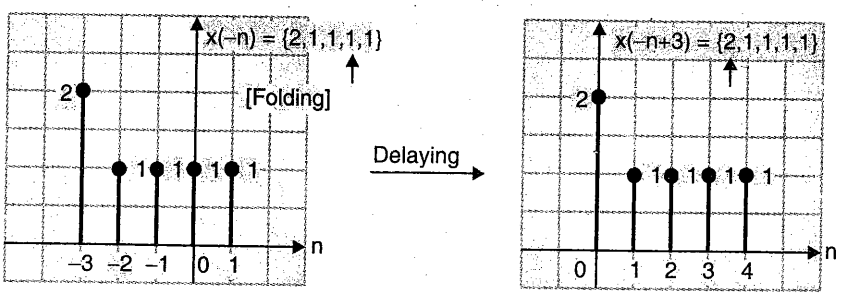


Fig. A-27(d) : Folding and delay operation, x (3 – n)

(d) x (n) u (n – 1) :

- This operation indicates multiplication of x (n) and u (n – 1).

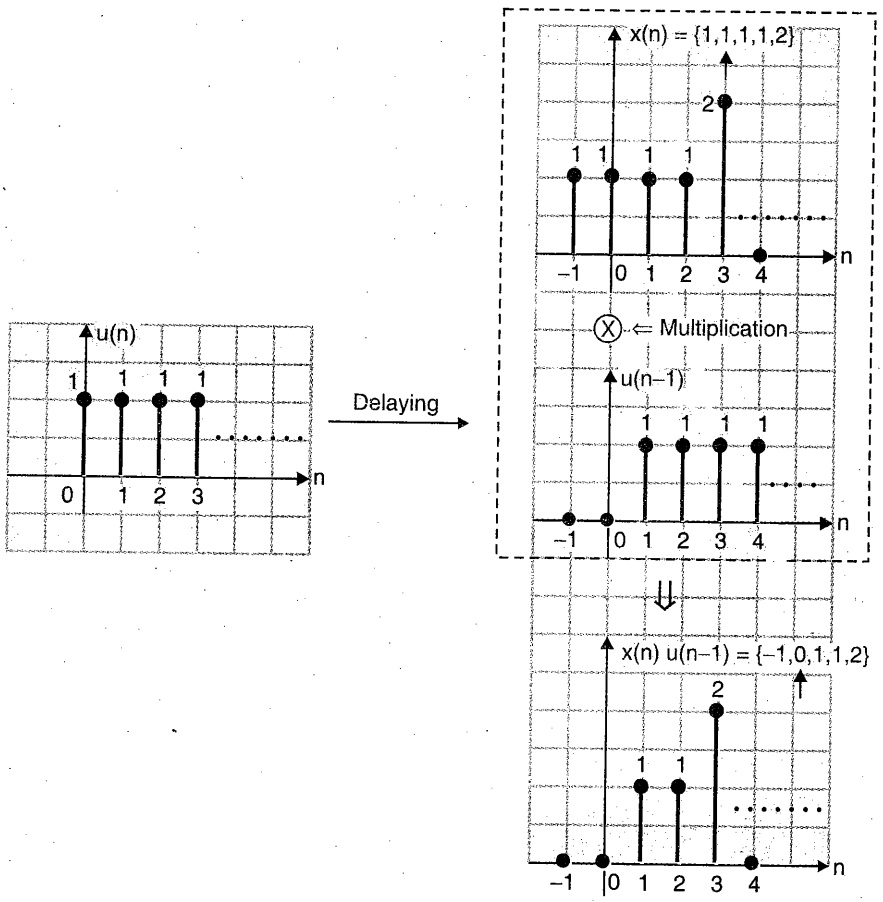


Fig. A-27(e) : Multiplication of x (n) and u (n – 1)

- x (n) is shown in Fig. A-27(a).
- u (n) means unit step and u (n – 1) ⇒ unit step delayed by 1 sample.
- Draw u (n – 1) and then take the multiplication, x (n) u (n – 1). The multiplication takes place sample to sample.
- This operation is shown in Fig. A-27(e).

(e) x (n – 1) δ (n – 1) :

- This operation indicates multiplication of x (n – 1) and δ (n – 1).
- x (n – 1) ⇒ delay of x (n) by 1 sample.
- δ (n) is unit impulse and δ (n – 1) is delayed unit impulse.
- The multiplication takes place sample to sample.
- This operation is shown in Fig. A-27(f).

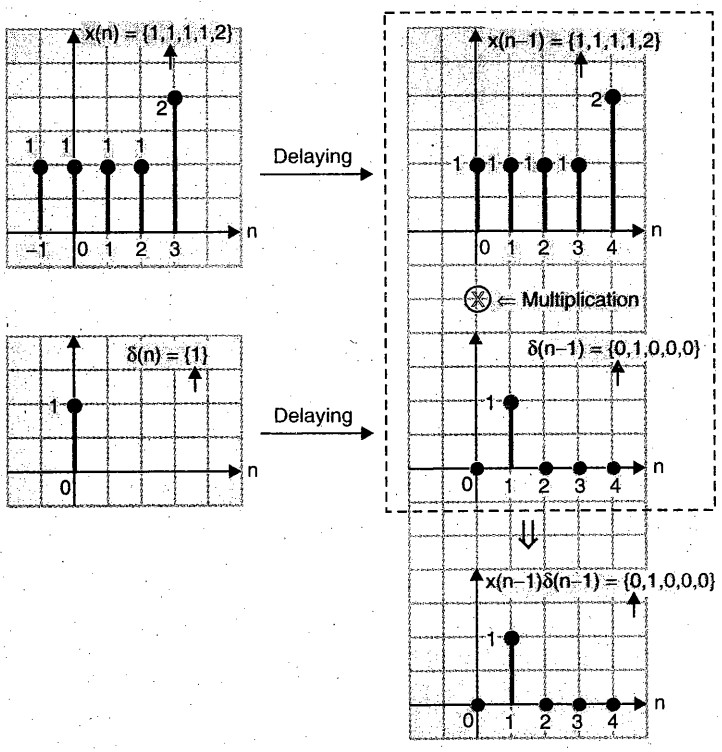


Fig. A-27(f) : Multiplication of x (n – 1) and δ (n – 1)

(f) Even samples of x (n) :

It means we have to find out x (2n).

Given sequence is, $x(n) = \{1, 1, 1, 1, 2\}$

↑

Here $x(-1) = 1$, $x(0) = 1$, $x(1) = 1$, $x(2) = 1$, $x(3) = 2$

We want x (2n). Putting different values of n we get,

at n = − 1, x (2n) = x (− 2) = 0
at n = 0, x (2n) = x (0) = 1
at n = 1, x (2 × 1) = x (2) = 1
at n = 2, x (2 × 2) = x (4) = 0

After this every value of x (2n) will be zero. This is because the given sequence is upto x (3).

$\therefore x(2n) = \{0, 1, 1, 0, 0 \ldots.\}$
$\uparrow$

This sequence is shown in Fig. A-27(g).

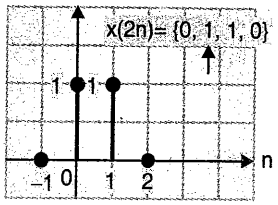


Fig. A-27(g) : Even samples of x (n)

(g) Odd samples of x (n) :

It means we have to find out x (2n + 1).

We want x (2n + 1). Putting different values of n we get,

at n = − 1, x (2n + 1) = x (− 2 + 1) = x (− 1) = 1
at n = 0, x (2n + 1) = x (0 + 1) = x (1) = 1
at n = 1, x (2n + 1)
= x (2 + 1) = x (3) = 2
at n = 2, x (2n + 1) = x (4 + 1)
= x (5) = 0

After this every value of x (2n + 1) will be zero.

$\therefore x(2n+1) = \{1, 1, 2, 0 \ldots\}$
$\uparrow$

This sequence is shown in Fig. A-27(h).

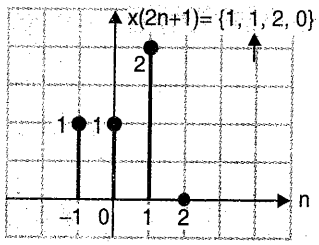


Fig. A-27(h) Odd samples of x (n)

Prob. 2 : Sketch a discrete time signal $x(n) = 2^{-n}$ for $-2 \le n \le 2$ and obtain :

(i) $y_1(n) = 2x(n) + \delta(n)$

(ii) $y_2(n) = x(n) \cdot u(2 - n)$

Soln. : The given discrete time signal is,

$x(n) = 2^{-n}$ for $-2 \le n \le 2$

That means the range of 'n' is from − 2 to + 2. Putting these values of 'n' in the equation of x (n) we get,

at n = − 2, $x(n) = 2^{-n} = 2^{+2} = 4$
at n = − 1, $x(n) = 2^{-n} = 2^{+1} = 2$
at n = 0, $x(n) = 2^{-n} = 2^{0} = 1$
at n = 1, $x(n) = 2^{-n} = 2^{-1} = \frac{1}{2}$
at n = 2, $x(n) = 2^{-n} = 2^{-2} = \frac{1}{4}$

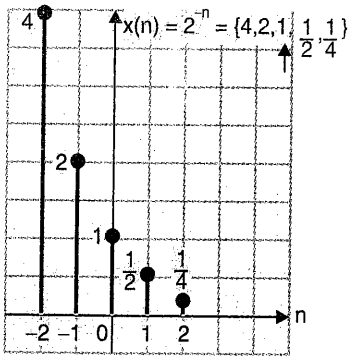


Fig. A-28(a) : Discrete time signal $x(n) = 2^{-n}$

Thus the sequence can be written as,

$$x(n) = \left\{ 4, 2, 1, \frac{1}{2}, \frac{1}{4} \right\}$$
$$\uparrow$$

This sequence is shown in Fig. A-28(a).

(i) $y_1(n) = 2x(n) + \delta(n)$:

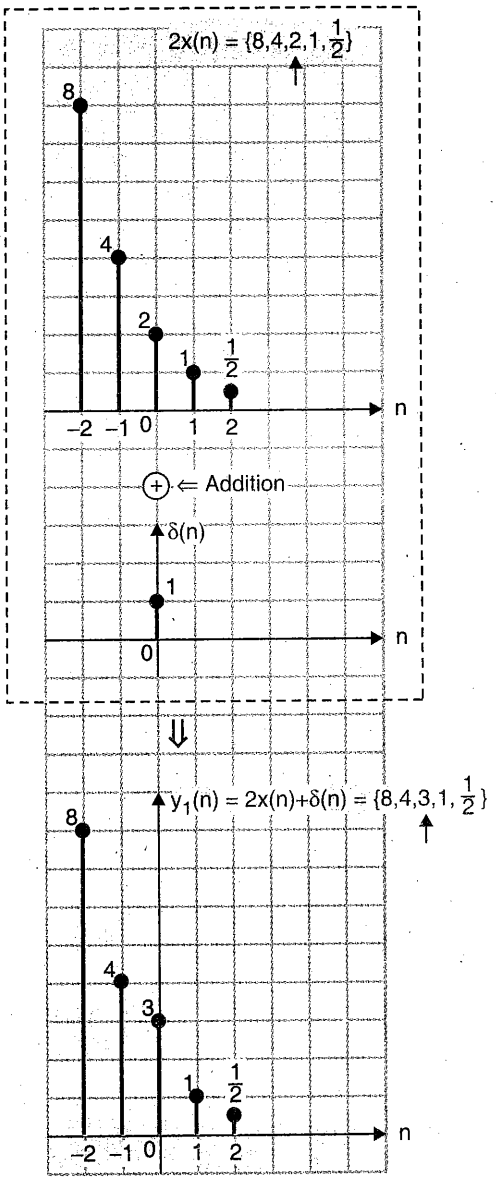


Fig. A-28(b) : $y_1(n) = 2x(n) + \delta(n)$

Here $\delta(n)$ is unit impulse. It is expressed as,

$$\delta(n) = \{1\}$$
$$\uparrow$$

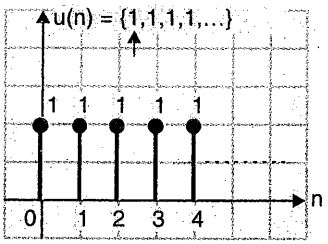
That means at $n = 0$, $\delta(n) = 1$; while for all other values of n, $\delta(n) = 0$. Now we have to calculate,

$$y_1(n) = 2x(n) + \delta(n)$$

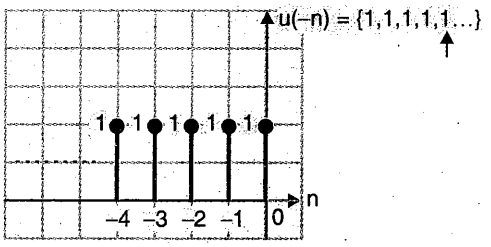
$2x(n)$ means, multiply each sample of $x(n)$ by 2. To obtain $y_1(n)$ add $2x(n)$ and $\delta(n)$. The addition takes place sample to sample. This operation is shown in Fig. A-28(b).

(ii) $y_2(n) = x(n) \cdot u(2-n)$:

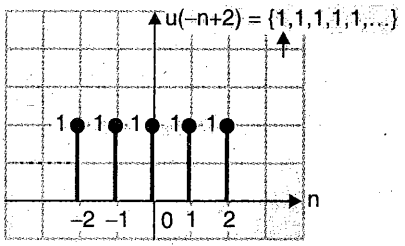
We know that $u(n)$ is the unit step. It is as shown in Fig. A-28(c). We want $u(2-n)$ that means $u(-n+2)$. Now $u(-n)$ indicates folded unit step as shown in Fig. A-28(d). While $u(-n+2)$ indicates that we have to delay $u(-n)$ by two samples. That means shift $u(-n)$ towards right by two samples. This is shown in Fig. A-28(e). $x(n) \cdot u(-n+2)$ indicates multiplication of $x(n)$ and $u(-n+2)$. This operation is shown in Fig. A-28(f).



(c) Unit step $u(n)$



(d) Folded unit step $u(-n)$



(e) $u(-n+2)$ delayed by two samples

Fig. A-28

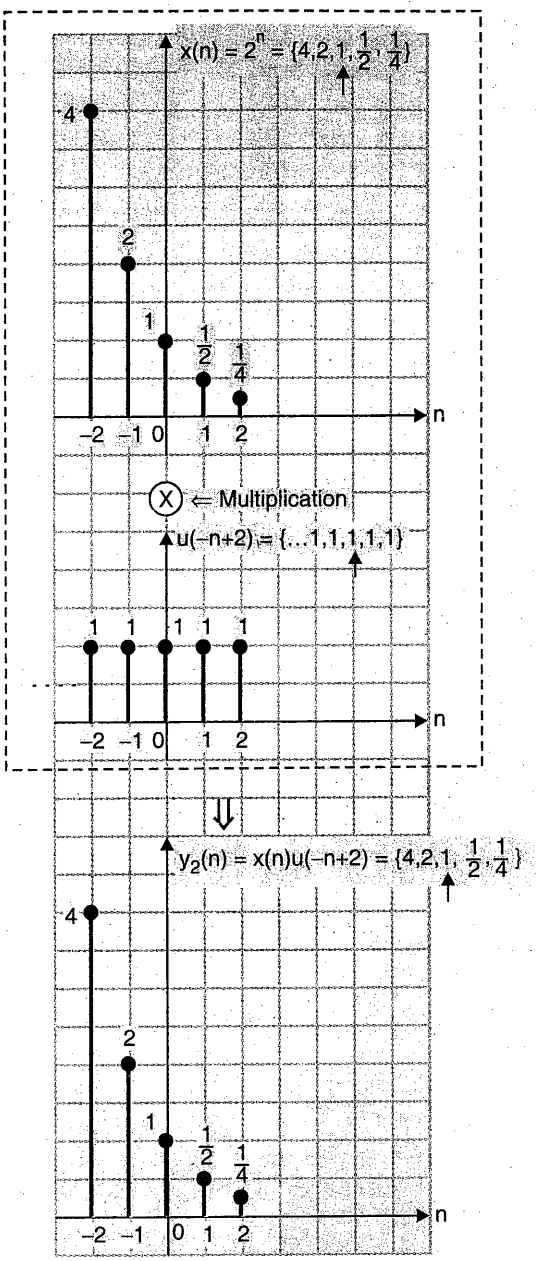


Fig. A-28(f) : $y_2(n) = x(n)\,u(-n+2)$

Prob. 3 : Generate a pulse signal (one sided) starting from n = 0 of duration N using

(i) Unit sample response.

(ii) Unit step sequences.

Soln. :

(i) We know that unit sample response is,

$$\delta(n) = 1 \quad \text{for } n = 0$$
$$= 0 \quad \text{otherwise}$$

Let the signal to be generated be $x(n)$. The duration of this signal is from $n = 0$ to $n = N$ and the amplitude of every sample is 1. That means the required sequence is,

$$x(n) = \{1, 1, 1, 1, \ldots. 1\}$$
$$\uparrow \qquad \uparrow$$
$$n = 0 \qquad n = N-1$$

Thus it can be expressed in terms of unit sample response as,

$$x(n) = \sum_{k=0}^{N-1} \delta(n-K)$$

(ii) Unit step starts from $n = 0$ and every sample is having magnitude equals to 1 till $n = \infty$. But we want the sequence having samples of magnitude 1 till $n = N$. This is obtained as shown in Fig. A-28(g).

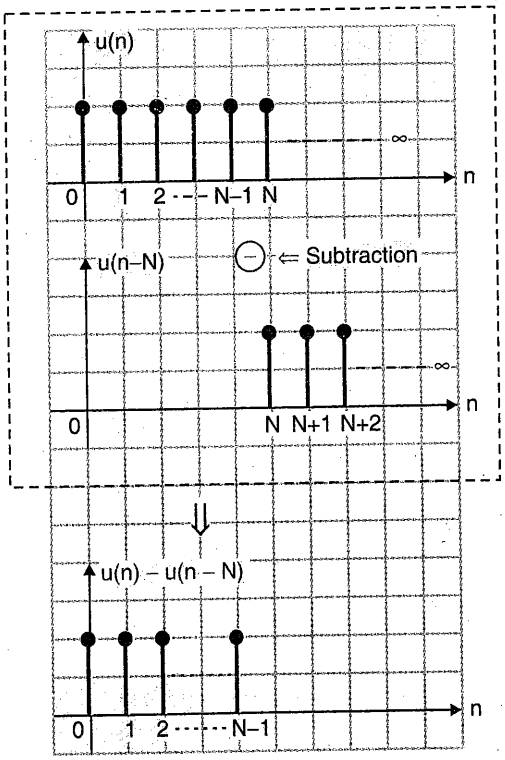


Fig. A-28(g) : $x(n) = u(n) - u(n-N)$

1.6 Discrete Time (DT) Systems :

Definition : A discrete time system is a physical device (or an algorithm) which performs required operation on a discrete time signal.

A discrete time system is represented as shown in Fig. A-29(a).

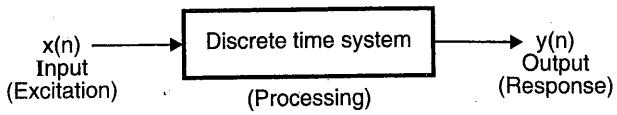


Fig. A-29(a) : Discrete time system

- Here x (n) is input discrete time signal applied to the system. It is also called as excitation.
- The system operates on discrete time signal. This is called as procession of input signal, x (n).
- Output of system is denoted by y (n). It is also called as response of system.

Notation :

When input signal is passed through the system then it is represented by following notation :

$$x(n) \xrightarrow{T} y(n) \quad \text{OR} \quad y(n) = T[x(n)]$$

Here T is called as transformation operation.

Example : A filter is a good example of a system. A signal containing noise is applied at the input of filter. This is an input signal to the system (filter). The filter cancels or attenuates noise signal. This is the processing of signal. A noise-free signal obtained at the output of filter is called as response of system.

1.6.1 Symbols used to Represent Discrete Time Systems :

The operation of discrete time system can be described by drawing a block schematic. The different building blocks are used to form a complete block schematic. These building blocks are as follows :

(1) Adder (2) Constant multiplier (3) Signal multiplier
(4) Unit delay (5) Unit advance

(1) Adder :

As the name indicates, adder block is used to perform addition of two input sequences. Symbol of an adder is as shown in Fig. A-29(b).

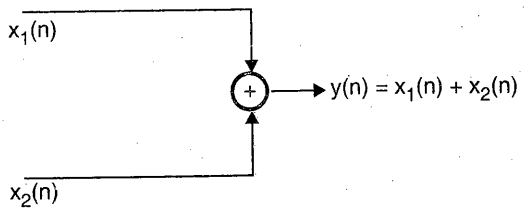


Fig. A-29(b) : An adder

Here $x_1(n)$ and $x_2(n)$ are two input sequences. Adder block adds $x_1(n)$ and $x_2(n)$. The corresponding output is $y(n)$.

$$\therefore \quad y(n) = x_1(n) + x_2(n)$$

Sometimes instead of addition sign ('+'), sigma (Σ) sign is used to represent the addition. The subtraction operation that means, $y(n) = x_1(n) - x_2(n)$ is represented as shown in Fig. A-29(c).

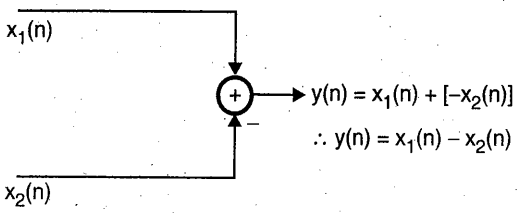


Fig. A-29(c) : Subtraction operation

In case of addition (or subtraction) operation, it is not required to store any one of the inputs in the memory. So this is memoryless operation.

(2) Constant multiplier :

This is used to multiply input sequence by some constant (a). That means this operation is used to change the amplitude of input sample. This is also memoryless operation. It is as shown in Fig. A-29(d).

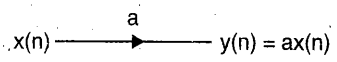


Fig. A-29(d) : Constant multiplier

(3) Signal multiplier :

As the name indicates, this block is used to multiply two input sequences. Here multiplication takes place sample to sample basis. Again this is memoryless operation. It is shown in Fig. A-29(e).

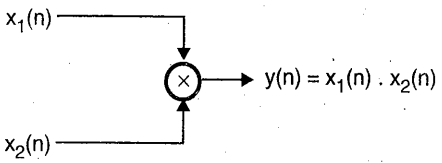


Fig. A-29(e) : Signal multiplier

(4) Unit delay :

This block is used to delay input sequence by one sample. It is represented as shown in Fig. A-29(f).

So input sequence $x(n)$ is delayed by one sample and the output is $y(n) = x(n-1)$. This indicates that the calculation of output at a particular instant, needs the previous input. Here $x(n)$ is present input and $x(n-1)$ is previous input.

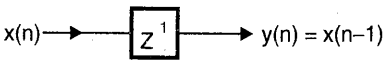


Fig. A-29(f) : Unit delay

For example, let us say we want to calculate output at n = 0. Then putting n = 0 in the equation of y (n) we get,

$$y(0) = x(0-1)$$

$$\therefore\ y(0) = x(-1)$$

So to calculate output at n = 0 i.e. y (0) we do not require input at n = 0, i.e. x (0). But we want x (− 1). Here x (− 1) is previous input compared to x (0). So in this operation every time it is required to store previous input in the memory. Thus this block requires the memory; it is not memoryless operation.

If we want to delay input sequence by two samples then two unit delay blocks should be connected in series as shown in Fig. A-29(g).

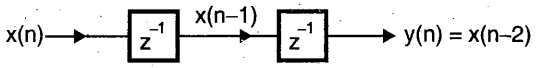


Fig. A-29(g) : Delay by two samples

(5) Unit advance :

As the name indicates; this block is used to advance input sequence by one sample. This block is represented as shown in Fig. A-29(h).



Fig. A-29(h) : Unit advance

Equation of output is,

$$y(n) = x(n+1)$$

To calculate the output at n = 0; put n = 0 in the equation of y (n).

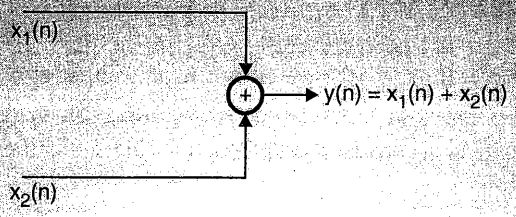
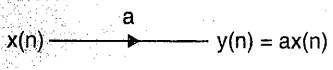
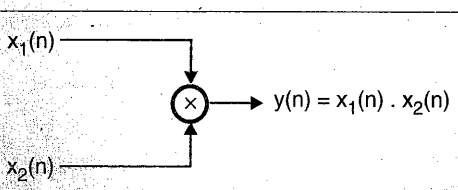
$$\therefore\ y(0) = x(1)$$

That means to calculate present output y (0), we are not making use of present input x (0). But we want input x (1). Here x (1) is future input corresponding to x (0). Thus in this operation, to calculate output y (n), a future input sample is required. This is not practically possible. So this operation is not realizable in practice.

Summary of symbols used for discrete time systems :

Table A-4 shows different symbols used to represent discrete time systems.

Table A-4

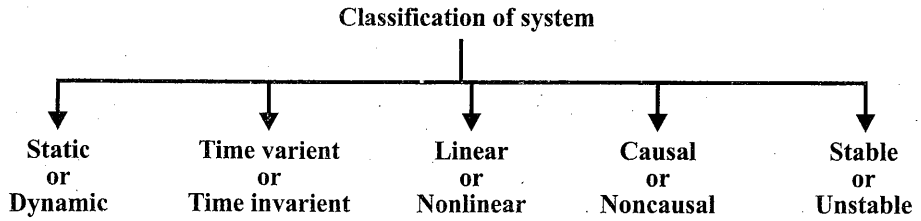
Sr. No.	Name	Symbol	Significance
(1)	Adder		It is memoryless operation
(2)	Constant multiplier		It is memoryless operation
(3)	Signal multiplier		It is memoryless operation
(4)	Unit delay		It requires memory to store the previous input
(5)	Unit advance		This operation is not realizable in practice

1.7 Properties of D.T. Systems and Classification :

Properties of systems are with respect to input and output signal. For the simplicity we will consider the system having single input and single output. But the same explanation is valid for the system having multiple inputs and multiple outputs. Important properties of systems are as follows :

- Dynamic or static nature
- Linearity
- Stability
- Time variance
- Causality

Now based on the properties, the systems can be classified as follows :



1.7.1 Static or Dynamic Systems (Dynamicity Property) :

i) Static systems :

Definition : It is a system in which output at any instant of time depends on input sample at the same time.

Example :

(i) $y(n) = 5x(n)$

Here 5 is constant which multiplies input $x(n)$. But output at n^{th} instant that means $y(n)$ depends on input at the same (n^{th}) time instant $x(n)$. So this is static system.

(ii) $y(n) = x^2(n) + 5x(n) + 10$

Here also output at n^{th} instant, $y(n)$ depends on the input at n^{th} instant. So this is static system.

Significance :

Observe the input-output relations of static system. Output does not depend on delayed $[x(n-k)]$ or advanced $[x(n+k)]$ input signals. It depends only on present (n^{th}) input signal. If output depends on delayed input signals then such signals should be stored in memory to calculate the output at n^{th} instant. This is not required in static systems. Thus for static systems, memory is not required. So static systems are memoryless systems.

ii) Dynamic systems :

Definition : It is a system in which output at any instant of time depends on input sample at the same time as well as at other times.

Here other times means, other than the present time instant. It may be past time or future time. Note that if $x(n)$ represents input signal at present instant then,

(a) $x(n-k)$; that means delayed input signal is called as past signal.

(b) $x(n+k)$; that means advanced input signal is called as future signal.

Thus in dynamic systems, output depends on present input as well as past or future inputs.

Examples :

(i) $y(n) = x(n) + 5x(n-1)$

Here output at n^{th} instant depends on input at n^{th} instant, $x(n)$ as well as $(n-1)^{th}$ instant $x(n-1)$ is previous (past) sample. So the system is dynamic.

(ii) $y(n) = 3x(n+2) + x(n)$

Here $x(n+2)$ indicates advanced version of input sample that means it is future sample; so this is dynamic system.

Significance :

Observe input-output relations of dynamic system. Since output depends on past or future input sample; we need a memory to store such samples. Thus dynamic system has a memory.

Note : Just by observing input-output relations; we can check whether the system is static or dynamic. If output $y(n)$ contains delayed or advanced terms of input then the system is dynamic, otherwise it is static system.

Prob. 1 : Check whether the following systems are static or dynamic

(1) $y(n) = n\,x(n)$,

(2) $y(n) = n\,x(n) + b\,x^2(n)$,

(3) $y(n) = 3x(n) + 5$,

(4) $y(n) = x(n^2)$,

(5) $y(n) = x(n)\cos\omega_0 n$,

(6) $y(n) = 3\,|x(n)|$

(7) $y(n) = x(n) - 3x(n-2)$

(8) $y(n) = \sum_{k=0}^{2} x(n-k)$

(9) $y(n) = x(-n)$

(10) $y(n) = A\,e^{x(n)}$

Soln. :

1) $\mathbf{y(n) = n\,x(n)}$

It is static system because output depends on present input only.

2) $\mathbf{y(n) = n\,x(n) + b\,x^2(n)}$

Note that $x^2(n)$ means, square the amplitude of input signal. So it is not related to time shifting. (It is not $x(n^2)$). Here 'b' is some constant. Since the equation does not contain any advanced or delayed input term; it is static system.

3) $\mathbf{y(n) = 3x(n) + 5}$

It is static system because output, $y(n)$ depends only on present input $x(n)$.

4) $\mathbf{y(n) = x(n^2)}$

It is dynamic system, because at n^{th} instant, system expects future inputs (n^2). For example,

at $n = 1$, $y(1) = x(1^2) = x(1)$

at $n = 2$, $y(2) = x(2^2) = x(4)$

Thus at $n = 2$, system requires future input, $x(4)$. So it is dynamic system.

5) $\mathbf{y(n) = x(n)\cos\omega_0 n}$

It is static system because output depends on present input only.

6) $\mathbf{y(n) = 3\,|x(n)|}$

It is static system because output depends on present input only.

7) $\mathbf{y(n) = x(n) - 3x(n-2)}$

It is dynamic system because output requires delayed input.

8) $y(n) = \sum_{k=0}^{2} x(n-k)$

First we will expand this summation.

$$\therefore \quad y(n) = x(n)+x(n-1)+x(n-2)$$

Since output contains delayed input terms; it is dynamic system.

9) $y(n) = x(-n)$

We know that $x(-n)$ means folded input signal. Now we will put different values of n

at $n = 0, y(0) = x(-0) = x(0)$

at $n = 1, y(1) = x(-1)$

at $n = 2, y(2) = x(-2)$

Here $x(-1)$, $x(-2)$ ... etc. indicates past input sample. So the system is dynamic.

10) $y(n) = A\, e^{x(n)}$

This is **staticsystem** because past or future input samples are not present.

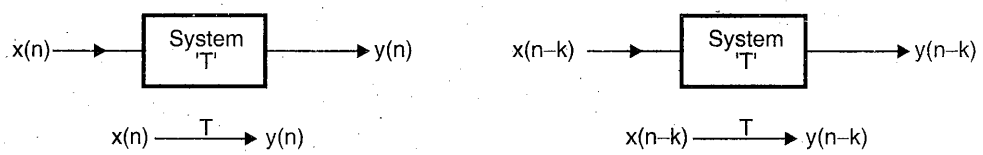
1.7.2 Time Varient or Time Invarient Systems (Time Variance Property) :

Definition : A system is time invarient if its input-output characteristics do not change with time.

Explanation :

A time invarient system means its input-output characteristics are not changing with time shifting. Let us consider we are applying input signal $x(n)$ to the system as shown in Fig. A-30(a). It produces output $y(n)$.

Now delay input by 'k' samples. That means input becomes $x(n-k)$. Apply this delayed input to the same system as shown in Fig. A-30(b). Let us say the system now gives output $y(n-k)$. Then this system is called as time or shift invarient system.



(a) (b)

Fig. A-30 : Time invarient system

Observe that initially the output is $y(n)$ with input $x(n)$. When input is delayed by 'k' samples then the output is also delayed by same 'k' samples. Thus the input-output characteristics are not changed. So the system is time invarient.

Theorem : A system is time invarient if and only if

$$x(n) \xrightarrow{T} y(n) \text{ implies that } x(n-k) \xrightarrow{T} y(n-k)$$

How to determine whether the system is time invarient or not ?

To check whether the system is time invarient (TIV) or time varient (TV) follow the following steps :

Step I : Delay the input $x(n)$ by 'k' units that means $x(n-k)$. Denote the corresponding output by y (n, k).

$$\text{That means } x(n-k) \xrightarrow{T} y(n,k)$$

Step II : In the given equation of system $y(n)$ replace 'n' by 'n – k' throughout. This output is $y(n-k)$.

Step III : If y (n, k) = $y(n-k)$ then system is time invarient (TIV) and if y (n, k) ≠ $y(n-k)$ then system is time varient (TV).

Solved Problems :

Prob. 1 : Determine whether the following systems is time invarient or not.

$y(n) = x(n) - x(n-1)$

Soln. :

Step I : Delay the input by 'k' samples and denote corresponding output by y (n, k)

$$\therefore \quad y(n, k) = x(n-k) - x(n-k-1) \qquad ...(1)$$

Step II : Replace 'n' by 'n – k' throughout the given equation.

$$\therefore \quad y(n-k) = x(n-k) - x(n-k-1) \qquad ...(2)$$

Step III : Compare Equations (1) and (2)

Here y (n, k) = $y(n-k)$. Thus the system is time invarient (TIV).

Prob. 2 : Determine whether the following systems is time invarient or not.

$T[x(n)] = e^{x(n)}$

Soln. : We know that $T[x(n)]$ means $x(n)$ is passed through the system to produce output $y(n)$. Thus given equation can be written as,

$$\therefore \quad y(n) = e^{x(n)} \text{ that means } x(n) \xrightarrow{T} y(n) = e^{x(n)}$$

Step I : Delay input by k samples and denote this output by y (n, k)

$$\therefore \quad y(n, k) = e^{x(n-k)} \qquad ...(1)$$

Step II : Replace 'n' by 'n – k' throughout the given equation.

$$\therefore \quad y(n-k) = e^{x(n-k)} \qquad ...(2)$$

Step III : Compare Equations (1) and (2)

Since y (n, k) = y (n – k); the system is time invarient (TIV).

Prob. 3 : Determine whether the following systems is time invarient or not.

$T[x(n)] = ax(n) + 6$

Soln. : The given equation can be written as,

$$y(n) = ax(n) + 6$$

Step I : Delay input by k samples and denote this output by y (n, k)

$$\therefore \quad y(n, k) = ax(n-k) + 6 \qquad ...(1)$$

Step II : Replace n by n – k throughout the equation.

$$\therefore \quad y(n-k) = ax(n-k) + 6 \qquad ...(2)$$

Step III : Compare Equations (1) and (2)

Here y (n, k) = y (n – k). The system is time invarient (TIV).

Prob. 4 : Determine whether the following systems is time invarient or not.

$y(n) = \cos x(n)$

Soln. :

Step I : Delay the input by 'k' units and denote this output by y (n, k)

$$\therefore \quad y(n, k) = \cos x(n-k) \qquad ...(1)$$

Step II : Replace 'n' by 'n – k' throughout the equation.

$$\therefore \quad y(n-k) = \cos x(n-k) \qquad ...(2)$$

Step III : Compare Equations (1) and (2)

Since y (n, k) = y (n – k); the system is time invarient (TIV).

Prob. 5 : Determine whether the following systems is time invarient or not.

$y(n) = |x(n)|$

Soln. :

Step I : Delay input by 'k' samples and denote this output by y (n, k)

$$\therefore \quad y(n, k) = |x(n-k)| \qquad ...(1)$$

Step II : Replace 'n' by 'n – k' throughout the equation

$$\therefore \quad y(n-k) = |x(n-k)| \qquad ...(2)$$

Step III : Compare Equations (1) and (2)

Since y (n, k) = y (n – k); the system is time invarient (TIV).

Prob. 6 : Determine whether the following systems is time invarient or not.

$y(n) = x(n) + n\,x(n-1)$

Soln. :

Step I : Delay input by 'k' units and denote this output by y (n, k)

$$\therefore \quad y(n, k) = x(n-k) + n\,x(n-k-1) \qquad ...(1)$$

Note that only input $x(n)$ is delayed. That means in the given equation wherever $x(n)$ is present, replace it by $x(n-k)$. You should not replace 'n' present in the second term by $n-k$. That means in this step do not write the equation as,

$$y(n, k) = x(n-k) + (n-k)\,x(n-k-1)$$

This is not valid since we are delaying input, $x(n)$ and not only n.

Step II : Replace 'n' by 'n − k' throughout the equation.

$$\therefore \quad y(n-k) = x(n-k) + (n-k)\,x(n-k-1) \qquad ...(2)$$

Step III : Compare Equations (1) and (2)

Since $y(n, k) \neq y(n-k)$; the system is time varient (TV).

Prob. 7 : Determine whether the following systems is time invarient or not.

$y(n) = x(n^2)$

Soln. :

Step I : Delay input and denote this output by y (n, k)

$$\therefore \quad y(n, k) = x[(n-k)^2] \qquad ...(1)$$

Step II : Replace n by n − k throughout the equation

$$\therefore \quad y(n-k) = x[(n-k)^2] \qquad ...(2)$$

Step III : Compare Equations (1) and (2)

Since $y(n, k) = y(n-k)$; the system is time invarient (TIV).

Prob. 8 : Determine whether the following systems is time invarient or not.

$y(n) = x(2n)$

Soln. :

Step I : Delay input by k units and denote this output by y (n, k)

$$\therefore \quad y(n, k) = x[2(n-k)] \qquad ...(1)$$

Step II : Replace 'n' by 'n − k' throughout the equation

$$\therefore \quad y(n-k) = x[2(n-k)] \qquad ...(2)$$

Step III : Compare Equations (1) and (2)

Since $y(n, k) = y(n-k)$; the system is time invarient (TIV).

Prob. 9 : Determine whether the following systems is time invarient or not.

$y(n) = y(n-4) + x(n-4)$

Soln. : Note that the term $y(n-4)$ denotes the delayed output. It also indicates that the feedback is taken from the output $y(n)$ and it is delayed by 4 units.

Step I : Delay input by k units and denote this output by y (n, k)

$$\therefore \quad y(n, k) = y(n-4) + x(n-k-4) \qquad ...(1)$$

Step II : Replace 'n' by 'n – k' throughout the equations

$$\therefore \quad y(n-k) = y(n-k-4)+x(n-k-4) \qquad ...(2)$$

Step III : Compare Equations (1) and (2)

Since $y(n, k) \neq y(n-k)$; the system is time varient (TV).

Prob. 10 : Determine whether the following systems is time invarient or not.

$y(n) = x(-n)$

Soln. :

Step I : Delay input by 'k' units and denote this output by y (n, k)

$$\therefore \quad y(n, k) = x(-n-k) \qquad ...(1)$$

Note that only 'n' should not be replaced by n – k. This is because we want to delay the whole input x (n) by 'k' units. Thus x (– n) should be delayed by k samples.

Step II : Replace 'n' by 'n – k' throughout the equation

$$\therefore \quad y(n-k) = x[-(n-k)] = x(-n+k) \qquad ...(2)$$

Step III : Compare Equations (1) and (2)

Since $y(n, k) \neq y(n-k)$; the system time varient (TV) system.

Prob. 11 : Show the following systems are time varying.

(i) $y(n) = hx(n)$ (ii) $y(n) = nx^2(n)$

Soln. :

(i) Given, $y(n) = nx(n)$

Step I : Delay input by k units and denote this output by y (n, k).

$$\therefore \quad y(n, k) = nx(n-k)$$

Step II : Replace n by n – k throughout the equation.

$$\therefore \quad y(n-k) = (n-k)x(n-k)$$

Step III : Since $y(n, k) \neq y(n-k)$; the system is time varient.

(ii) Given $y(n) = nx^2(n)$

Step I : Delay input by k units and denote this output by y (n, k).

$$\therefore \quad y(n, k) = n[x(n-k)]^2$$

Step II : Replace n by n – k throughout the equation.

$$\therefore \quad y(n-k) = n-kx^2(n-k)$$

Step III : Since $y(n, k) \neq y(n-k)$

the system is time varient.

Prob. 12 : Determine whether the following systems are time invarient or not.

i) $y(n) = g(n)x(n)$

ii) $y(n) = x(n)+3u(n+1)$

Soln. :

(i) Given,

$$y(n) = g(n)\, x(n)$$

Note that $x(n)$ denotes input and $y(n)$ denotes output.

Step I : Delay input by k samples and denote this output by $y(n, k)$

$$\therefore \quad y(n,k) = g(n)\, x(n-k) \qquad ...(1)$$

Step II : Replace n by $(n-k)$ throughout the equation.

$$\therefore \quad y(n-k) = g(n-k) \cdot x(n-k) \qquad ...(2)$$

Step III : Since $y(n, k) \neq y(n-k)$; the system is time varient.

ii) Given ,

$$y(n) = x(n) + 3u(n+1)$$

Step I : Delay input by k samples and denote this output by $y(n, k)$

$$\therefore \quad y(n,k) = x(n-k) + 3u(n+1) \qquad ...(1)$$

Step II : Replace n by $(n-k)$ throughout the equation.

$$\therefore \quad y(n-k) = x(n-k) + 3u(n-k+1) \qquad ...(2)$$

Step III : Since $y(n, k) \neq y(n-k)$; the system is time varient.

Prob. 13 : Find out if the following systems are shift invarient or not

i) $y(n) = x(n-n_0)$, n_0 is constant

ii) $y(n) = ax(n) + b$, a and b are constants

iii) $y(n) = x^2(n)$

iv) $y(n) = x(n) \cdot n(n-n_0)$, n_0 is constant.

Soln. :

(i) $y(n) = x(n-n_0)$

Step I : Delay input by k samples and denote output by $y(n, k)$

$$\therefore \quad y(n,k) = x(n-k-n_0) \qquad ...(1)$$

Step II : Replace n by $(n-k)$ throughout the equation.

$$\therefore \quad y(n-k) = x(n-k-n_0) \qquad ...(2)$$

Step III : Since $y(n, k) = y(n-k)$; the system is shift invarient.

(ii) $y(n) = ax(n) + b$

Step I : Delay input by k samples

$$\therefore \quad y(n,k) = ax(n-k) + b \qquad ...(1)$$

Step II : Replace n by $(n-k)$ throughout the equation.

$$\therefore \quad y(n-k) = ax(n-k) + b \qquad ...(2)$$

Step III : Since $y(n, k) = y(n-k)$; the system is shift invarient.

(iii) $y(n) = x^2(n)$

Step I : Delay input by k samples.

$$\therefore \quad y(n,k) = x^2(n-k)$$

Step II : Replace n by $(n-k)$ throughout the equation.

$$y(n-k) = x^2(n-k)$$

Step III : Since $y(n, k) = y(n-k)$; the system is shift invarient.

(iv) $y(n) = x(n) \cdot n(n-n_0)$

Step I : Delay input by k samples

$$\therefore \quad y(n,k) = x(n-k) \cdot n(n-n_0)$$

Step II : Replace n by $(n-k)$ throughout the equation

$$\therefore \quad y(n-k) = x(n-k) \cdot (n-k)[n-k-n_0]$$

Step III :

Since $y(n, k) \neq y(n-k)$; the system is shift varient.

Prob. 14 : Determine whether the following system is shift invarient or not

$y(n) = x(n) \cos(\omega_0 n)$

Soln. :

Step I : Delay input by k samples.

$$\therefore \quad y(n,k) = x(n-k) \cos(\omega_0 n)$$

Step II : Replace n by $(n-k)$ throughout the equation.

$$\therefore \quad y(n-k) = x(n-k) \cos[\omega_0 (n-k)]$$

Step III : Since $y(n, k) \neq y(n-k)$ the system is shift varient.

Prob. 15 : Determine whether the following system is time varient or not $y(n) = x(n) \cos \omega_0 n$

Soln. :

Step I : Delay input by 'k' units and denote this output by $y(n, k)$

$$\therefore \quad y(n,k) = x(n-k) \cos \omega_0 n$$

Step II : Replace 'n' by 'n – k' throughout the equation.

$$\therefore \quad y(n-k) = x(n-k) \cos \omega_0 (n-k)$$

Step III : Since $y(n, k) \neq y(n-k)$;

The system is Time Varient (TV).

Prob. 16 : Determine whether the following system is Time Invarient or not.

$$y(n) = \frac{1}{3}[x(n) + x(n-1) + x(n-2)]$$

Soln. :

Step I : Delay input by 'k' units and denote the output by $y(n, k)$

$$\therefore \quad y(n,k) = \frac{1}{3}[x(n-k)+x(n-k-1)+x(n-k-2)]$$

Step II : Replace 'n' by 'n – k' throughout the equation.

$$\therefore \quad y(n-k) = \frac{1}{3}[x(n-k)+x(n-k-1)+x(n-k-2)]$$

Step III : Since $y(n, k) = y(n-k)$;

The given system is Time Invarient (TIV).

1.7.3 Linear or Non-linear Systems (Linearity Property) :

A linear system is a system which follows superposition principle. Consider a system as shown in Fig. A-31(a); whose response is T. Its input is $x(n)$ and it produces output $y(n)$.

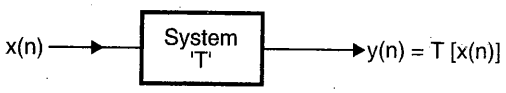


Fig. A-31(a) : A system with response 'T'

Let input $x_1(n)$ produces output $y_1(n)$ and input $x_2(n)$ produces output $y_2(n)$. Consider two arbitrary constants a_1 and a_2. Then simply we can say that input $a_1 x_1(n)$ produces output $a_1 y_1(n)$ and input $a_2 x_2(n)$ produces output $a_2 y_2(n)$.

Now we can write the theorem for linear systems as follows :

Theorem :

A system is said to be linear if the combined response of $a_1 x_1(n)$ and $a_2 x_2(n)$ is equal to the addition of individual responses.

That means,

$$T[a_1 x_1(n)+a_2 x_2(n)] = a_1 T[x_1(n)]+a_2 T[x_2(n)] \qquad ...(1)$$

Equation (1) is called as superposition theorem. The graphical representation of L.H.S. and R.H.S. of Equation (1) is as shown in Fig. A-31(b) and Fig. A-31(c) respectively.

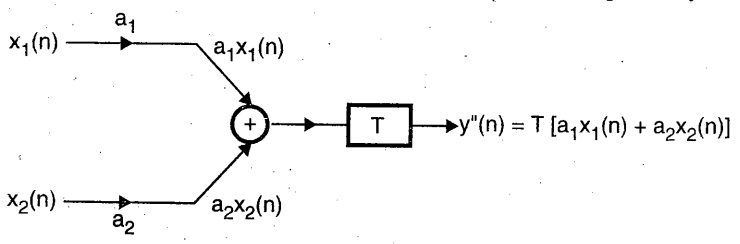


Fig. A-31(b) : L.H.S. of equation (1)

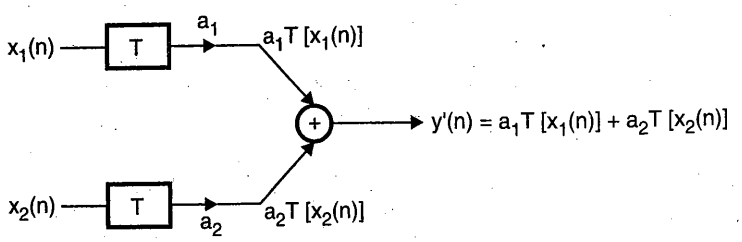


Fig. A-31(c) : R.H.S. of equation (1)

Important characteristic :

An important characteristics of linear system is that when input is zero then it produces zero output. But if zero input gives some value at the output (non-zero) then it is non-linear system. If this condition is satisfied then apply the superposition theorem to check whether the system is linear or not.

How to decide whether the system is linear or not ?

Step I : Apply zero input and check the output. If output is zero then system is linear. If this step is satisfied then follow the remaining steps.

Step II : Apply individual input to the system and note down corresponding outputs. Then add all outputs. Denote this addition by $y'(n)$. This is R.H.S. of Equation (1) as shown in Fig. A-31(c).

Step III : Combine all inputs, apply it to the system and find out output $y''(n)$. This is L.H.S. of Equation (1) as shown in Fig. A-31(b).

Step IV : If $y'(n) = y''(n)$ then the system is linear otherwise it is non-linear system.

Note : (1) For the simplicity here we have assumed arbitrary constants $a_1 = a_2 = 1$.
(2) If step I is satisfied then check remaining steps. But if step I is not satisfied then at this stage only we can say system is non-linear.

Solved Problems :

Q. Determine whether the following systems are linear or not.

Prob. 1 : $y(n) = n\,x(n)$

Soln. :

Step I : When input $x(n)$ is zero then output $y(n) = n.0 = 0$. Thus the system is linear. Now first step is satisfied so we will check remaining steps for linearity.

Step II : Let there are two inputs $x_1(n)$ and $x_2(n)$.

The given equation of output is $y(n) = n\,x(n)$ that means $x(n) \xrightarrow{T} n\,x(n)$. So the function of system is to multiply input $x(n)$ by 'n'. Now for two inputs $x_1(n)$ and $x_2(n)$ the corresponding outputs are $y_1(n)$ and $y_2(n)$.

$$\therefore \quad x_1(n) \xrightarrow{T} y_1(n) = n\,x_1(n)$$

$$\text{and} \quad x_2(n) \xrightarrow{T} y_2(n) = n\,x_2(n)$$

Now add these output to get $y'(n)$

$$\therefore \quad y'(n) = y_1(n) + y_2(n) = n\,x_1(n) + n\,x_2(n)$$

$$\therefore \quad y'(n) = n[x_1(n) + x_2(n)] \qquad \ldots(1)$$

Step III : Now add inputs $x_1(n)$ and $x_2(n)$ and then pass it through the system.

$$\therefore \quad [x_1(n) + x_2(n)] \xrightarrow{T} y''(n) = T[x_1(n) + x_2(n)] = n[x_1(n) + x_2(n)]$$

We know that the function of system is to multiply input by 'n'.

Here $[x_1(n) + x_2(n)]$ acts as one input to the system. So the corresponding output is,

$$y''(n) = n[x_1(n) + x_2(n)] \qquad \ldots(2)$$

Step IV : Compare Equations (1) and (2)

Since $y'(n) = 12y''(n)$; the system is linear.

Prob. 2 : $T[x(n)] = e^{x(n)}$

Soln. : We can write the given equation as,

$$y(n) = e^{x(n)}$$

Step I : If input $x(n)$ is zero then, output $y(n) = e^0 = 1$. Thus the system is non-linear. But we will also check the system for superposition principle.

Step II : Here the function of system is to take exponential of $x(n)$. Now let there are two inputs, $x_1(n)$ and $x_2(n)$

$$\therefore \quad x_1(n) \xrightarrow{T} y_1(n) = e^{x_1(n)}$$

$$\text{and} \quad x_2(n) \xrightarrow{T} y_2(n) = e^{x_2(n)}$$

Add these output to get $y'(n)$

$$\therefore \quad y'(n) = e^{x_1(n)} + e^{x_2(n)} \qquad \ldots(1)$$

Step III : First add input $x_1(n)$ and $x_2(n)$ and then apply it to the system. We know that the system produces exponential of input.

$$\therefore \quad [x_1(n)+x_2(n)] \xrightarrow{T} y''(n) = e^{[x_1(n)+x_2(n)]}$$

$$\therefore \quad y''(n) = e^{[x_1(n)+x_2(n)]}$$

$$\therefore \quad y''(n) = e^{x_1(n)} \cdot e^{x_2(n)} \qquad ...(2)$$

Step IV : Compare Equations (1) and (2)

Since $y'(n) \neq y''(n)$; the system is non-linear.

Prob. 3 : $T\{x(n)\} = ax(n)+6$.

Soln. : The given equation can be written as,

$$y(n) = ax(n)+6$$

Step I : If input $x(n)$ is zero then,

$$y(n) = 0+6 \qquad \therefore \quad y(n) = 6$$

Step II : Consider two inputs $x_1(n)$ and $x_2(n)$. We will apply these two inputs separately to the system.

$$\therefore \quad x_1(n) \xrightarrow{T} y_1(n) = a x_1(n)+6$$

and $$x_2(n) \xrightarrow{T} y_2(n) = a x_2(n)+6$$

Add these outputs to get $y'(n)$

Note that the given equation contains the coefficients 'a' and 6. So to check the linearity we will add the outputs as follows :

$$y'(n) = a_1 y_1(n)+a_2 y_2(n)$$

$$\therefore \quad y'(n) = a_1[a x_1(n)+6]+a_2[a x_2(n)+6] \qquad ...(1)$$

Here 'a_1' and 'a_2' are arbitrary constants used to check the linearity.

Step III : We will add two inputs and then we will apply this signal to the system. Again we will use arbitrary constants a_1 and a_2.

$$\therefore \quad [a_1 x_1(n)+a_2 x_2(n)] \xrightarrow{T} a[a_1 x_1(n)+a_2 x_2(n)]+6$$

$$\therefore \quad y''(n) = a[a_1 x_1(n)+a_2 x_2(n)]+6 \qquad ...(2)$$

Note that in this case the function of system is to multiply input by constant 'a' and then add 6. The term $a_1 x_1(n)+a_2 x_2(n)$ acts as combined input signal. Thus the output $y''(n)$ is obtained as given by Equation (2).

Step IV : Compare Equations (1) and (2)

Since $y'(n) \neq y''(n)$; the system is non-linear.

Note : Whenever the given equation contains some constants and the equation contains addition of two or more terms; then to check the linearity, use arbitrary constants a_1 and a_2. In other cases we are assuming $a_1 = a_2 = 1$.

Prob. 4 : $y(n) = \cos x(n)$

Soln. :

Step I : If input $x(n)$ is zero then,

$$y(n) = \cos(0)$$

$$\therefore \quad y(n) = 1$$

So the system is non-linear . It is not necessary to apply remaining steps but we will check the system for superposition theorem.

Step II : Consider two inputs $x_1(n)$ and $x_2(n)$. Apply these two inputs separately to the system. Observe that the function of system is to take 'cos' of input signal.

$$\therefore \quad x_1(n) \xrightarrow{T} y_1(n) = \cos x_1(n)$$

$$\text{and} \quad x_2(n) \xrightarrow{T} y_2(n) = \cos x_2(n)$$

Now add two outputs to get $y'(n)$

$$\therefore \quad y'(n) = y_1(n) + y_2(n)$$

$$\therefore \quad y'(n) = \cos x_1(n) + \cos x_2(n) \qquad \ldots(1)$$

Step III : Add two inputs and pass it through the system.

$$\therefore \quad [x_1(n) + x_2(n)] \xrightarrow{T} \cos[x_1(n) + x_2(n)]$$

$$\therefore \quad y''(n) = \cos[x_1(n) + x_2(n)] \qquad \ldots(2)$$

Use trignometric identity,

$$\cos[A + B] = \cos A \cos B + \sin A \sin B$$

Thus Equation (2) becomes,

$$y''(n) = \cos x_1(n) \cos x_2(n) + \sin x_1(n) \sin x_2(n) \qquad \ldots(3)$$

Step IV : Compare Equations (1) and (2)

Since $y'(n) \neq y''(n)$; the system is non-linear.

Prob. 5 : $y(n) = |x(n)|$

Soln. :

Step I : Apply zero input to the system that means $x(n) = 0$.

$$\therefore \quad y(n) = |x(n)| = 0$$

Thus step I is satisfied so we will check the remaining steps.

Step II : Consider two inputs $x_1(n)$ and $x_2(n)$ and apply it to the system.

$$\therefore \quad x_1(n) \xrightarrow{T} y_1(n) = |x_1(n)|$$

$$\text{and} \quad x_2(n) \xrightarrow{T} y_2(n) = |x_2(n)|$$

Add these two outputs to get $y'(n)$

$$\therefore \quad y'(n) = y_1(n) + y_2(n)$$

$$\therefore \quad y'(n) = |x_1(n)| + |x_2(n)| \qquad ...(1)$$

Step III : Add two inputs and apply it to the system. Observe that the function of system is to take magnitude of input signal.

$$\therefore \quad [x_1(n) + x_2(n)] \xrightarrow{T} |x_1(n) + x_2(n)|$$

$$\therefore \quad y''(n) = |x_1(n) + x_2(n)| \qquad ...(2)$$

Step IV : Compare Equations (1) and (2)

Since $y'(n) \neq y''(n)$; the system is non-linear.

Prob. 6 : Explain the following system with respect to linearity property

$y(n) = x(n) + n\,x(n+1)$

Soln. :

Step I : Apply input $x(n) = 0$ to the system. The output $y(n) = 0$ so we will check remaining conditions.

Step II : Consider two inputs $x_1(n)$ and $x_2(n)$. Apply two inputs separately to the system.

$$\therefore \quad x_1(n) \xrightarrow{T} y_1(n) = x_1(n) + n\,x_1(n+1)$$

$$\text{and} \quad x_2(n) \xrightarrow{T} y_2(n) = x_2(n) + n\,x_2(n+1)$$

Add these two outputs to get $y'(n)$

$$\therefore \quad y'(n) = y_1(n) + y_2(n)$$

$$\therefore \quad y'(n) = x_1(n) + n x_1(n+1) + x_2(n) + n x_2(n+1)$$

$$\therefore \quad y'(n) = x_1(n) + x_2(n) + n[x_1(n+1) + x_2(n+1)] \qquad ...(1)$$

Step III : Combine two inputs and apply it to the system.

$$\therefore \quad [x_1(n) + x_2(n)] \xrightarrow{T} [x_1(n) + x_2(n)] + n[x_1(n+1) + x_2(n+1)]$$

$$\therefore \quad y''(n) = [x_1(n) + x_2(n)] + n[x_1(n+1) + x_2(n+1)] \qquad ...(2)$$

Step IV : Compare Equations (1) and (2)

Since $y'(n) = y''(n)$; the system is linear.

Prob. 7 : Explain the following system with respect to linearity property
$y(n) = x(2n)$

Soln. :

Step I : When input is zero, output $y(n)$ is also zero. So we will check remaining conditions.

Step II : Consider two separate inputs $x_1(n)$ and $x_2(n)$. Apply it to the system.

$$\therefore \quad x_1(n) \xrightarrow{T} y_1(n) = x_1(2n)$$

$$\text{and} \quad x_2(n) \xrightarrow{T} y_2(n) = x_2(2n)$$

Add $y_1(n)$ and $y_2(n)$ to get $y'(n)$

$$\therefore \quad y'(n) = x_1(2n) + x_2(2n) \qquad ...(1)$$

Step III : Combine two inputs and apply it to the system.

$$\therefore \quad [x_1(n) + x_2(n)] \xrightarrow{T} x_1(2n) + x_2(n)$$

$$\therefore \quad y''(n) = x_1(2n) + x_2(2n) \qquad ...(2)$$

Step IV : Compare Equations (1) and (2)

Since $y'(n) = y''(n)$; the system is linear.

Prob. 8 : Find whether the following system is linear
$y(n) = x(-n+2)$

Soln. :

Step I : Apply zero input to the system. It produces zero output. So we will check the remaining conditions.

Step II : Consider two inputs $x_1(n)$ and $x_2(n)$. Apply these inputs separately to the system.

$$\therefore \quad x_1(n) \xrightarrow{T} y_1(n) = x_1(-n+2)$$

$$\text{and} \quad x_2(n) \xrightarrow{T} y_2(n) = x_2(-n+2)$$

Add these two outputs to get $y'(n)$

$$\therefore \quad y'(n) = y_1(n) + y_2(n)$$

$$\therefore \quad y'(n) = x_1(-n+2) + x_2(-n+2) \qquad ...(1)$$

Step III : Add two inputs and apply it to the system.

$$\therefore \quad x_1(n) + x_2(n) \xrightarrow{T} x_1(-n+2) + x_2(-n+2)$$

$$\therefore \quad y''(n) = x_1(-n+2) + x_2(-n+2) \qquad ...(2)$$

Step IV : Since $y'(n) = y''(n)$; the system is linear.

Prob. 9 : $y(n) = x(n^2)$

Soln. :

Step I : For zero input output is zero.

Step II : Consider there are two inputs $x_1(n)$ and $x_2(n)$. Apply these inputs separately to the system.

$$\therefore \quad x_1(n) \xrightarrow{T} y_1(n) = x_1(n^2)$$

$$\text{and} \quad x_2(n) \xrightarrow{T} y_2(n) = x_2(n^2)$$

Add these outputs to get $y'(n)$

$$\therefore \quad y'(n) = y_1(n) + y_2(n)$$

$$\therefore \quad y'(n) = x_1(n^2) + x_2(n^2) \qquad ...(1)$$

Step III : Add two inputs and apply it to the system.

$$\therefore \quad [x_1(n) + x_2(n)] \xrightarrow{T} [x_1(n^2) + x_2(n^2)]$$

$$\therefore \quad y''(n) = x_1(n^2) + x_2(n^2) \qquad ...(2)$$

Note that the function of system it to square n^{th} sample. It is not squaring of input that means $x^2(n)$.

Step IV : Compare Equations (1) and (2)

Since $y'(n) = y''(n)$; the system is linear.

Prob. 10 : $y(n) = x^2(n)$

Soln. :

Step I : When input $x(n)$ is zero; output is zero.

Step II : Consider two inputs $x_1(n)$ and $x_2(n)$. Apply each input separately to the system.

$$\therefore \quad x_1(n) \xrightarrow{T} y_1(n) = x_1^2(n)$$

$$\text{and} \quad x_2(n) \xrightarrow{T} y_2(n) = x_2^2(n)$$

Add these outputs to get $y'(n)$

$$\therefore \quad y'(n) = \left[x_1^2(n) + x_2^2(n)\right] \qquad ...(1)$$

Step III : Combine two inputs and apply it to the system.

$$\therefore \quad [x_1(n) + x_2(n)] \xrightarrow{T} [x_1(n) + x_2(n)]^2$$

$$\therefore \quad y''(n) = [x_1(n) + x_2(n)]^2 \qquad ...(2)$$

Step IV : Compare Equations (1) and (2)

Since $y'(n) \neq y''(n)$; the system is non-linear.

Prob. 11 : $y(n) = n x^2(n)$

Soln. :

Step I : When input $x(n)$ is zero; output is zero.

Step II : Consider two inputs $x_1(n)$ and $x_2(n)$. Apply each input separately to the system.

$$\therefore \quad x_1(n) \xrightarrow{T} y_1(n) = n x_1^2(n)$$

$$\text{and} \quad x_2(n) \xrightarrow{T} y_2(n) = n x_2^2(n)$$

Add these outputs to get $y'(n)$

$$\therefore \quad y'(n) = n\left[x_1^2(n) + x_2^2(n)\right] \qquad ...(1)$$

Step III : Combine two inputs and apply it to the system.

$$\therefore \quad [x_1(n)+x_2(n)] \xrightarrow{T} n[x_1(n)+x_2(n)]^2$$

$$\therefore \quad y''(n) = n[x_1(n)+x_2(n)]^2 \qquad ...(2)$$

Step IV : Compare Equations (1) and (2)

Since $y'(n) \neq y''(n)$; the system is non-linear.

Prob. 12 : $y(n) = g(n)x(n)$

Soln. :

Step I : When input $x(n) = 0$; output is zero.

Step II : Combine two inputs $x_1(n)$ and $x_2(n)$. Apply each input separately to the system.

$$\therefore \quad x_1(n) \xrightarrow{T} g(n)x_1(n)$$

$$\text{and} \quad x_2(n) \xrightarrow{T} g(n)x_2(n)$$

$$\therefore \quad y'(n) = g(n)x_1(n)+g(n)x_2(n) \qquad ...(1)$$

Step III : Combine two inputs and apply it to the system.

$$\therefore \quad [x_1(n)+x_2(n)] \xrightarrow{T} g(n)[x_1(n)+x_2(n)]$$

$$\therefore \quad y''(n) = g(n)x_1(n)+g(n)x_2(n) \qquad ...(2)$$

Step IV : Since $y'(n) = y''(n)$; the system is linear.

Prob. 13 : $y(n) = x(n-n_0)$, n_0 is constant.

Soln. :

Step I : When $x(n) = 0$; output $y(n)$ which indicates delay of $x(n)$ by n_0 is also zerc.

Step II : Apply each input separately to the system.

$$\therefore \quad x_1(n) \xrightarrow{T} x_1(n-n_0)$$

$$\text{and} \quad x_2(n) \xrightarrow{T} x_2(n-n_0)$$

$$\therefore \quad y'(n) = x_1(n-n_0)+x_2(n-n_0) \qquad ...(1)$$

Step III : Combine all inputs and apply it to the system.

$$\therefore \quad [x_1(n)+x_2(n)] \xrightarrow{T} x_1(n-n_0)+x_2(n-n_0)$$

$$\therefore \quad y''(n) = x_1(n-n_0)+x_2(n-n_0)$$

Step IV : Since $y'(n) = y''(n)$; the system is linear.

Prob. 14 : $y(n) = ax(n)+b$

Soln. :

Step I : When input $x(n) = 0$ then $y(n) = b$

Step II : Apply each input separately to the system.

$$\therefore \quad x_1(n) \xrightarrow{T} ax_1(n)+b$$

$$\text{and} \quad x_2(n) \xrightarrow{T} ax_2(n)+b$$

$$\therefore \quad y'(n) = ax_1(n)+ax_2(n)+2b \qquad ...(1)$$

Step III : Combine two inputs and apply it to the system.

$$\therefore \quad [x_1(n)+x_2(n)] \xrightarrow{T} a[x_1(n)+x_2(n)]+b$$

$$\therefore \quad y''(n) = ax_1(n)+ax_2(n)+b \qquad ...(2)$$

Step IV : Since $y'(n) \neq y''(n)$; the system is non-linear.

Prob. 15 : $y(n) = x(n) \cdot n(n-n_0)$

Soln. :

Step I : When input is zero then ,output $y(n) = 0$

Step II : Apply each input separately to the system.

$$\therefore \quad x_1(n) \xrightarrow{T} x_1(n) \cdot n(n-n_0)$$

$$\text{and} \quad x_2(n) \xrightarrow{T} x_2(n) \cdot n(n-n_0)$$

$$\therefore \quad y'(n) = [x_1(n)+x_2(n)]n(n-n_0) \qquad ...(1)$$

Step III : Combine all inputs and apply it to the system

$$\therefore \quad x_1(n)+x_2(n) \xrightarrow{T} [x_1(n)+x_2(n)]\cdot n(n-n_0)$$

$$\therefore \quad y''(n) = [x_1(n)+x_2(n)]\cdot n(n-n_0) \qquad ...(2)$$

Step IV : Since $y'(n) = y''(n)$; the system is linear.

Prob. 16 : Check whether the following discrete time system is linear or not $y(n) = x(n)\cos\omega_0 n$.

Soln. :

Step I : If input $x(n) = 0$ then,

$$y(n) = 0$$

Step II : Consider two inputs $x_1(n)$ and $x_2(n)$. Apply these two inputs separately to the system.

$$\therefore \quad x_1(n) \xrightarrow{T} y_1(n) = x_1(n)\cos\omega_0 n$$

$$\text{and} \quad x_2(n) \xrightarrow{T} y_2(n) = x_2(n)\cos\omega_0 n$$

Now add two outputs to get $y'(n)$.

$$\therefore \quad y'(n) = y_1(n)+y_2(n)$$

$$\therefore \quad y'(n) = x_1(n)\cos\omega_0 n + x_2(n)\cos\omega_0 n \qquad ...(1)$$

Step III : Add two inputs and pass it through the system

$$\therefore \quad [x_1(n)+x_2(n)] \xrightarrow{T} [x_1(n)+x_2(n)]\cos\omega_0 n$$

$$\therefore \quad y''(n) = [x_1(n)+x_2(n)]\cos\omega_0 n$$

$$\therefore \quad y''(n) = x_1(n)\cos\omega_0 n + x_2(n)\cos\omega_0 n \qquad ...(2)$$

Step IV : From Equations (1) and (2) we see that $y'(n) = y''(n)$

The given system is linear.

Prob. 17 : Examine the following system for linearity.

$$y(n) = x(-n)$$

Soln. :

Step I : Apply zero input to the system; it produces zero output. Now we will check the remaining conditions.

Step II : Consider two inputs $x_1(n)$ and $x_2(n)$. Apply these inputs separately to the system.

$$\therefore \quad x_1(n) \xrightarrow{T} y_1(n) = x_1(-n)$$

$$\text{and} \quad x_2(n) \xrightarrow{T} y_2(n) = x_2(-n)$$

Add two outputs $y_1(n)$ and $y_2(n)$ to get $y'(n)$.

$$\therefore \quad y'(n) = x_1(-n) + x_2(-n) \qquad ...(1)$$

Step III : Combine two inputs and apply it to the system.

$$\therefore \quad [x_1(n) + x_2(n)] \xrightarrow{T} x_1(-n) + x_2(-n)$$

$$\therefore \quad y''(n) = x_1(-n) + x_2(n) \qquad ...(2)$$

Step IV : From Equations (1) and (2) we observe that $y'(n) = y''(n)$;
The given system is linear.

Prob. 18 : Determine whether the following system is linear or not

$$y(n) = \frac{1}{3}[x(n) + x(n-1) + x(n-2)]$$

Soln. :

Step I : Here $x(n)$ is input sequence while $x(n-1)$ and $x(n-2)$ represents delayed input sequences. When input is zero its delayed version is also zero. Thus by applying zero input to the system, output $y(n) = 0$. Now we will check the system for superposition theorem.

Step II : Consider two inputs $x_1(n)$ and $x_2(n)$. Apply these two inputs separately to the system.

$$\therefore \quad x_1(n) \xrightarrow{T} y_1(n) = \frac{1}{3}\left[x_1(n) + x_1(n-1) + x_1(n-2)\right]$$

$$\text{and} \quad x_2(n) \xrightarrow{T} y_2(n) = \frac{1}{3}[x_2(n) + x_2(n-1) + x_2(n-2)]$$

Add these two outputs to get $y'(n)$.

$$\therefore y'(n) = \frac{1}{3}\left[x_1(n) + x_1(n-1) + x_1(n-2)\right] + \frac{1}{3}\left[x_2(n) + x_2(n-1) + x_2(n-2)\right] \qquad ...(1)$$

Step III : Combine two inputs and apply it to the system.

$$\therefore [x_1(n) + x_2(n)] \xrightarrow{T} \frac{1}{3}\left\{[x_1(n) + x_2(n)] + [x_1(n-1) + x_2(n-1)] + [x_1(n-2) + x_2(n-2)]\right\}$$

$$\therefore y''(n) = \frac{1}{3}[x_1(n)+x_2(n)]+\frac{1}{3}[x_1(n-1)+x_2(n-1)]+\frac{1}{3}[x_1(n-2)+x_2(n-2)] \quad ...(2)$$

$$\therefore y''(n) = \frac{1}{3}x_1(n)+\frac{1}{3}x_1(n-1)+\frac{1}{3}x_1(n-2)+\frac{1}{3}x_2(n)+\frac{1}{3}x_2(n-1)+\frac{1}{3}x_2(n-2)$$

$$\therefore y''(n) = \frac{1}{3}[x_1(n)+x_1(n-1)+x_1(n-2)]+\frac{1}{3}[x_2(n)+x_2(n-1)+x_2(n-2)]$$

Step IV : From Equations (1) and (2) we see that $y'(n) = y''(n)$;

The given system is linear.

1.7.4 Causal or Anticausal Systems (Causality Property) :

A) Causal systems :

Definition : A system is said to be causal system if output at any instant of time depends only on present and past inputs. But the output does not depend on future inputs.

Examples :

The output of system depends on present and past inputs that means output, $y(n)$ is a function of $x(n)$, $x(n-1)$, $x(n-2)$ ... etc. some examples of causal systems are :

(i) $y(n) = x(n)+x(n-1)$

(ii) $y(n) = 3x(n)$

(iii) $y(n) = x(n)+4x(n-1)$

Significance : Since causal system does not include future input samples; such system is practically realizable. That means such system can be implemented practically. Generally all real time systems are causal systems; because in real time applications only present and past samples are present.

Since future samples are not present; causal system is memoryless system.

B) Anticausal or non-causal systems :

Definition : A system is said to be anticausal system if its output depends not only on present and past inputs but also on future inputs.

Examples :

In this case, output $y(n)$ is function of $x(n)$, $x(n-1)$, $x(n-2)$ .... etc. as well as it is function of $x(n+1)$, $x(n+2)$ ... etc. some examples of non-causal systems are :

(i) $y(n) = x(n)+x(n+1)$

(ii) $y(n) = Bx(n+2)$

(iii) $y(n) = x(n) + nx(n+1)$

Significance : Since non-causal system contains future samples; a non-causal system is practically not realizable. That means in practical cases it is not possible to implement a non-causal system.

- But if the signals are stored in the memory and at a later time they are used by a system then such signals are treated as advanced or future signal. Because such signals are already present, before the system has started its operation. In such cases it is possible to implement a non-causal system.

- Some practical examples of non-causal systems are as follows :
  (i) Population growth (ii) Weather forecasting (iii) Planning commission etc.

Solved Problems :

Prob. 1 : Determine if the systems described by following equations are causal or non-causal.

(i) $y(n) = x(n) + x(n-2)$ (ii) $T[x(n)] = e^{x(n)}$

(iii) $T[x(n)] = ax(n) + 6$ (iv) $y(n) = \cos x(n)$

(v) $y(n) = |x(n)|$ (vi) $y(n) = x(n) + nx(n-1)$

(vii) $y(n) = x(n) + nx(n+1)$, (viii) $y(n) = x(2n)$

(ix) $y(n) = x(-n+2)$ (x) $y(n) = x(n^2)$

Soln. :

(i) $\mathbf{y(n) = x(n) + x(n-2)}$

It is a causal system because output depends only on present input i.e. $x(n)$ and past input i.e. $x(n-2)$.

(ii) $\mathbf{T[x(n)] = e^{x(n)}}$

The given equation is,

$$y(n) = e^{x(n)}$$

This is causal system since output depends on present input, $x(n)$.

(iii) $\mathbf{T[x(n)] = ax(n) + 6}$

The given equation is,

$$y(n) = ax(n) + 6$$

This is causal system because output depends on present input $x(n)$. There is no future input signal.

(iv) $\mathbf{y(n) = \cos x(n)}$

This is a causal system because the function of system is to obtain cosine value of present input.

(v) $\mathbf{y(n) = |x(n)|}$

This is a causal system because output depends on present input.

(vi) $\mathbf{y(n) = x(n) + n\,x(n-1)}$

This is a causal system because output depends on present and past input but not on the future input.

(vii) $\mathbf{y(n) = x(n) + n\,x(n+1)}$

Here output depends on future input i.e. $x(n+1)$. So this is non-causal system.

(viii) $y(n) = x(2n)$

This is non-causal system because output expects future input. This can be verified by putting different values of n.

at $n = 0 \Rightarrow y(0) = x(0)$ Here present output excepts present input only.

at $n = 1 \Rightarrow y(1) = x(2)$ Here present output i.e. at n = 1, expects future value of input i.e. $x(2)$

(ix) $y(n) = x(-n+2)$

This is non-causal system. This is because at $n = -1$ we get,

$$y(-1) = x[-(-1)+2] = x[1+2] = x(3)$$

Thus present output at $n = -1$, expects future input i.e. $x(3)$.

(x) $y(n) = x(n^2)$

This is non-causal system. This is because at n = 2 we get,

$$y(2) = x(2^2) = x(4).$$

Thus present output at n = 2, expects future input $x(4)$.

1.7.5 Stable or Unstable Systems (Stability Property) :

To define stability of a system we will use the term 'BIBO'. It stands for Bounded Input Bounded Output. The meaning of word 'bounded' is some finite value. So bounded input means input signal is having some finite value. i.e. input signal is not infinite. Similarly bounded output means, the output signal attains some finite value i.e. the output is not reaching to infinite level.

Definition of stable system : An initially relaxed system is BIBO stable if and only if every bounded input produces bounded output.

Here a relaxed system means when input to the system is zero then the output of system is also zero.

Mathematical representation :

Let us consider some finite number M_x whose value is less than infinity. That means $M_x < \infty$, so its a finite value. Then if input is bounded, we can write,

$$|x(n)| \leq M_x < \infty \quad \ldots(1)$$

Similarly consider some finite number M_y whose value is less than infinity. That means $M_y < \infty$, so its a finite value. Then if output is bounded, we can write,

$$|y(n)| \leq M_y < \infty \quad \ldots(2)$$

Definition of unstable system :

An initially relaxed system is said to be unstable if bounded input produces unbounded (infinite) output.

Significance :

- Unstable system shows erratic and extreme behaviour.

- When unstable system is practically implemented then it causes overflow.

Solved Problems :

Prob. 1 : Determine whether the following discrete-time systems are stable or not.

(i) $T[x(n)] = e^{x(n)}$ (ii) $T[x(n)] = ax(n)+6$

(iii) $y(n) = \cos[x(n)]$ (iv) $y(n) = x(-n)$

(v) $y(n) = x(2n)$

Soln. :

(i) $T[x(n)] = e^{x(n)}$

The given equation is,

$$y(n) = e^{x(n)}$$

We have to check the stability of a system by applying bounded input. That means the value of $x(n)$ should be finite (bounded). The value of 'e' is 2.718. So for bounded input, the output $y(n)$ will be bounded. Thus this is stable system.

(ii) $T[x(n)] = ax(n)+6$.

The given equation is, $y(n) = ax(n)+6$

Here 'a' is some arbitrary constant. So as long as $x(n)$ is bounded the output $y(n)$ is also bounded. Thus it is stable system.

(iii) $y(n) = \cos[x(n)]$.

For every bounded value of $x(n)$, its cosine value is always bounded (finite). So given **system is stable.**

(iv) $y(n) = x(-n)$

Here $x(-n)$ means folding of input sequence. That means taking mirror image of $x(n)$. But in case of folding operation; the amplitude of signal is not changed. So if input is bounded the output $y(n) = x(-n)$ will be bounded. Thus it is stable system.

(v) $y(n) = x(2n)$

If input $x(n)$ is bounded then output $y(n)$ is also bounded. So it is a stable system.

1.7.6 More Solved Problems on D.T. Systems :

Prob. 1 : Explain the given systems with respect to following properties :
(i) Dynamicity (ii) Time invariance (iii) Linearity (iv) Causality (v) Stability.

(1) $y(n) = \sum_{k=-\infty}^{n} x(k)$,

(2) $y(n) = \text{sgn}[x(n)]$ (3) $y(n) = \text{Trunc}[x(n)]$

Soln. :

(1) $y(n) = \sum_{k=-\infty}^{n} x(k)$

(i) ***Dynamicity :*** Expand the given summation by putting different values of k.

$$\therefore y(n) = x(-\infty) + ... + x(-3) + x(-2) + x(-1) + x(0) + x(1) + x(2) + ... + x(n) \quad ...(1)$$

Observe this equation. This equation contains past terms $x(-3)$, $x(-2)$ ... etc. These terms should be stored in memory. So it is dynamic system.

(ii) ***Time invariance :***

Time invariance property states that if we delay the input sequence then output should be delayed by same amount. Observe Equation (1). It is summation of all input terms. So it is clear that if we delay input by some value then output $y(n)$ will be delayed by same amount. Thus this is time invariant system.

(iii) ***Linearity :*** Equation (1) indicates the addition of individual terms and the addition operation is linear. So given system is linear.

(iv) ***Causality :*** From Equation (1) we can say that the present output $y(n)$ depends on (a) present input $x(n)$ and (b) Past inputs like $x(-3)$, $x(-2)$, $x(-1)$ .... But there is no future or advanced input term like $x(n+1)$. Thus this system is causal.

(v) ***Stability :*** We know that the system is stable if bounded input produces bounded output. Now, given equation is,

$$y(n) = \sum_{k=0}^{n} x(k)$$

Here $x(k)$ is the input signal. To check the stability of a system, we have to apply bounded input signal. So let us assume that we are applying unit step as an input signal.

That means $x(k) = u(k)$ ... An unit step.

We know that unit step is present from $n = 0$ to $n = \infty$ and its magnitude is always 1. So its value is bounded (less than ∞). This unit step is shown in Fig. A-32.

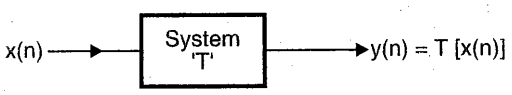


Fig. A-32 : Bounded input, u (k)

Mathematically it is expressed as,

$$u(k) = \begin{cases} 1 & \text{for } k \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Since input x (k) = u (k) we can write given equation y (n) as,

$$y(n) = \sum_{k=0}^{n} u(k)$$

Let us calculate y (n) for few values of n.

$$\text{for } n = 2 \Rightarrow y(2) = \sum_{k=0}^{2} u(k) = u(0)+u(1)+u(2) = 1+1+1 = 3$$

$$\text{For } n = 5 \Rightarrow y(5) = \sum_{k=0}^{5} u(k) = u(0)+u(1)+u(2)+u(3)+u(4)+u(5)$$

$$= 1+1+1+1+1+1 = 6$$

$$\text{For } n = 10 \Rightarrow y(10) = \sum_{k=0}^{10} u(k)$$

$$= u(0)+u(1)+u(2)+u(3)+u(4)+u(5)+u(6)+u(7)+u(8)+u(9)+u(10)$$

$$= 11$$

It is clear that as $n \to \infty$, $y(n) \to \infty$, even if input u (k) is bounded. Thus the system is unstable.

(2) y (n) = sgn [x (n)]

The signum (sgn) function is defined as,

$$\text{sgn}[x(n)] = \begin{cases} 1 & \text{for } n \geq 0 \\ -1 & \text{for } n < 0 \end{cases}$$

It is shown in Fig. A-33.

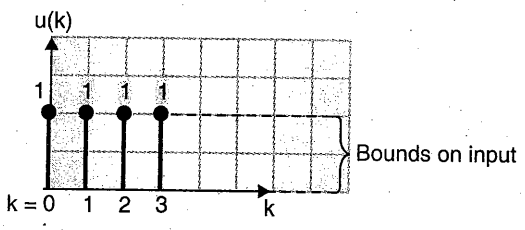


Fig. A-33 : Signum function sgn [x (n)]

(i) Dynamicity :

Observe the given equation. The output contains only present input x (n). Thus the given system is static.

(ii) Time invariance :

Step I : Delay the input $x(n)$ by 'k' samples and apply it to the system. The corresponding output is $y(n,k)$.

$$\therefore \quad y(n,k) = \text{sgn}[x(n-k)] \qquad ...(1)$$

Step II : Replace 'n' by $n-k$ throughout the system and denote the corresponding output by $y(n-k)$.

$$\therefore \quad y(n-k) = \Delta\,\text{gn}[x(n-k)] \qquad ...(2)$$

Step III : Compare Equations (1) and (2).

Since $y(n,k) = y(n-k)$; the system is time invarient.

(iii) Linearity :

Step I : For zero input, and zero output.

Step II : Consider that there are two inputs $x_1(n)$ and $x_2(n)$. Apply these input separately to the system.

$$\therefore \quad x_1(n) \xrightarrow{T} y_1(n) = \text{sgn}[x_1(n)]$$

$$\text{and} \quad x_2(n) \xrightarrow{T} y_2(n) = \text{sgn}[x_2(n)]$$

Add these two outputs to get $y'(n)]$

$$\therefore \quad y'(n) = y_1(n) + y_2(n)$$

$$\therefore \quad y'(n) = \text{sgn}[x_1(n)] + \text{sgn}[x_2(n)] \qquad ...(1)$$

Step III : Combine two inputs $x_1(n)$ and $x_2(n)$ and then apply it to the system.

$$\therefore \quad [x_1(n) + x_2(n)] \xrightarrow{T} \text{sgn}[x_1(n) + x_2(n)]$$

$$\therefore \quad y''(n) = \text{sgn}[x_1(n) + x_2(n)] \qquad ...(2)$$

Step IV : Compare Equations (1) and (2).

In Equation (1),

$$\text{sgn}[x_1(n)] = \begin{cases} 1 & \text{for } n \geq 0 \\ -1 & \text{for } n < 0 \end{cases} ; \text{ and } \text{sgn}[x_2(n)] = \begin{cases} 1 & \text{for } n \geq 0 \\ -1 & \text{for } n < 0 \end{cases}$$

While in Equation (2),

$$\text{sgn}[x_1(n) + x_2(n)] = \begin{cases} 1 & \text{for } n \geq 0 \\ -1 & \text{for } n < 0 \end{cases}$$

Thus Equations (1) and (2) are not same. So given system is non-linear.

(iv) Causality :

Observe the given equation of y (n). The output depends only on present input; it does not contain any future input term. So given system is causal.

(v) Stability :

From Fig. A-33(b); we can say that the maximum value of y (n) = ± 1 for any value of n. This is definitely less than infinity. Thus for bounded input system produces bounded output. Thus it is a stable system.

(3) y (n) = Trunc [x (n)]

Here 'Trunc' stands for truncation. So if x (n) = 3.4354 then,

$$\text{Trunc } [x (n)] = 3$$

(i) Dynamicity :

Truncation operation contains only present samples, not the past sample. So memory is not required. Thus it is a static system.

(ii) Time Invarience :

If input is delayed by some factor then truncation is not affected. Because this operation is related to amplitude change and not the change in time period. So this is time invarient system.

(iii) Linearity :

Consider two inputs $x_1 (n)$ and $x_2 (n)$.

$$\text{Let} \quad x_1 (n) = 3.4567 \text{ and } x_2 (n) = 4.3456$$

$$\text{Now} \quad y_1 (n) = \text{Trunc } [x_1 (n)] = \text{Trunc } [3.4567] \approx 3$$

$$\text{and} \quad y_2 (n) = \text{Trunc } [x_2 (n)] = \text{Trunc } [4.3456] \approx 4$$

$$\therefore \quad y' (n) = y_1 (n) + y_2 (n) = 3 + 4 = 7$$

$$\text{Now} \quad \text{Trunc } [x_1 (n) + x_2 (n)] = \text{Trunc } [3.4567 + 4.3456]$$

$$\therefore \quad y'' (n) = \text{Trunc } [7.8023]$$

$$\therefore \quad y'' (n) = 8$$

$$\text{Since} \quad y' (n) \neq y'' (n); \text{ the system is non-linear.}$$

(iv) Causality :

The truncation operation depends only on present input. It does not contain any future sample. So the system is causal.

(v) Stability :

If input x (n) is bounded (less than infinity) then the result of truncation is also bounded. So it is BIBO stable system.

Note : The properties of rounding operation are similar to truncation operation. So on the same lines we can show that rounding operation is static, non-linear, time invarient, causal and stable.

1.8 Linear Time Invarient (LTI or LSI) Systems :

LSI stands for linear shift invarient system. It is also called as LTI system. We have already studied different characteristics of LTI system like linearity, causality, stability etc. we have studied the basic operation of discrete time (DT) systems. These basic operations are required to perform linear convolution.

1.9 Impulse Response :

In this sub-section we will show that LTI system can be completely characterized by its impulse response. We will be using following notations related to discrete time systems.

$x(n) \Rightarrow$ Input sequence

$h(n) \Rightarrow$ Impulse response of system

$y(n) \Rightarrow$ Output sequence

Consider input sequence,

$$x(n) = \{1, 2, 1, 2, 1\} \quad \text{...(1)}$$
$$\uparrow$$

This sequence is represented as shown in Fig. A-34.

The different values of $x(n)$ are as follows :

$x(-2) = 1,\ x(-1) = 2,\ x(0) = 1,\ x(1) = 2,\ x(2) = 1$

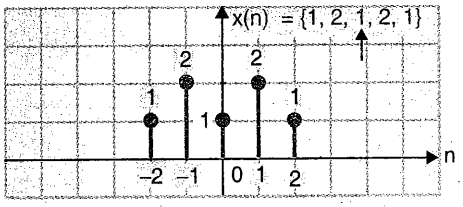


Fig. A-34(a) : Input sequence x (n)

Now consider unit impulse $\delta(n)$. It is expressed as,

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{elsewhere} \end{cases} \quad \text{...(2)}$$

That means its value is 1 at n = 0 and its value is zero for all other values of 'n'. Such unit impulse is as shown in Fig. A-34(b).

If we want to delay this unit impulse by '2' samples then it is denoted by $\delta(n-2)$

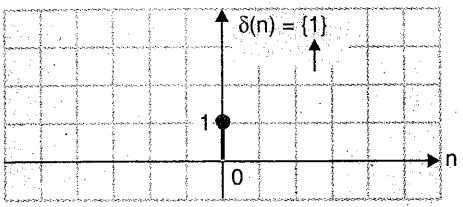


Fig. A-34(b) : Unit impulse, $\delta(n)$

$$\text{And} \quad \delta(n-2) = \begin{cases} 1 & \text{for } n = 2 \\ 0 & \text{otherwise} \end{cases} \qquad \ldots(3)$$

This sequence is shown in Fig. A-34(c).

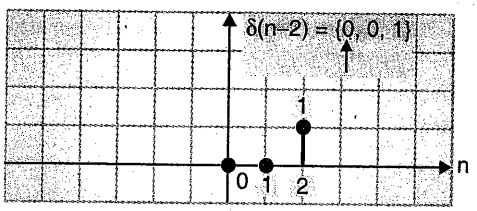


Fig. A-34(c) : Impulse sequence $\delta(n-2)$

Similarly if we want to advance this unit impulse by '2' samples then it is denoted by $\delta(n+2)$.

$$\therefore \quad \delta(n+2) = \begin{cases} 1 & \text{for } n = 2 \\ 0 & \text{otherwise} \end{cases} \qquad \ldots(4)$$

This sequence is shown in Fig. A-34(d)

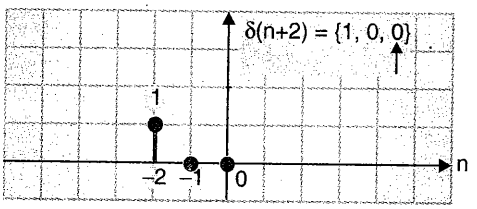


Fig. A-34(d) : Impulse sequence $\delta(n+2)$

In general, if $\delta(n)$ is delayed by 'k' samples then we can write,

$$\delta(n-k) = \begin{cases} 1 & \text{for } n = k \\ 0 & \text{otherwise} \end{cases} \qquad \ldots(5)$$

Decomposition of input sequence x (n) :

The given input sequence is,

$$x(n) = \{1, 2, 1, 2, 1\}$$
$$\uparrow$$

Its range is from -2 to $+2$. Now decomposition means obtaining a particular sample from given sequence $x(n)$. Suppose we want to obtain sample at $n = -2$ that means $x(-2)$. Then take the multiplication of $x(n)$ and $\delta(n+2)$. We know that the value of $\delta(n+2)$ is one only at $n = -2$, while for all other values of n, it is zero. Thus if we take multiplication of $x(n)$ and $\delta(n+2)$ then we will get the sample $x(-2)$. Because all other terms become zero, after multiplication. This is shown in Fig. A-34(e).

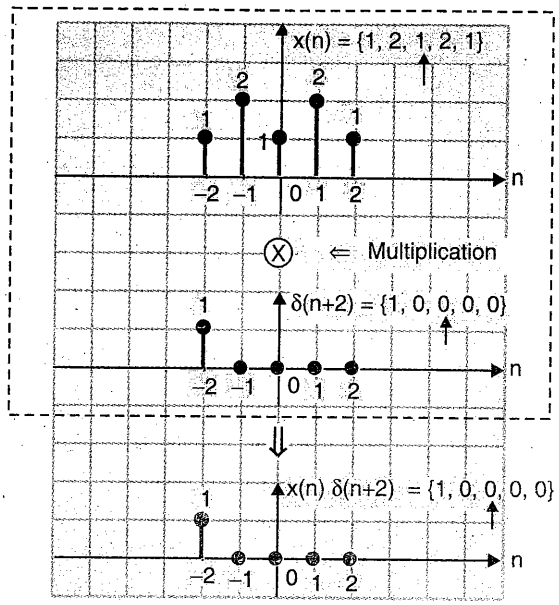


Fig. A-34(e) : Decomposed input signal sample at n = – 2

Similarly other decomposed input signal samples are obtained.

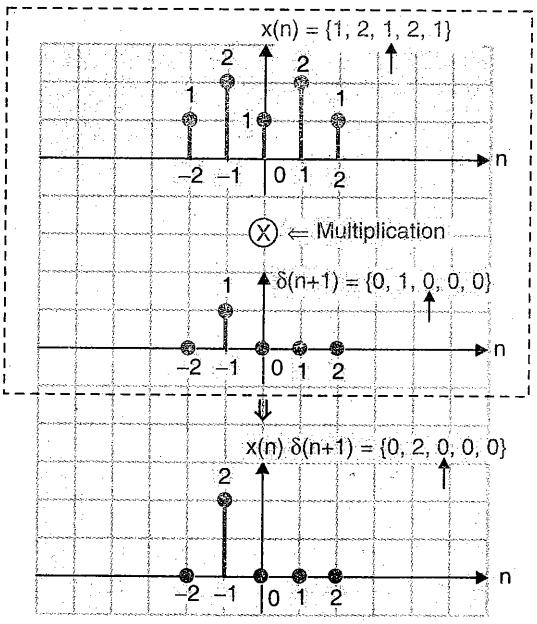


Fig. A-34(f) : Decomposed sample at n = – 1

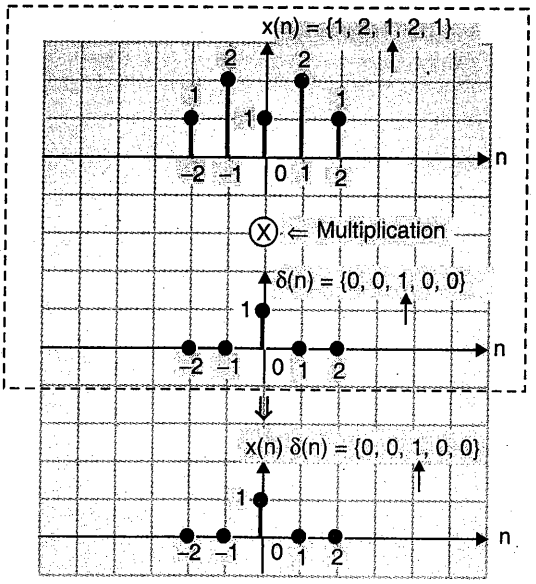


Fig. A-34(g) : Decomposed sample at n = 0

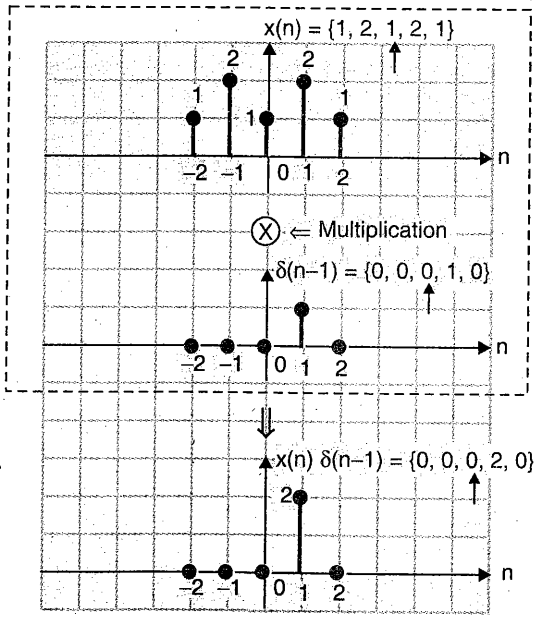


Fig. A-34(h) : Decomposed sample at n = + 1

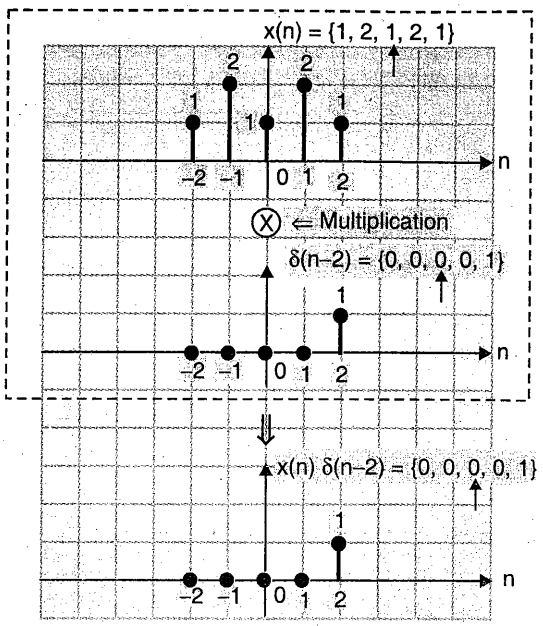


Fig. A-34(i) : Decomposed sample at n = + 2

Now the original sequence $x(n)$ can be obtained simply by adding all decomposed samples as shown in Fig. A-35.

Mathematically we can write,

$$x(n) = x(n) \cdot \delta(n+2) + x(n) \cdot \delta(n+1) + x(n) \cdot \delta(n) + x(n) \cdot \delta(n-1) + x(n) \cdot \delta(n-2) \quad ...(6)$$

Observe every decomposed sample. We get a decomposed sample only at $n = k$. For example sample $x(-2)$ is obtained by multiplication of $x(n)$ and $\delta(n+2)$ as shown in Fig. A-34(e). This sample is present at $n = k = -2$. Thus for simplicity replacing $x(n)$ by $x(k)$ in R.H.S. of Equation (6) we get,

$$x(n) = x(k)\,\delta(n+2) + x(k)\,\delta(n+1) + x(k)\,\delta(n) + x(k)\,\delta(n-1) + x(k)\,\delta(n-2) \quad ...(7)$$

In terms of summation we can write,

$$x(n) = \sum_{k=-2}^{2} x(k)\,\delta(n-k) \quad ...(8)$$

In Equation (7), the amplitude of every 'δ' term is 1. We are multiplying every unit impulse (advanced or delayed) by corresponding amplitude of input sample. This is called as scaling of unit samples. Thus Equations (7) and (8) indicates that input sequence $x(n)$ is represented as sum of weighted (or scaled) unit impulses.

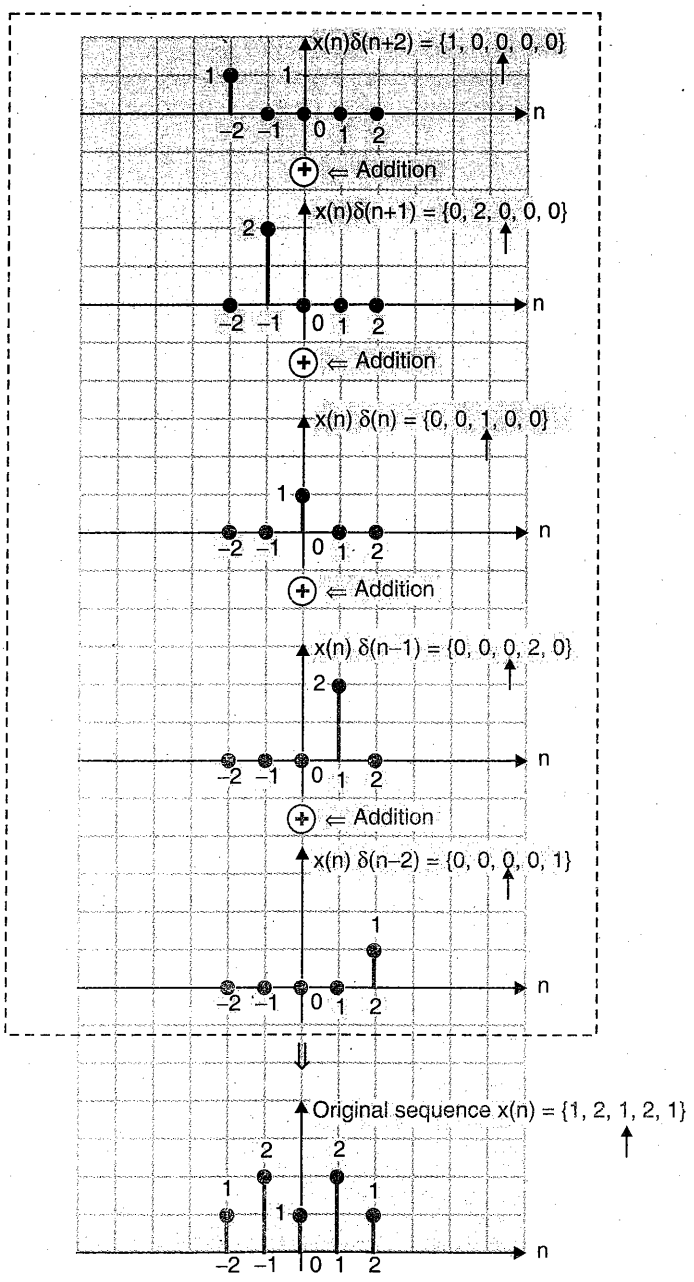


Fig. A-35 : x (n) = Addition of all decomposed samples

In Equation (8) the limits of summation are from -2 to $+2$; this is because input sequence x (n) is present from $n = -2$ to $n = +2$. In general we can write Equation (8) as,

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k) \qquad \ldots(9)$$

1.10 Linear Convolution :

Proof of linear convolution Or Proof of "LTI system is completely characterized by unit impulse response h (n)" :

Consider relaxed LSI (LTI) system. A relaxed system means if input x (n) is zero then output y (n) is zero. Let us say unit impulse $\delta(n)$ is applied to this system then its output is denoted by h (n). "h (n)" is called as impulse response of system.

Step I : $\delta(n) \xrightarrow{T} y(n) = h(n)$

Since the system is time invariant, if we delay input by 'k' samples then output should be delayed by same amount.

Step II : $\delta(n-k) \xrightarrow{T} y(n) = h(n-k)$

Now multiplying both sides by x (k) we get,

Step III : $x(k)\,\delta(n-k) \xrightarrow{T} y(n) = x(k)\,h(n-k)$

Since the system is linear; we can apply superposition theorem. So taking summation at both sides we get,

Step IV :

$$\sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k) \xrightarrow{T} y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

Thus we have,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) \qquad \ldots(10)$$

This summation is called as convolution sum. In this case output y (n) is obtained by taking convolution of x (n) and h (n). This is called as linear convolution and it is denoted by symbol '*'.

$$\therefore \quad x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

In this case we can say that the behaviour of LTI system is completely characterized by its impulse response.

Prob. 1 : Determine impulse response of the DT-LTI system in terms of unit impulse input $\delta(n)$ for $h(n) = 2^{-n}u(n) - 2^{-n}u(n-3)$.

Soln. :

Given : $h(n) = 2^{-n}u(n) - 2^{-n}u(n-3)$

We have to express $h(n)$ in terms of unit impulse response $\delta(n)$.

$$\therefore \quad h(n) = 2^{-n}[u(n) - u(n-3)]$$

The sequence $u(n) - u(n-3)$ can be written as,

$$u(n) - u(n-3) = \{1, 1, 1\}$$
$$\uparrow$$

Since every sample is having unity magnitude,

we can write,

$$u(n) - u(n-3) = \{1, 1, 1\} = \delta(n) + \delta(n-1) + \delta(n-2)$$
$$\uparrow$$

$$\therefore \quad h(n) = 2^{-n}[\delta(n) + \delta(n-1) + \delta(n-2)]$$

$$\therefore \quad h(n) = 2^{-n}\delta(n) + 2^{-n}\delta(n-1) + 2^{-n}\delta(n-2)$$

This gives impulse response in terms of $\delta(n)$.

1.10.1 Computation of Linear Convolution :

The different methods used for the computation of linear convolution are as follows :

1) Graphical method
2) Using mathematical equation of convolution
3) Tabulation method
4) Multiplication method

1) Graphical method :

The linear convolution of two sequence is given by,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \qquad \ldots(1)$$

We will have to calculate output for different values of 'n'.

For n = 0 we get,

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)\,h(-k) \qquad ...(2)$$

Note that $h(-k)$ indicates folding of $h(k)$.

For n = 1 we get,

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)\,h(1-k) \qquad ...(3)$$

Here the term $h(1-k)$ can be written as $h(-k+1)$. Thus Equation (3) becomes,

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)\,h(-k+1) \qquad ...(4)$$

Here $h(-k+1)$ indicates shifting of folded signal $h(-k)$. It indicates that $h(-k)$ is delayed by '1' sample. Similarly for other values of 'n' output $y(n)$ is calculated.

Thus different operations involved in the calculation of linear convolution are as follows :

i) Folding operation : It indicates folding of sequence $h(k)$.

ii) Shifting operation : It indicates time shifting of $h(-k)$ eg. $h(-k+1)$.

iii) Multiplication : It indicates multiplication of $x(k)$ and $h(n-k)$.

iv) Summation : It indicates addition of all product terms obtained because of multiplication of $x(k)$ and $h(n-k)$

Solved Problems :

Prob. 1 : Obtain linear convolution of following sequences

$x(n) = \{1, 2, 1, 2\}$ and $h(n) = \{1, 1, 1\}$

Soln. : If the sequences are given in terms of 'n' then obtain $x(k)$ and $h(k)$ by replacing 'n' by 'k'.

$$\therefore \quad x(k) = \{1, 2, 1, 2\} \qquad \text{and} \qquad h(k) = \{1, 1, 1\}$$

Arrow is not mentioned in any sequences. So by default it is at first position.

$$\therefore \quad x(k) = \underset{\uparrow}{\{1, 2, 1, 2\}} \qquad \text{and} \qquad h(k) = \underset{\uparrow}{\{1, 1, 1\}}$$

Now we have equation of convolution,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) \qquad ....(1)$$

Now we have to decide range of 'n' and 'k'.

Range of 'n' :

Basically to calculate y (n) we should know from which value of 'n'. We should start (lower range of n) and what is ending value of 'n' (higher range of n). We will use following notations to decide range of y (n).

y_l = Lowest range of y (n)

y_h = Highest range of y (n)

x_l = Lowest range of x (k)

x_h = Highest range of x (k)

h_l = Lowest range of h (k)

h_h = Highest range of h (k)

Now use following formulae to calculate range of y (n)

$$y_l = x_l + h_l \text{ and } y_h = x_h + h_h$$

From the given sequences we have,

$$x_l = 0, \quad x_h = 3, \quad h_l = 0 \text{ and } \quad h_h = 2$$

$$\therefore \quad y_l = x_l + h_l = 0 + 0 = 0$$

and $$y_h = x_h + h_h = 3 + 2 = 5$$

Thus range of y (n) is from y (0) to y (5).

Range of 'k' : The value of k in the summation sign will be always same as the sequence x (k). Thus in this case the range of k is from k = 0 to 3. So Equation (1) becomes,

$$y(n) = \sum_{k=0}^{3} x(k)\, h(n-k) \qquad \ldots(2)$$

Now we will calculate output y (n) by putting different values of n from 0 to 5.

Calculation of y (0) :

Putting n = 0 in Equation (2) we get,

$$y(0) = \sum_{k=0}^{3} x(k)(0-k) = \sum_{k=0}^{3} x(k)\, h(-k) \qquad \ldots(3)$$

Now we have to perform calculations using graphical method. Here h (– k) indicates folded version of h (k).

Step I : Sketch x (k).

Sketch of x (k) is as shown in Fig. A-36(a)

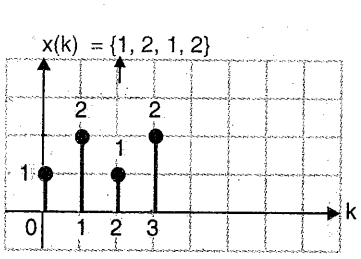


Fig. A-36(a) : Sequence x (k)

Step II : Sketch h (k)

Sketch of h (k) is shown in Fig. A-36(b).

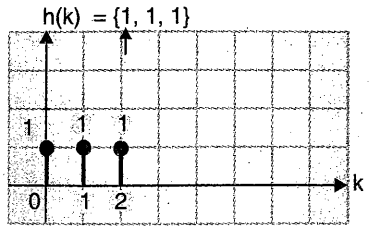


Fig. A-36(b) : Sequence h (k)

Step III : Fold h (k) to obtain h (– k).

Sketch of h (– k) is shown in Fig. A-36(c)

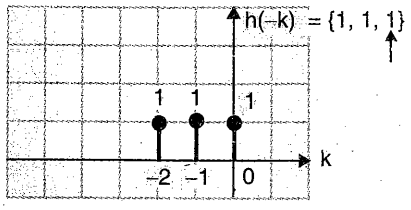


Fig. A-36(c) : Folded sequence h (– k)

Step IV : Obtain multiplication of x (k) and h (– k). This multiplication takes place on sample to sample basis as shown in Fig. A-36(d).

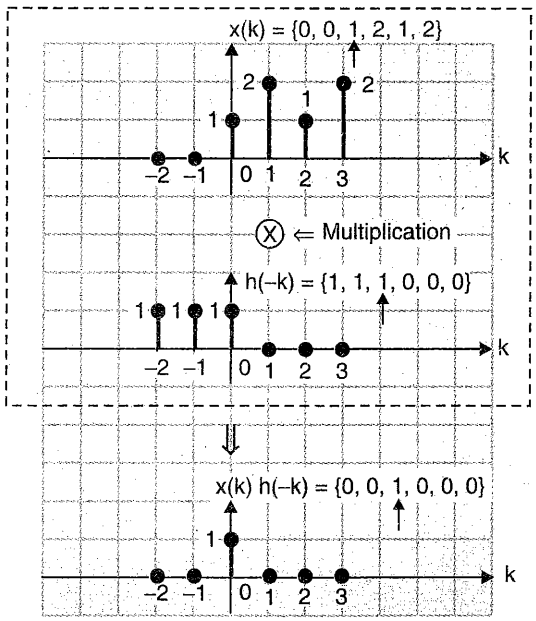


Fig. A-36(d) : Product of x (k) and h (– k)

Step V : Take the summation of all product terms.

In this case as result of multiplication we have only one sample at n = 0.

$$\therefore \quad y(0) = 1$$

Calculation of y (1) : Putting n = 1 in Equation (2) we get,

$$y(n) = \sum_{k=0}^{3} x(k)\,h(1-k) \qquad \ldots(4)$$

h (1 − k) can be written as h (− k + 1). Thus Equation (4) becomes,

$$y(n) = \sum_{k=0}^{3} x(k)\,h(-k+1) \qquad \ldots(5)$$

Step I : Now h (− k + 1) indicates delay of h (− k) by '1' sample. So shift the sequence in Fig. A-36(c) towards right by '1' sample. This sequence is shown in Fig. A-37(a).

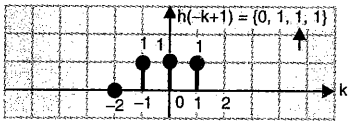


Fig. A-37(a) : Sequence h (− k + 1)

Step II : Obtain multiplication of x (k) and h (− k + 1). It is shown in Fig. A-37(b).

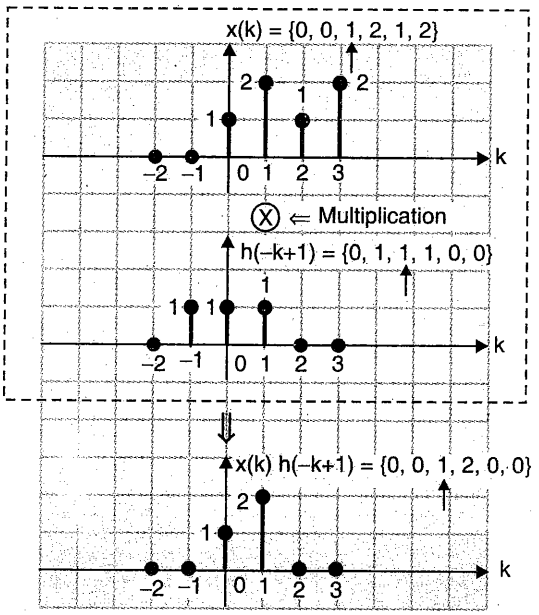


Fig. A-37(b) : Product of x (k) and h (− k + 1)

Step III : Now according to Equation (5) we have to add all product terms shown in Fig. A-37(b) to obtain y (n).

$$\therefore \quad y(1) = (1\times 1)+(2\times 1) = 1 + 2$$

$$\therefore \; y(1) = 3$$

Calculation of y (2) :

Putting n = 2 in Equation (2) we get,

$$y(2) = \sum_{k=0}^{3} x(k)\,h(2-k) \qquad ...(6)$$

The term h (2 – k) is same as h (– k + 2).

$$\therefore \quad y(2) = \sum_{k=0}^{3} x(k)\,h(-k+2) \qquad ...(7)$$

Step I : Here h (– k + 2) indicates delay of h (– k) by '2' samples. This is obtained by shifting sequence in Fig. A-36(c) towards right by '2' samples. This sequence is shown in Fig. A-38(a).

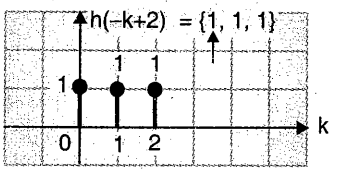


Fig. A-38(a) : Sequence h (– k + 2)

Step II : Obtain multiplication of x (k) and h (– k + 2). It is shown in Fig. A-38(b).

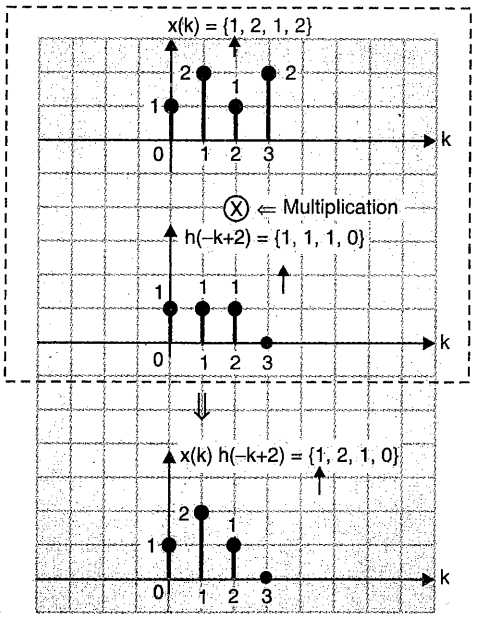


Fig. A-38(b) : Product of x (k) and h (– k + 2)

Step III : Add all product terms to obtain y (2).

$$\therefore \quad y(2) = (1\times1)+(1\times2)+(1\times1) = 1+2+1$$

$$\therefore \quad y(2) = 4$$

Calculation of y (3) : Putting n = 3 in Equation (2) we get,

$$y(3) = \sum_{k=0}^{3} x(k)\,h(3-k) \qquad ...(8)$$

The term h (3 – k) is same as h (– k + 3)

$$\therefore \quad y(3) = \sum_{k=0}^{3} x(k)\,h(-k+3) \qquad ...(9)$$

Step I : Here h (– k + 3) indicates delay of h (– k) by 3 samples. It is shown in Fig. A-39(a).

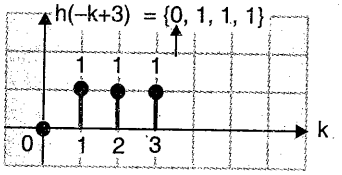


Fig. A-39(a) : Sequence h (– k + 3)

Step II : Obtain multiplication of x (k) and h (– k + 3). This sequence is shown in Fig. A-39(b).

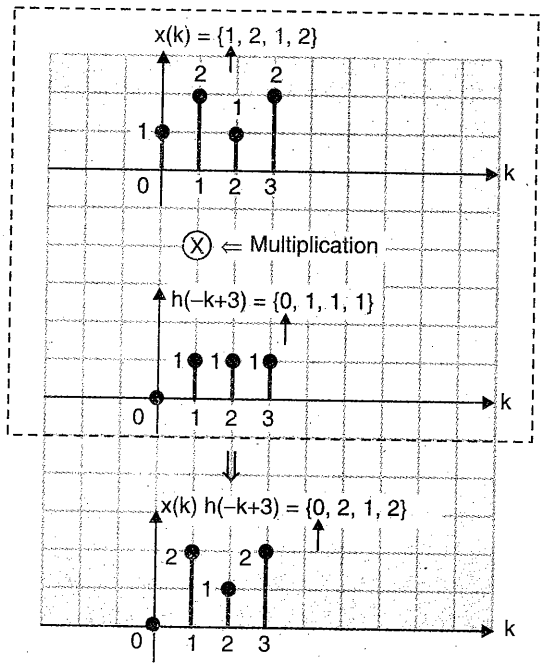


Fig. A-39(b) : Product of x (k) and h (– k + 3)

Step III : Add all product terms to obtain y (3).

$$\therefore \quad y(3) = (0\times1)+(1\times2)+(1\times1)+(2\times1) = 0+2+1+2$$

$\therefore \quad y(3) = 5$

Calculation of y (4) : Putting n = 4 in Equation (2) we get,

$$y(4) = \sum_{k=0}^{4} x(k)\,h(4-k) \qquad \ldots(10)$$

The term $h(4-k)$ is same as $h(-k+4)$

$$\therefore \quad y(4) = \sum_{k=0}^{4} x(k)\,h(-k+4) \qquad \ldots(11)$$

Step I : Here $h(-k+4)$ indicates delay of $h(-k)$ by '4' samples. It is shown in Fig. A-40(a).

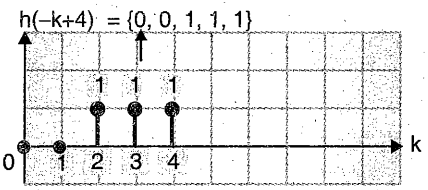


Fig. A-40(a) : Sequence $h(-k+4)$

Step II : Obtain multiplication of $x(k)$ and $h(-k+4)$. This sequence is shown in Fig. A-40(b).

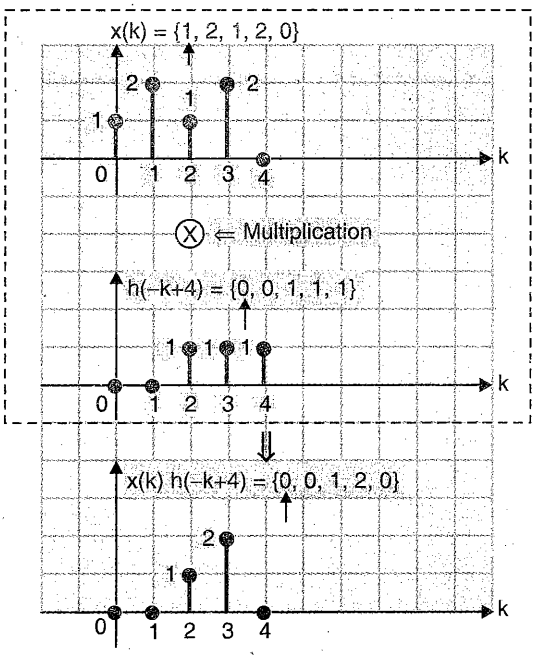


Fig. A-40(b) : Product of $x(k)$ and $h(-k+4)$

Step III : Add all product terms to obtain $y(4)$.

$\therefore \quad y(4) = (1\times0)+(2\times0)+(1\times1)+(2\times1)+(0\times1) = 0+0+1+2+0$

$\therefore$ y (4) = 3

Calculation of y (5) : Putting n = 5 in Equation (2) we get,

$$y(5) = \sum_{k=0}^{3} x(k)\, h(5-k) \qquad \ldots(12)$$

The term h (5 – k) is same as h (– k + 5)

$$\therefore \quad y(5) = \sum_{k=0}^{3} x(k)\, h(-k+5) \qquad \ldots(13)$$

Step I : Here h (– k + 5) indicates delay of h (– k) by '5' samples. It is shown in Fig. A-41(a).

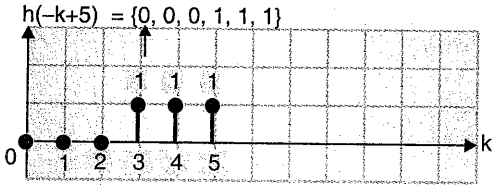


Fig. A-41(a) : Sequence h (– k + 5)

Step II : Obtain multiplication of x (k) and h (– k + 5). This sequence is shown in Fig. A-41(b).

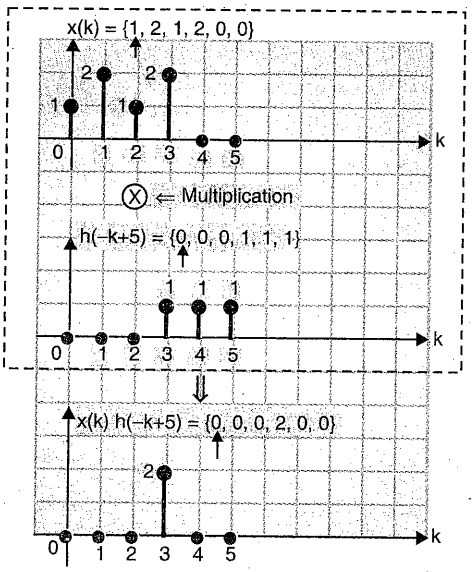


Fig. A-41(b) : Product of x (k) and h (– k + 5)

Step III : Add all product terms to obtain y (5).

$$\therefore \quad y(5) = (1\times0)+(2\times0)+(1\times0)+(2\times1)+(0\times1)+(0\times1)$$

The result of convolution, y (n) is represented graphically as shown in Fig. A-42.

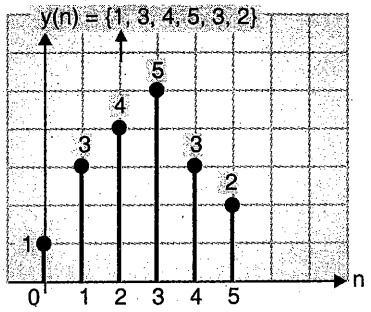


Fig. A-42 : Sketch of y (n) = { 1, 3, 4, 5, 3, 2 }
↑

Prob. 2 : Prove and explain graphically the difference between relations :

(i) $x(n)\,\delta(n-n_0) = x(n_0)$ (ii) $x(n) * \delta(n-n_0) = x(n-n_0)$

Soln. :

(i) $\mathbf{x(n)\,\delta(n-n_0) = x(n_0)}$

Here x (n) is input sequence. Let the input sequence be given by,

$$x(n) = \{1, 1, 2, 1, 1\}$$
↑

This sequence is plotted as shown in Fig. A-43(a).

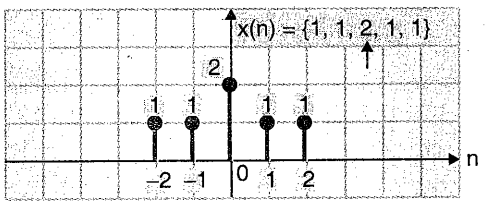
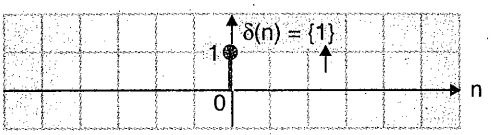


Fig. A-43(a) : Sequence x (n)

We know that δ (n) represent unit impulse. It is defined as,

$$\delta(n) = 1 \quad \text{for } n = 0$$
$$= 0 \quad \text{elsewhere}$$



sequence δ (n) is shown in Fig. A-43(b).

Fig. A-43(b) : Unit impulse, δ (n)

Now $\delta(n-n_0)$ indicates delay of $\delta(n)$ by no units. So sequence $\delta(n-n_0)$ can be represented as,

$$\delta(n-n_0) = 1 \quad \text{for } n = n_0$$
$$= 0 \quad \text{for } n \neq n_0$$

For example, say $n_0 = 1$. Thus we get,

$$\delta(n-1) = 1 \quad \text{for } n = 1$$
$$= 0 \quad \text{for } n \neq n_0$$

The sequence $\delta(n-1)$ is plotted as shown in Fig. A-43(c).

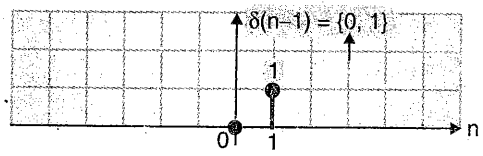


Fig. A-43(c) : Sequence $\delta(n-1)$

Now multiplication of $x(n)$ and $\delta(n-n_0)$ i.e. $\delta(n-1)$ is shown in Fig. A-43(d). Here we are getting the resultant sample only at $n = n_0$. We have assumed $n_0 = 1$. Thus result of multiplication is the sample at $n = 1$. This value is 1; which is the value of $x(1)$.

$$\therefore \quad x(n)\delta(n-1) = x(1)$$

Thus in general we can say,

$$x(n)\delta(n-n_0) = x(n_0)$$

Hence proved.

(ii) $x(n) * \delta(n-n_0) = x(n-n_0)$

Here '*' indicates convolution. So we have to calculate convolution of $x(n)$ and $\delta(n-n_0)$. According to the definition of convolution,

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \qquad ...(1)$$

$$\text{Let} \quad h(n) = \delta(n-n_0) \qquad ...(2)$$

Now assume two sequence as follows,

$$x(n) = \{1, \underset{\uparrow}{1}, 1\} \quad \text{and } h(n) = \delta(n-n_0) = \{0, \underset{\uparrow}{1}\}$$

also assume that value of $n_0 = 1$

Note : We cannot assume any arbitrary sequence for h (n). Since h (n) = δ (n – n_0), it should be in terms of delayed unit impulse.

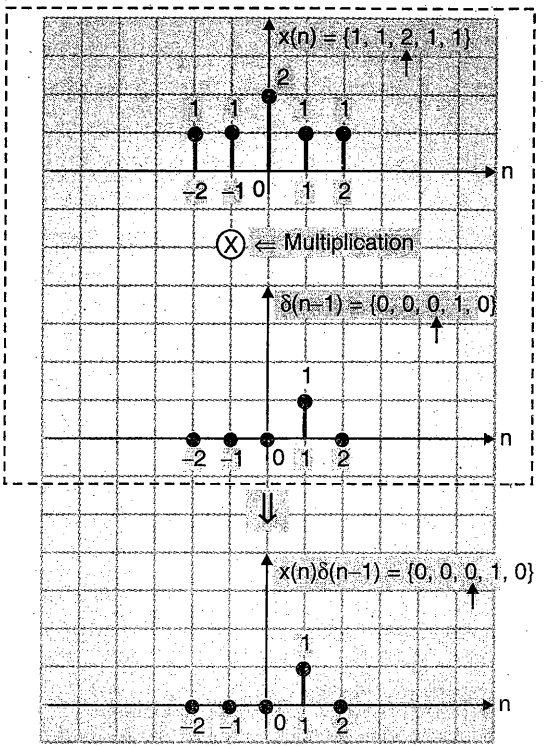


Fig. A-43(d) : Product of x (n) δ (n – n_0)

Step I : Sequences x (k) and h (k) can be written by directly replacing 'n' by 'k'.

$$\therefore \quad x(k) = \underset{\uparrow}{\{1, 1, 1\}} \quad \text{and} \quad h(k) = \delta(k - n_0) = \underset{\uparrow}{\{0, 1\}}$$

Thus different samples of x (k) and h (k) can be written as,

$x(-1) = 1$ similarly, $h(0) = 0$

$x(0) = 1$ and $h(1) = 1$

and $x(1) = 1$

Plot of x (k) and h (k) is shown in Fig. A-44.

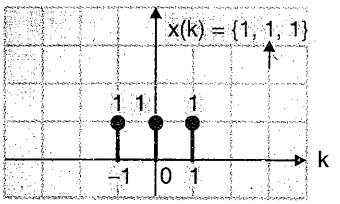


Fig. A-44(a) : Sequence x (k)

Step II : Now range of 'n' is calculated as follows :

lower value of $n \Rightarrow y_l = x_l + h_l = -1 + 0 = -1$

and higher value of $n \Rightarrow y_h = x_h + h_h = 1 + 1 = 2$

Thus range of 'n' is from – 1 to 2. That means we have to calculate sequence y (n) from y (– 1) to y (2).

Range of 'k' is same as x (k). Thus 'k' varies from – 1 to + 1.

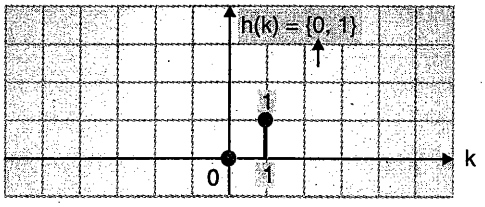


Fig. A-44(b) : Sequence h (k) .

Step III : Putting the value of 'k' Equation (1) can be written as,

$$y(n) = \sum_{k=-1}^{1} x(k)\, h(n-k) \qquad ...(3)$$

To calculate y (n); put values of 'n' from – 1 to + 2.

Calculation of y (– 1) :

Putting n = – 1 in Equation (3) we get,

$$y(-1) = \sum_{k=-1}^{1} x(k)\, h(-1-k) \qquad ...(4)$$

Here h (– 1 – k) is same as h (– k – 1) · h (– k) is folded version of h (k) and h (– k – 1) indicates advance of h (– k) by 1 sample. This sequence is shown in Fig. A-44(c).

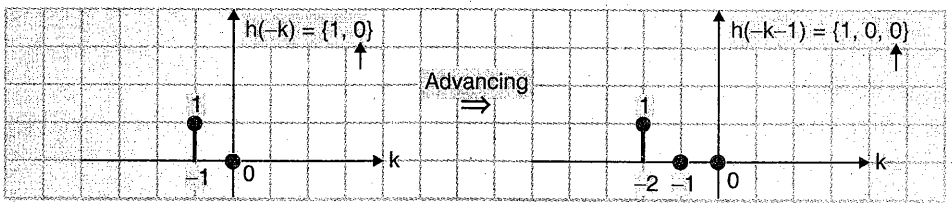


Fig. A-44(c) : Sequence h (– k – 1)

The multiplication of x (k) and h (– k – 1) is shown in Fig. A-44(d).

y (– 1) is obtained by adding all product terms.

$$\therefore \quad y(-1) = 0$$

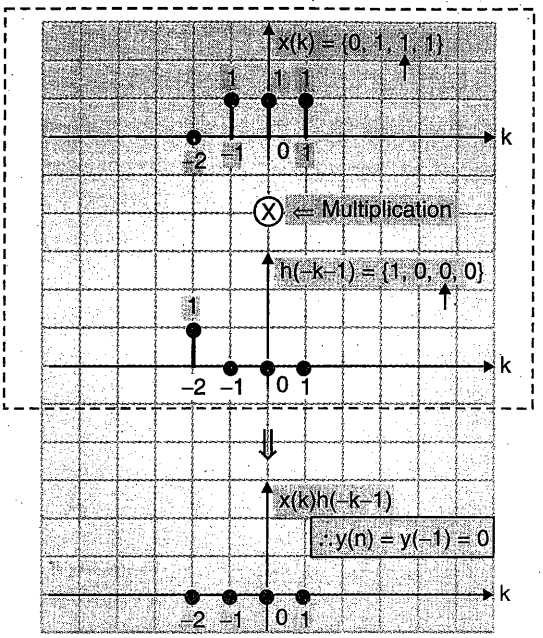


Fig. A-44(d) : Product of x (k) and h (− k − 1)

Calculation of y (0) :

Putting n = 0 in Equation (3) we get,

$$y(0) = \sum_{k=-1}^{1} x(k)\, h(-k) \qquad \ldots(5)$$

Sequence h (− k) is shown in Fig. A-44(c). The product of x (k) and h (− k) shown in Fig. A-44(e).

y (0) is obtained by adding all product terms.

$$\therefore \; y(0) = 1$$

Calculation of y (1) :

Putting n = 1 in Equation (3) we get,

$$y(1) = \sum_{k=-1}^{1} x(k)\, h(1-k) \qquad \ldots(6)$$

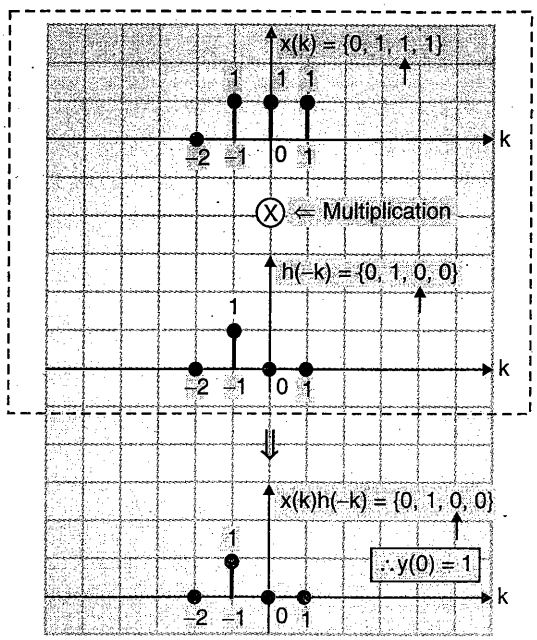


Fig. A-44(e) : Product of x (k) h (– k)

Here h (1 – k) is same as h (– k + 1). It indicates delay of h (– k) by 1 sample. It is obtained by shifting h (– k) towards right by 1 sample. The sequence h (– k + 1) is shown in Fig. A-45. The product of x (k) and h (– k + 1) is shown in Fig. A-46.

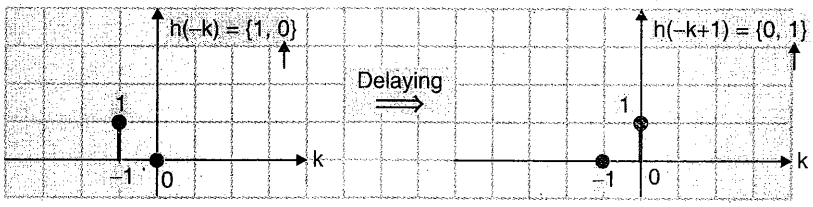


Fig. A-45 : Sequence h (– k + 1)

∴ y (1) = 1

Calculation of y (2) :

Putting n = 2 in Equation (3) we get,

$$y(2) = \sum_{k=-1}^{1} x(k)\, h(2-k) \qquad \ldots(7)$$

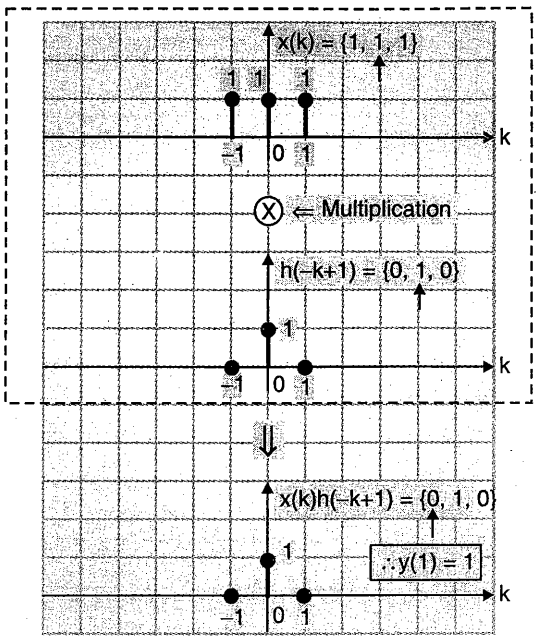


Fig. A-46 : Product of x (k) and h (– k + 1)

Here h (2 – k) is same as h (– k + 2). It indicates delay of h (– k) by two samples. It is obtained by shifting h (– k) towards right by two samples. The sequence h (– k + 2) is shown in Fig. A-47(a). The product of x (k) and h (– k + 2) is shown in Fig. A-47(b).

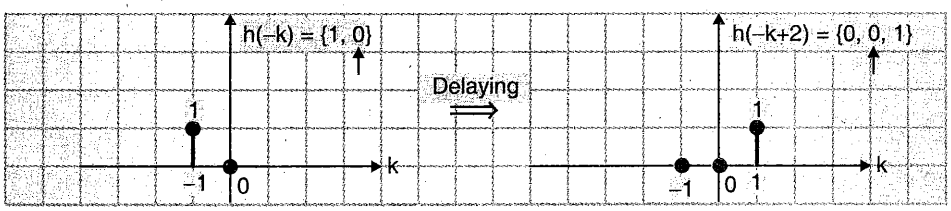


Fig. A-47(a) : Sequence h (– k + 2)

y (2) is obtained by adding all product terms.

$$\therefore \quad y(2) = 1$$

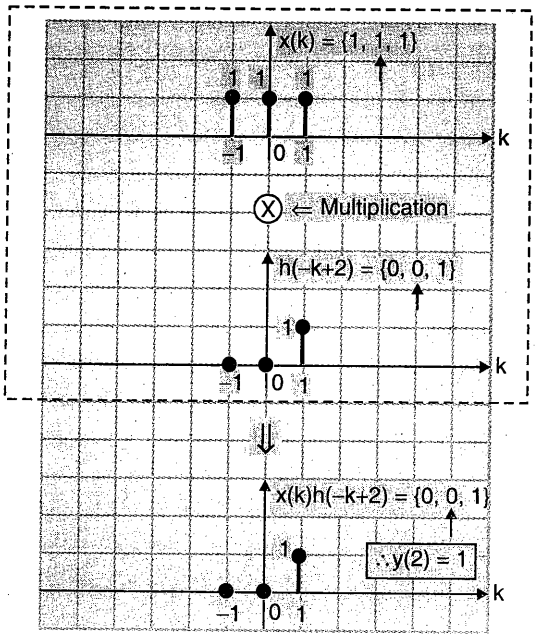


Fig. A-47(b) : Product of x (k) and h (– k + 2)

The output sequence y (n) is,

$$y(n) = \{y(-1), y(0), y(1), y(2)\}$$

$$\therefore \quad y(n) = \{0, 1, 1, 1\}$$
$$\uparrow$$

This sequence is plotted in Fig. A-48.

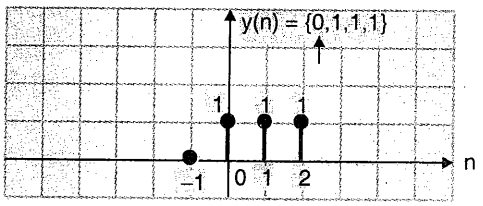


Fig. A-48 : Output y (n)

Now the output sequence y (n) can also be written as,

$$y(n) = \{1, 1, 1\}$$
$$\uparrow$$

Observe graph of x (k) shown in Fig. A-44(a) and graph of y (n) shown in Fig. A-48. It shows that output y (n) is obtained by delaying x (k) by '1' sample.

$$\therefore \quad y(n) = x(k-1)$$

But x (k) is same as x (n)

$$\therefore \quad y(n) = x(n-1)$$

Initially we have assumed $n_0 = 1$

$$\therefore \quad y(n) = x(n - n_0)$$

$$\therefore \quad y(n) = x(n) * \delta(n - n_0) = x(n - n_0)$$

Hence proved.

Note : This indicates that the convolution of input sequence x (n) with delayed unit impulse is equivalent to delay of input sequence by corresponding samples (n_0).

Prob. 3 : Obtain the expression for convolution of unit step sequence with finite duration sequence.

Soln. : We will assume some sequence x (n) as,

$$x(n) = \underset{\uparrow}{\{2, 2, 1\}} \qquad ...(1)$$

Note that this is finite duration sequence.

We have to obtain convolution of x (n) with unit step u (n). So let,

$$h(n) = u(n) \qquad ...(2)$$

The equations of h (k) and x (k) can be directly written as,

$$x(k) = \underset{\uparrow}{\{2, 2, 1\}} \quad \text{and} \quad h(k) = u(k) = \{1, 1, 1, \ldots \infty\}$$

According to the definition of convolution we have,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \qquad ...(3)$$

To obtain range of y (n) :

Lowest index of $y(n) \Rightarrow y_l = x_l + h_l = 0 + 0 = 0$

Highest index of $y(n) \Rightarrow y_h = x_h + h_h = 2 + \infty = \infty$

Thus range of y (n) is from '0' to '∞'. That means we have to obtain y (n) for different values of n from 0 to ∞.

Now the range of k is same as x (k). So range of k is from 0 to 2.

Calculation of y (0) :

y (0) is calculated by putting n = 0 in Equation (3).

$$\therefore \quad y(0) = \sum_{k=0}^{2} x(k)\,h(-k) \qquad \ldots(4)$$

The sequence $x(k)$ is shown in Fig. A-49(a). Sequence $h(k) = u(n)$ is shown in Fig. A-49(b). Now $h(-k)$ is folded version of $h(k)$. It is obtained by folding unit step $u(n)$. Sequence $h(-k)$ is shown in Fig. A-49(c).

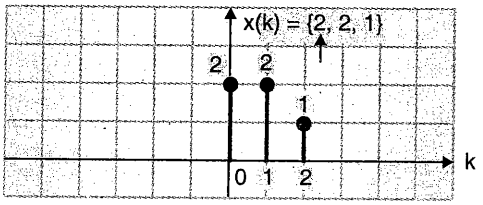


Fig. A-49(a) : Sequence $x(k) = \{2, 2, 1\}$

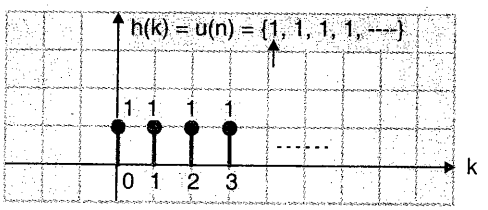


Fig. A-49(b) : Sequence $h(k) = u(n)$

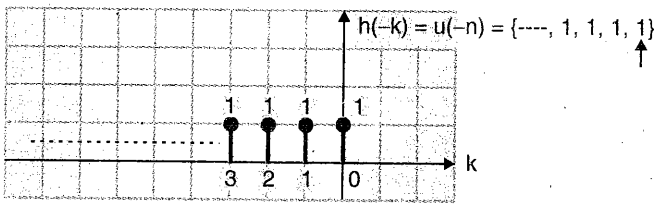


Fig. A-49(c) : Sequence $h(-k) = u(-n)$

According to Equation (4) $y(0)$ is obtained by multiplying $x(k)$ by $h(-k)$ and then adding all product terms. This is shown in Fig. A-49(d).

$$\therefore y(0) = 2$$

Calculation of $y(1)$:

Putting $n = 1$ in Equation (3) we get,

$$y(n) = \sum_{k=0}^{2} x(k)\,h(1-k) \qquad \ldots(5)$$

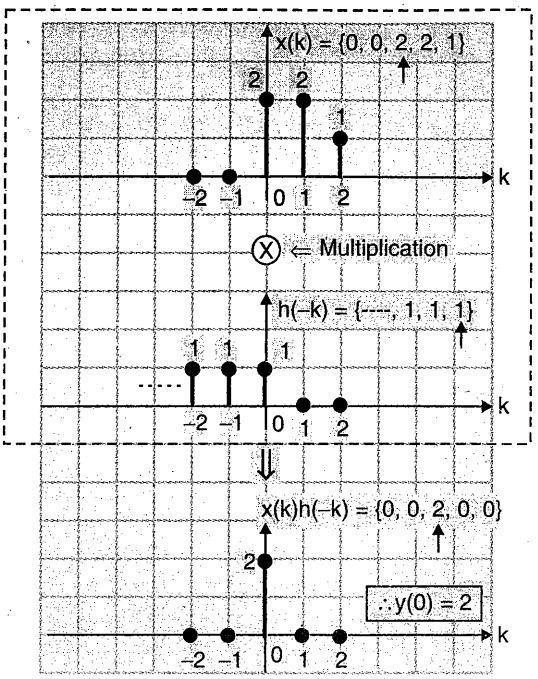


Fig. A-49(d) : Product of x (k) and h (– k)

Here h (1 – k) is same as h (– k + 1). It indicates delay of h (– k) by '1' sample. This sequence is shown in Fig. A-50(a).

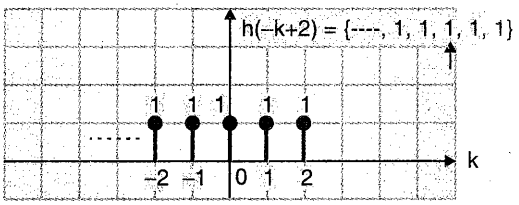


Fig. A-50(a) : Sequence h (– k + 1)

The multiplication of x (k) and h (– k + 1) is shown in Fig. A-50(b).

Now y (1) is obtained by adding all product terms.

$$\therefore \quad y(1) = (2 \times 1) + (2 \times 1) = 2 + 2 = 4$$

$$\therefore \; y(1) = 4$$

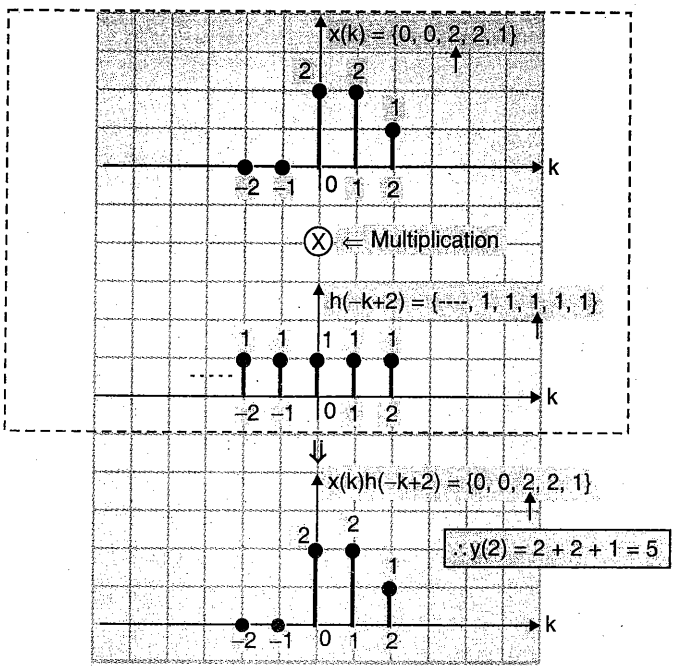


Fig. A-50(b) : Product of x (k) and h (– k + 1)

Calculation of y (2) : Putting n = 2 in Equation (3) we get,

$$y(2) = \sum_{k=0}^{2} x(k)\, h(2-k) \qquad \ldots(6)$$

$h(2-k)$ is same as $h(-k+2)$. It is obtained by shifting $h(-k)$ towards right by '2' samples as shown in Fig. A-50(c).

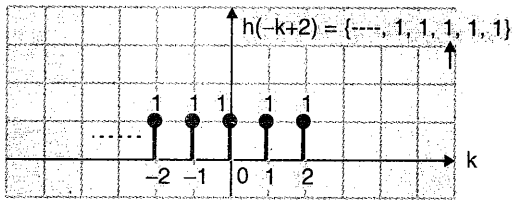


Fig. A-50(c) : Sequence h (– k + 2)

The product of x (k) and h (– k + 2) is shown in Fig. A-50(d).

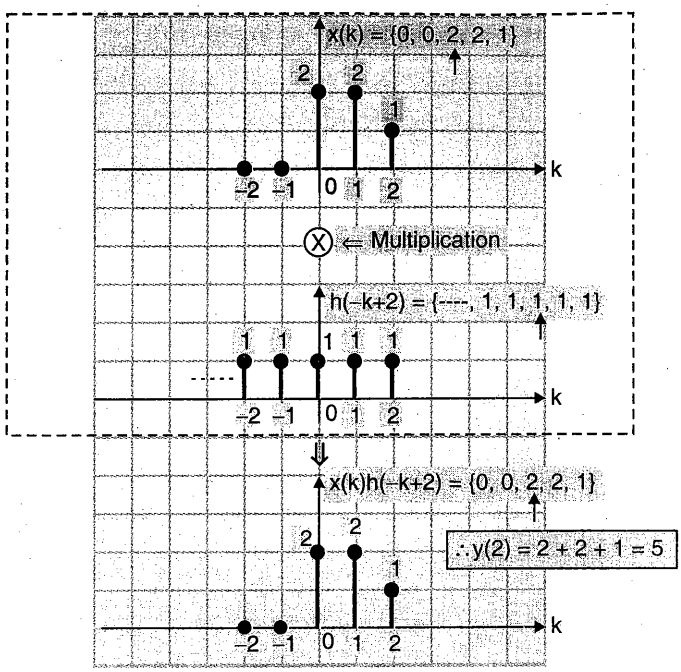


Fig. A-50(d) : Product of x (k) and h (– k + 2)

$$\therefore \quad y(2) = (2\times1)+(2\times1)+(1\times1) = 2+2+1 = 5$$

$$\therefore \quad y(2) = 5$$

Calculation of y (3) :

Putting n = 3 in Equation (3) we get,

$$y(3) = \sum_{k=0}^{2} x(k)\,h(3-k) \qquad ...(7)$$

Sequence h (3 – k) is same as h (– k + 3). It is obtained by shifting h (– k) towards right by 3 samples. This sequence is shown in Fig. A-51(a).

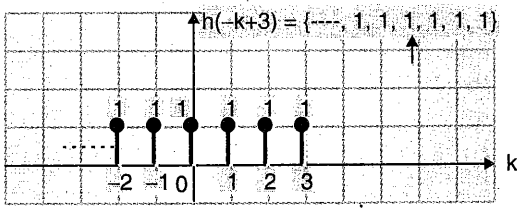


Fig. A-51(a) : Sequence h (– k + 3)

The product of x (k) and h (– k + 3) is shown in Fig. A-51(b).

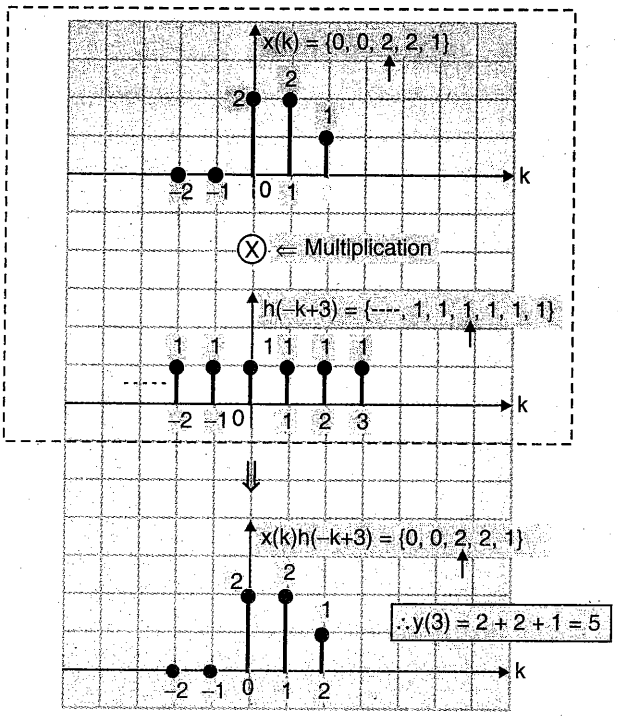


Fig. A-51(b) : Product of x (k) and h (– k + 3)

$$\therefore \quad y(3) = (2 \times 1) + (2 \times 1) + (1 \times 1) = 2 + 2 + 1 = 5$$

$$\therefore\ y(3) = 5$$

Since unit step is infinite sequence; it is clear that all the next samples of y (n) will have magnitude equal to 5. This is because every time we will shift h (– k) towards right by 1 sample. So for every value of n the product of x (k) and delayed h (– k) will remain same.

$$\therefore \quad y(4) = 5,\ y(5) = 5,\ y(6) = 5 \ldots.$$

Now the resultant output y (n) can be written as,

$$y(n) = \{y(0), y(1), y(2), y(3), \ldots\ldots \infty\}$$

$$\therefore \quad y(n) = \underset{\uparrow}{\{2, 4, 5, 5, 5, \ldots.\}} \qquad \ldots.(8)$$

General expression :

The given sequence x (n) is ,

$$x(n) = \underset{\uparrow}{\{2, 2, 1\}} \quad \text{and } h(n) = u(n) = \underset{\uparrow}{\{1, 1, 1, 1, \ldots\}}$$

$$\therefore \quad x(0) = 2 \qquad \therefore \quad h(0) = u(0) = 1$$

$$x(1) = 2 \qquad h(1) = u(1) = 1$$

$$\text{and} \quad x(2) = 1 \qquad h(2) = u(2) = 1$$

$$h(3) = u(3) = 1 \ldots$$

Now observe Equations (6) and (7) carefully. We can write the generalized expression to obtain Equation (8) as follows,

$$y(n) = x(0)u(n)+x(1)u(n-1)+x(2)u(n-2)+\ldots \qquad \ldots(9)$$

Let us verify this equation.

To calculate y (0) put n = 0 in Equation (9).

$$\therefore \quad y(0) = x(0)u(0)+x(1)u(-1)+x(2)u(-2)$$

$$\therefore \quad y(0) = (2\times1)+(2\times0)+(1\times0) = 2$$

This is because u (0) = 1, u (− 1) = 0 and u (− 2) = 0. This can be observed from unit step shown in Fig. A-52.

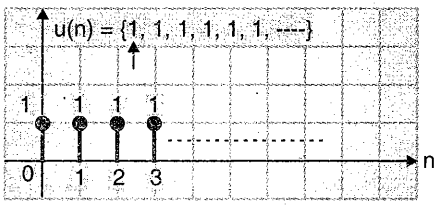


Fig. A-52 : Unit step u (n)

Similarly putting n = 1 in Equation (9) we get,

$$y(1) = x(0)u(1)+x(1)u(0)+x(2)u(-1)$$

$$\therefore \quad y(1) = (2\times1)+(2\times1)+(1\times0) = 2+2+0$$

$$\therefore \quad y(1) = 4$$

Similarly other values can be verified.

Prob. 4 : Obtain linear convolution of two discrete time signals given as,

$$x(n) = u(n)$$

$$h(n) = a^n u(n),\ a<1$$

Soln. :

Given x (n) = u (n) i.e. unit step which is shown in Fig. A-52. h (n) = a^n u (n) is an exponential signal. Here a^n is exponential signal which is multiplied by u (n). We know that u (n) is present only for positive values of 'n'. That means from n = 0 to n = ∞. So this multiplication indicates that a^n is present for positive values of 'n'. It will not affect the magnitude of a^n, because magnitude of u (n) is 1. This sequence a^n is shown in Fig. A-53.

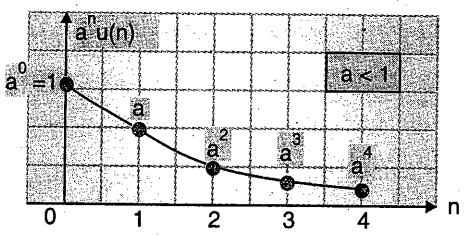


Fig. A-53 : Exponential sequence a^n

Recall the equation of convolution,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad ...(1)$$

In this equation $h(n-k)$ is delayed version of $h(k)$. Instead of that we can delay $x(k)$ and keep $h(k)$ as it is.

$$\therefore \quad y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) \quad ...(2)$$

This is commutative property of convolution which will be discussed later in this chapter. We will use Equation (2) in this example because it is easy to delay $u(n)$ than $a^n u(n)$.

Now we have derived generalized expression of convolution for finite sequence with an infinite sequence. This equation is,

$$y(n) = x(0)u(n)+x(1)u(n-1)+x(2)u(n-2)+\ldots \quad ...(3)$$

When both sequences are infinite then Equation (3) can be easily modified as,

$$y(n) = \ldots..x(-2)u(n+2)+x(-1)u(n+1)+x(0)u(n)$$
$$+x(1)u(n-1)+x(2)u(n-2)+\ldots . \quad ...(4)$$

Here $x(n)$ is some infinite sequence and $h(n) = u(n)$.

Equation (4) is based on Equation (1). In this case input $x(k)$ is kept as it is and $h(k)$ is delayed. But if we want to use Equation (2) to obtain linear convolution then,

$$y(n) = \ldots..h(-2)u(n+2)+h(-1)u(n+1)+h(0)u(n)$$
$$+h(1)u(n-1)+h(2)u(n-2)+\ldots \quad ...(5)$$

In Equation (5) we are taking $x(n) = u(n)$.

The given equation of $h(n)$ is,

$$h(n) = a^n u(n) \quad ...(6)$$

$$\therefore \quad h(n) = \begin{cases} a^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases} \quad ...(7)$$

Sequence $h(n)$ can be written by putting values of n in Equation (7).

That means,

$$\text{For } n = 0 \Rightarrow h(n) = h(0) = a^0 = 1$$

$$\text{For } n = 1 \Rightarrow h(n) = h(1) = a^1 = a$$

$$\text{For } n = 2 \Rightarrow h(n) = h(2) = a^2$$

$$\text{For } n = 3 \Rightarrow h(n) = h(3) = a^3 \ldots.$$

$$\therefore \quad h(n) = \{1, a, a^2, a^3 \ldots.\} \qquad \ldots.(8)$$
$$\uparrow$$

Putting values of $h(n)$ is Equation (5) we get,

$$y(n) = 1u(n) + au(n-1) + a^2 u(n-2) + a^3 u(n-3) + \ldots. \qquad \ldots(9)$$

Equation (9) gives the convolution of given sequences. To calculate sequence $y(n)$, put different values of n in Equation (9).

$$\therefore \quad y(0) = u(0) + au(-1) + a^2 u(-2) + a^3 u(-3)$$

$$\text{Here} \quad u(0) = 1, \; u(-1) = u(-2) = u(-3) = 0$$

$$\therefore \; y(0) = 1$$

Putting $n = 1$ in Equation (9) we get,

$$y(1) = 1u(1) + au(0) + a^2 u(-1) + a^3 u(-2)$$

$$\text{Here} \quad u(1) = u(0) = 1 \text{ and } u(-1) = u(-2) = 0$$

$$\therefore \; y(1) = 1 + a$$

Now putting $n = 2$ in Equation (9) we get,

$$y(2) = u(2) + au(1) + a^2 u(0) + a^3 u(-1)$$

$$\text{Here} \quad u(2) = u(1) = u(0) = 1 \text{ and } u(-1) = 0$$

$$\therefore \; y(2) = 1 + a + a^2$$

Similarly for the n^{th} sample, $y(n)$ can be written as,

$$y(n) = 1 + a + a^2 + a^3 + \ldots. + a^n \qquad \ldots(10)$$

In terms of summation Equation (10) can be written as,

$$y(n) = \sum_{k=0}^{n} a^k \qquad \ldots(11)$$

Now use standard equation of summation,

$$\sum_{n=N_1}^{N_2} a^n = \begin{cases} \dfrac{a^{N_1} - a^{N_2+1}}{1-a} & \text{For } a \neq 1 \\ N_2 - N_1 + 1 & \text{For } a = 1 \end{cases} \quad \ldots(12)$$

In the present example $a < 1$ thus $a \neq 1$.

Now from Equation (11),

N_1 = lower limit of summation = 0

N_2 = upper limit of summation = n

Putting these values in the standard equation of summation (Equation (12)) we get,

$$y(n) = \frac{a^0 - a^{n+1}}{1-a}$$

$$\therefore \quad y(n) = \frac{1 - a^{n+1}}{1-a}$$

Prob. 5 : Use discrete convolution to find the response to the input $x(n) = a^n u(n)$ of the LTI system with impulse response $h(n) = b^n u(n)$.

Soln. :

Given : $x(n) = a^n u(n)$ and $h(n) = b^n u(n)$.

Sequences $x(k)$ and $h(k)$ can be directly written as,

$x(k) = a^k u(k)$ and $h(k) = b^k u(k)$

According to definition of linear convolution we have,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \quad \ldots(1)$$

We have $h(k) = b^k u(k)$ ...(2)

Sequence $h(n-k)$ is obtained by replacing k by $n-k$ in Equation (2)

$$\therefore \quad h(n-k) = b^{n-k} u(n-k) \quad \ldots(3)$$

Thus Equation (1) becomes,

$$y(n) = \sum_{k=-\infty}^{\infty} a^k u(k) \cdot b^{n-k} u(n-k)$$

$$\therefore \quad y(n) = \sum_{k=-\infty}^{\infty} a^k \cdot b^{n-k} \cdot u(k)\, u(n-k) \quad \ldots(4)$$

Sequence $u(k)$ is unit step. While $u(n-k) = u(-k+n)$ indicates delay of $u(-k)$ by 'n' samples. This sequence is obtained as shown in Fig. A-54.

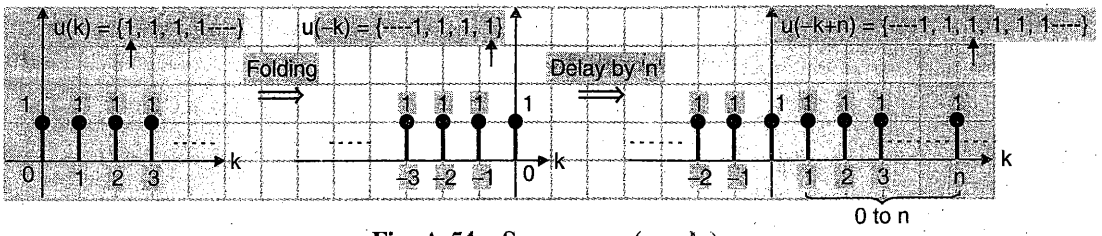


Fig. A-54 : Sequence u (n – k)

Now the product of u (k) and u (n – k) is shown in Fig. A-55.

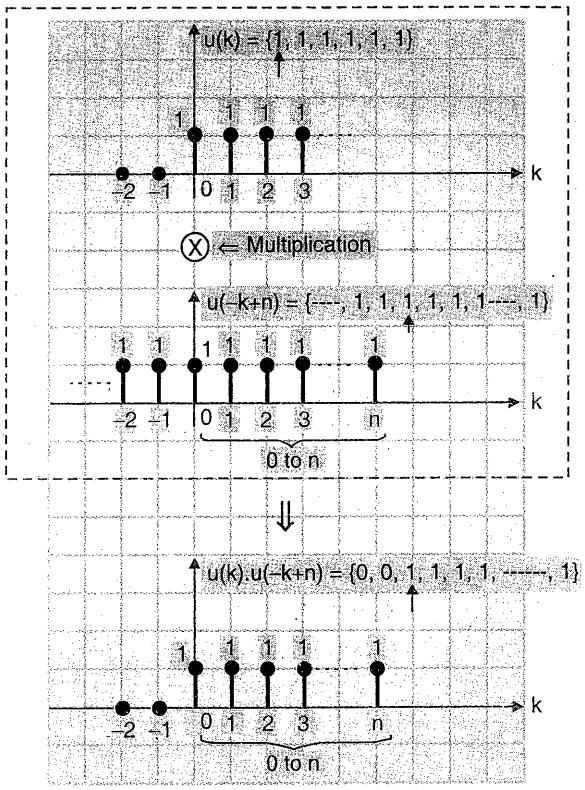


Fig. A-55 : Product of u (k) and u (n – k)

This multiplication i.e. $u(k) \cdot u(n-k)$ indicates that the sequence is present only for the range 0 to n. So we will change the limits of summation in Equation (4). New limits of k will be from 0 to n.

$$\therefore \quad y(n) = \sum_{k=0}^{n} a^k b^{n-k} u(k) \cdot u(n-k) \qquad ...(5)$$

But the term $u(k) \cdot u(n-k) = 1$.

$$\therefore \quad y(n) = \sum_{k=0}^{n} a^k b^{n-k} \qquad \ldots(6)$$

$$\text{Now } b^{n-k} = b^n \cdot b^{-k} = \frac{b^n}{b^k}$$

$$\therefore \quad y(n) = \sum_{k=0}^{n} a^k \frac{b^n}{b^k} \qquad \ldots(7)$$

Since the summation is in terms of 'k' we can take the term b^n outside the summation.

$$\therefore \quad y(n) = b^n \sum_{k=0}^{n} \frac{a^k}{b^k}$$

$$\therefore \quad y(n) = b^n \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k \qquad \ldots(8)$$

Use standard equation of summation,

$$\sum_{n=N_1}^{N_2} a^n = \begin{cases} \dfrac{a^{N_1} - a^{N_2+1}}{1-a} & a \neq 1 \\ N_2 - N_1 + 1 & a = 1 \end{cases} \qquad \ldots(9)$$

Compare Equations (8) and (9).

Here $N_1 = 0$, $N_2 = n$ and $a = \left(\frac{a}{b}\right)$.

$$\therefore \quad y(n) = b^n \cdot \left[\frac{\left(\frac{a}{b}\right)^0 - \left(\frac{a}{b}\right)^{n+1}}{1 - \frac{a}{b}}\right] \qquad \ldots \text{ for } \frac{a}{b} \neq 1$$

$$\therefore \quad y(n) = b^n \left[\frac{1 - \left(\frac{a}{b}\right)^{n+1}}{1 - \frac{a}{b}}\right] \qquad \ldots(10)$$

Equation (10) gives convolution of $a^n u(n)$ and $b^n u(n)$.

Prob. 6 : Impulse response of an LTI system is given as : $h(n) = u(n) - u(n-N)$. Find the output $y(n)$ of the system to an input $x(n) = a^n u(n)$.

Soln. : The sequences $x(k)$ and $h(k)$ can be written as,

$$x(k) = a^k u(k) \quad \text{and} \quad h(k) = u(k) - u(k-N).$$

According to definition of convolution,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \qquad \ldots(1)$$

Equation (1) can also be written as,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) \qquad \ldots(2)$$

$x(n-k)$ is obtained by replacing 'k' by 'n – k'

$$\therefore \quad x(n-k) = a^{n-k} u(n-k) \qquad \ldots(3)$$

Putting this value in Equation (2) we get,

$$y(n) = \sum_{k=-\infty}^{\infty} [a^{n-k} u(n-k)] \cdot [u(k) - u(k-N)] \qquad \ldots(4)$$

Consider the term $u(k) - u(k-N)$. Here 'N' is some constant. $u(k)$ is unit step and $u(k-N)$ is, unit step delayed by N samples. This sequence $u(k-N)$ is shown in Fig. A-56(a).

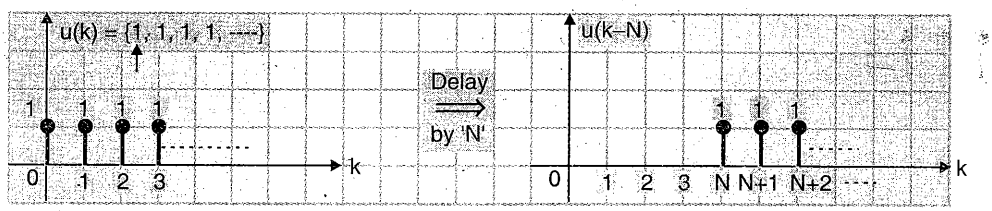


Fig. A-56(a) : Sequence $u(k-N)$

While the sequence $u(k) - u(k-N)$ is shown in Fig. A-56(b).

Note that this sequence is present from '0' to 'N'.

Now consider first term of Equation (4). Here $u(n-k)$ is same as $u(-k+n)$. It is obtained as shown in Fig. A-56(c).

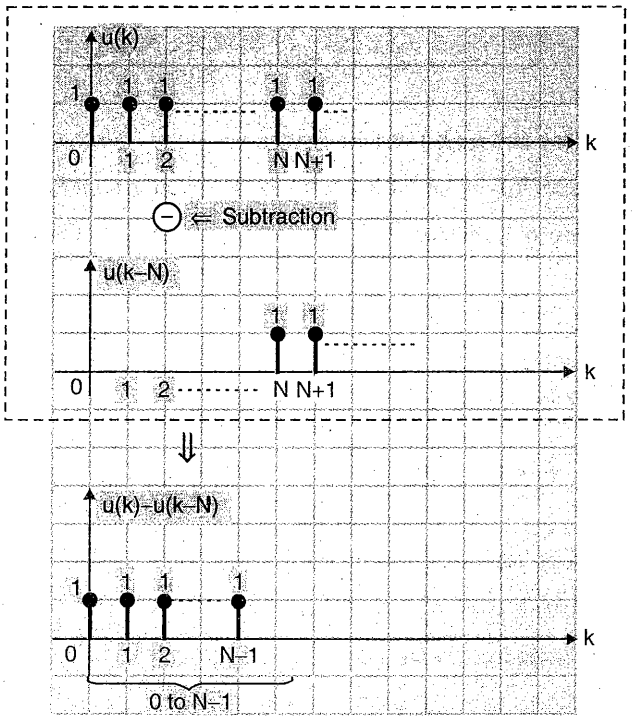


Fig. A-56(b) : Sequence u (k) – u (k – N)

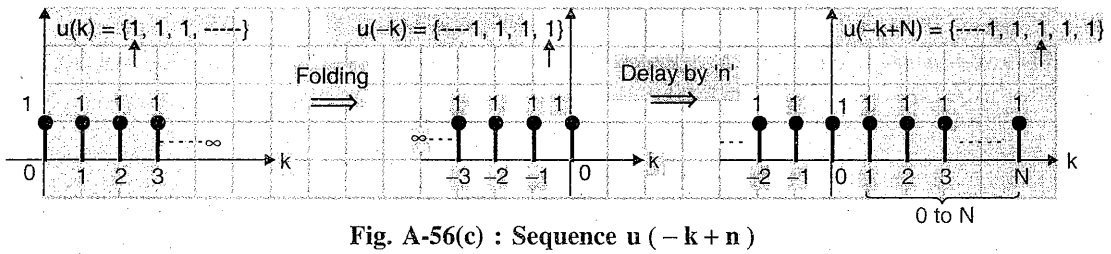


Fig. A-56(c) : Sequence u (– k + n)

According to Equation (1) we have to obtain product of [u (n – k)] and [u (k) – u (k – N)]. These two terms are basically unit steps shifted in discrete domain. This product is shown in Fig. A-56(d).

Note that we have assumed N < n. So the product term is present from 0 to N; having magnitude 1. Thus by changing the limits of Equation (4) we get,

$$y(n) = \sum_{k=0}^{N} a^{n-k}$$

$$\therefore \quad y(n) = \sum_{k=0}^{N} a^{n} \cdot (a^{-1})^{k} \qquad \ldots(5)$$

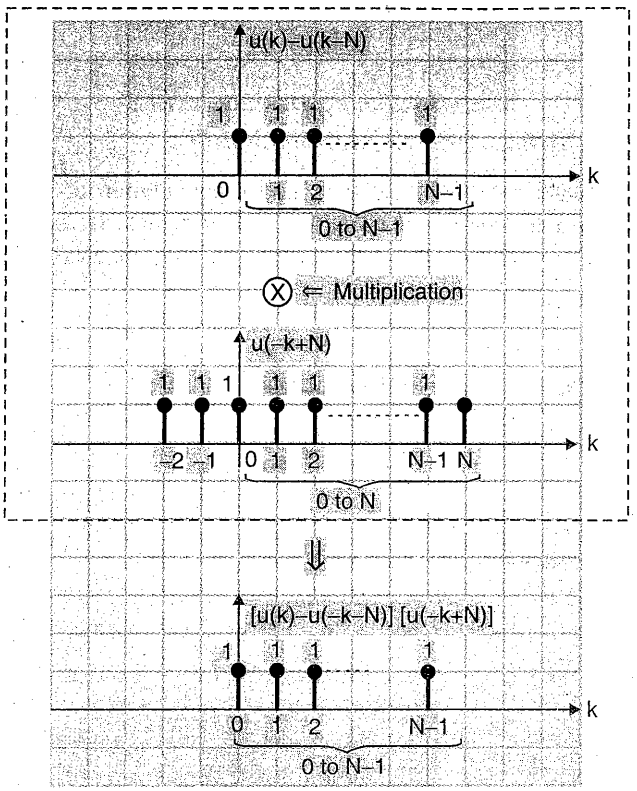


Fig. A-56(d) : Product of $u(-k+N)$ and $[u(k)-u(k-N)]$

Since summation is in terms of 'k'; a^n can be taken out of summation sign.

$$\therefore \quad y(n) = a^n \sum_{k=0}^{N} (a^{-1})^k \qquad ..(6)$$

Use standard equation of summation,

$$\sum_{n=N_1}^{N_2} a^n = \begin{cases} \dfrac{a^{N_1} - a^{N_2+1}}{1-a} & a \neq 1 \\ N_2 - N_1 + 1 & a = 1 \end{cases} \qquad ...(7)$$

Compare Equations (6) and (7)

Here $a = a^{-1}$, $N_1 = 0$, $N_2 = N$. Then if $a^{-1} \neq 1$ we have,

$$y(n) = a^n \left[\frac{[a^{-1}]^0 - [a^{-1}]^{N+1}}{1-a^{-1}} \right]$$

$$\therefore \quad y(n) = a^n \left[\frac{1 - a^{-N-1}}{1 - a^{-1}} \right]$$

2) Calculation using basic equation of convolution :

In the graphical method, we studied how to obtain convolution of two signals, using graphs. In that method, every time it is required to draw the plot of signal. So that is tedious method. One more simple method is to calculate the convolution of two signals using basic equation of convolution.

How to solve ?

According to the definition of convolution we have,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \qquad \ldots(1)$$

Step I : Write down the equations of x (k) and h (k)

Step II : Decide range of 'n'

We have studied, how to decide range of 'n'. Let us recall those equations.

Lowest index of 'n' for y (n) ⇒ Lowest index of x (n) + Lowest index of h (n)

$$\therefore \quad y_l = x_l + h_l \qquad \ldots(2)$$

and Highest index of 'n' for y (n) ⇒ Highest index of x (n) + Highest index of h (n)

$$\therefore \quad y_h = x_h + h_h \qquad \ldots(3)$$

Decide range of 'k' :

Range of k is same as range of x (k).

Step III : Once the range of 'n' and 'k' is decided; then put different values of 'n' in the equation of y (n).

Examples (5), (6) and (7) are solved with the help of basic equation of convolution. Let us solve some more examples.

Prob. 7 : Compute linear convolution of the following :

$x(n) = \{ \underset{\uparrow}{1}, 1, 1, 1 \}$ and $h(n) = \{ \underset{\uparrow}{1}, 1, 1, 1 \}$

Soln. : We have,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \qquad \ldots(1)$$

Step I : The sequences $x(k)$ and $h(k)$ can be written as,

$x(k) = \{1, 1, 1, 1\}$ and $h(k) = \{1, 1, 1, 1\}$
$\uparrow$ $\uparrow$

$\therefore$ $x(0) = 1$ and $h(0) = 1$
$x(1) = 1$ $h(1) = 1$
$x(2) = 1$ $h(2) = 1$
$x(3) = 1$ $h(3) = 1$

Step II : Range of 'n' for $y(n)$ is,

Lowest index : $y_l = x_l + h_l = 0 + 0 = 0$

Highest index : $y_h = x_h + h_h = 3 + 3 = 6$

Thus range of $y(n)$ is from $y(0)$ to $y(6)$.

Now range of 'k' is same as $x(k)$.

Thus range of 'k' is from $k = 0$ to $k = 3$.

Step III : Putting the value of 'k' in Equation (1) we get,

$$y(n) = \sum_{k=0}^{3} x(k)\,h(n-k) \qquad ...(2)$$

Calculation of y (0) :

Putting $n = 0$ in Equation (2) we get,

$$y(0) = \sum_{k=0}^{3} x(k)\,h(0-k) = \sum_{k=0}^{3} x(k)\,h(-k)$$

Expanding the summation by putting values of 'k' we get,

$$y(0) = x(0)\,h(-0) + x(1)\,h(-1) + x(2)\,h(-2) + x(3)\,h(-3)$$

$$\therefore \quad y(0) = (1\times1) + (1\times0) + (1\times0) + (1\times0)$$

.... as $h(-1) = h(-2) = h(-3) = 0$

$\therefore$ $y(0) = 1$

Calculation of y (1) :

Putting $n = 1$ in Equation (2) we get,

$$y(1) = \sum_{k=0}^{3} x(k)\,h(1-k)$$

Expanding the summation we get,

$$y(1) = x(0)\,h(1) + x(1)\,h(0) + x(2)\,h(-1) + x(3)\,h(-2)$$

$$\therefore \quad y(1) = (1\times1) + (1\times1) + (1\times0) + (1\times0)$$

$\therefore \quad y(1) = 1+1$ ..... as $h(-1) = h(-2) = 0$

$\therefore \quad y(1) = 2$

Calculation of y (2) :

Putting n = 2 in Equation (2) we get,

$$y(2) = \sum_{k=0}^{3} x(k)h(2-k)$$

Expanding the summation we get,

$$y(2) = x(0)h(2)+x(1)h(1)+x(2)h(0)+x(3)h(-1)$$

$$\therefore \quad y(2) = (1\times1)+(1\times1)+(1\times1)+(1\times0)$$

$\therefore \quad y(2) = 1+1+1$ ... as $h(-1) = 0$

$\therefore \quad y(2) = 3$

Calculation of y (3) :

Putting n = 3 in Equation (2) we get,

$$y(3) = \sum_{k=0}^{3} x(k)h(3-k)$$

Expanding the summation we get,

$$y(3) = x(0)h(3)+x(1)h(2)+x(2)h(1)+x(3)h(0)$$

$$\therefore \quad y(3) = (1\times1)+(1\times1)+(1\times1)+(1\times1)$$

$$\therefore \quad y(3) = 1+1+1+1$$

$\therefore \quad y(3) = 4$

Calculation of y (4) :

Putting n = 4 in Equation (2) we get,

$$y(4) = \sum_{k=0}^{3} x(k)h(4-k)$$

Expanding the summation we get,

$$y(4) = x(0)h(4)+x(1)h(3)+x(2)h(2)+x(3)h(1)$$

$$\therefore \quad y(4) = (1\times0)+(1\times1)+(1\times1)+(1\times1)$$

$\therefore \quad y(4) = 0+1+1+1$ .... as $h(4) = 0$

$\therefore y(4) = 3$

Calculation of y (5) :

Putting n = 5 in Equation (2) we get,

$$y(5) = \sum_{k=0}^{3} x(k)h(5-k)$$

Expanding the summation we get,

$$y(5) = x(0)h(5)+x(1)h(4)+x(2)h(3)+x(3)h(2)$$

$$\therefore \quad y(5) = (1\times0)+(1\times0)+(1\times1)+(1\times1)$$

$$\therefore \quad y(5) = 0+0+1+1 \quad \text{.... as } h(5) = h(4) = 0$$

$\therefore y(5) = 2$

Calculation of y (6) :

Putting n = 6 in Equation (2) we get,

$$y(6) = \sum_{k=0}^{3} x(k)h(6-k)$$

Expanding the summation we get,

$$y(6) = x(0)h(6)+x(1)h(5)+x(2)h(4)+x(3)h(3)$$

$$\therefore \quad y(6) = (1\times0)+(1\times0)+(1\times0)+(1\times1)$$

$$\therefore \quad y(6) = 0+0+0+1 \quad \text{... as } h(6) = h(5) = h(4) = 0$$

$\therefore y(6) = 1$

Step IV : The result of convolution y (n) is given by,

$$y(n) = \{ y(0), y(1), y(2), y(3), y(4), y(5), y(6) \}$$

$$\therefore y(n) = \{ 1, 2, 3, 4, 3, 2, 1 \}$$
$$\uparrow$$

Note that arrow is marked at the sample y (0).

Prob. 8 : Compute the linear convolution of following :

$$x(n) = 1 \text{ and } h(n) = \{2, 1, 2, 1\}$$

Soln. :

Step I : The sequences $x(k)$ and $h(k)$ can be written as,

$$x(k) = \underset{\uparrow}{\{1\}} \text{ and } h(k) = \underset{\uparrow}{\{2, 1, 2, 1\}}$$

$$\therefore \quad x(0) = 1 \qquad \therefore \quad h(0) = 2$$
$$h(1) = 1$$
$$h(2) = 2$$
$$h(3) = 1$$

Note that arrow is not present in the given sequences of $x(n)$ and $h(n)$. So by default it is at first (starting) position.

Step II : Range of 'n' for $y(n)$ is,

Lower index : $y_l = x_l + h_l = 0 + 0 = 0$

Highest index : $y_h = x_h + h_h = 0 + 3 = 3$

Thus range of $y(n)$ is from $y(0)$ to $y(3)$.

Step III : According to definition of convolution we have,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \qquad \ldots(1)$$

In this case $h(n-k)$ indicates folding and delaying $h(-k)$. Since $x(k)$ is smaller sequence than $h(k)$; it will be convenient to use following equation,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) \qquad \ldots(2)$$

Here range of 'k' will be as per the sequence $h(k)$. Thus range of 'k' is from 0 to 3.

Calculation of y (0) :

Putting $n = 0$ in Equation (2) we get,

$$y(0) = \sum_{k=0}^{3} h(k)\, x(-k)$$

Expanding the summation we get,

$$y(0) = h(0)\,x(0) + h(1)\,x(-1) + h(2)\,x(-2) + h(3)\,x(-3)$$

$$\therefore \quad y(0) = (2 \times 1) + (1 \times 0) + (2 \times 0) + (1 \times 0)$$

$$\therefore \quad y(0) = 2 + 0 + 0 + 0 \qquad \text{..as } x(-1) = x(-2) = x(-3) = 0$$

$$\therefore \quad y(0) = 2$$

Calculation of y (1) :

Putting n = 1 in Equation (2) we get,

$$y(1) = \sum_{k=0}^{3} h(k)x(1-k)$$

Expanding the summation we get,

$$y(1) = h(0)x(1)+h(1)x(0)+h(2)x(-1)+h(3)x(-2)$$

$$\therefore \quad y(1) = (2\times0)+(1\times1)+(2\times0)+(1\times0)$$

$$\therefore \quad y(1) = 0+1+0+0 \qquad \text{...as } x(+1) = x(-1) = x(-2) = 0$$

$$\therefore \quad y(1) = 1$$

Calculation of y (2) :

Putting n = 2 in Equation (2) we get,

$$y(2) = \sum_{k=0}^{3} h(k)x(2-k)$$

Expanding the summation we get,

$$y(2) = h(0)x(2)+h(1)x(1)+h(2)x(0)+h(3)x(-1)$$

$$\therefore \quad y(2) = (2\times0)+(1\times0)+(2\times1)+(1\times0) \qquad \text{...as } x(2) = x(1) = x(-1) = 0$$

$$\therefore \quad y(2) = 0+0+2+0$$

$$\therefore \quad y(2) = 2$$

Calculation of y (3) :

Putting n = 3 in Equation (2) we get,

$$y(3) = \sum_{k=0}^{3} h(k)x(3-k)$$

Expanding the summation we get,

$$y(3) = h(0)x(3)+h(1)x(2)+h(2)x(1)+h(3)x(0)$$

$$\therefore \quad y(3) = (2\times0)+(1\times0)+(2\times0)+(1\times0)$$

$$\therefore \quad y(3) = 0+0+0+1 \qquad \text{... as } x(3) = x(2) = x(1) = 0$$

$$\therefore \quad y(3) = 1$$

Step IV : Now the result of convolution can be written as,

$$y(n) = \{y(0), y(1), y(2), y(3)\}$$

$$\therefore \quad y(n) = \{2, 1, 2, 1\}$$
$$\uparrow$$

Note : Here given sequence $x(n) = \{1\}$. We know that $\delta(n)$ is the unit impulse and is expressed as, $\delta(n) = \{1\}$. That means we have calculated convolution of $\{2, 1, 2, 1\}$ with unit impulse.
$\uparrow$ $\uparrow$
Observe the result of convolution. It is the same as $\{2, 1, 2, 1\}$. Thus we can conclude that the convolution of any sequence with unit impulse produces the same sequence.
$\uparrow$

3) Tabulation method of linear convolution :

This is the simplest method of performing linear convolution.

Let $x(n) = \{x(0), x(1), x(2)\}$ and $h(n) = \{h(0), h(1), h(2)\}$
$\uparrow$ $\uparrow$

Step I : Form the matrix as shown in Fig. A-57.

		x(n)		
		x(0)	x(1)	x(2)
h(n)	h(0)			
	h(1)			
	h(2)			

Fig. A.57 : Matrix of x (n) and h (n)

Note that we can interchange the positions of x (n) and h (n).

Step II : Multiply the corresponding elements of x (n) and h (n) as shown in Fig. A-58.

	x (0)	x (1)	x (2)
h (0)	h (0) x (0)	h (0) x (1)	h (0) x (2)
h (1)	h (1) x (0)	h (1) x (1)	h (1) x (2)
h (2)	h (2) x (0)	h (2) x (1)	h (2) x (2)

Fig. A-58 : Multiplication of elements x (n) and h (n)

Step III : Separate out the elements diagonally as shown in Fig. A-59.

	x(0)	x(1)	x(2)
h(0)	h(0)x(0)	h(0)x(1)	h(0)x(2)
h(1)	h(1)x(0)	h(1)x(1)	h(1)x(2)
h(2)	h(2)x(0)	h(2)x(1)	h(2)x(2)

Fig. A-59 : Diagonally separation of elements

Step IV : Simply add the elements in that particular block. This addition gives corresponding values of y (n).

Here range of 'n' is :

$$y_l = x_l + h_l = 0 + 0 = 0$$

and $$y_h = x_h + h_h = 2 + 2 = 4$$

Thus we get,
$$y(0) = h(0)x(0)$$
$$y(1) = h(1)x(1) + h(0)x(1)$$
$$y(2) = h(2)x(0) + h(1)x(1) + h(0)x(2)$$
$$y(3) = h(2)x(1) + h(1)x(2)$$

and
$$y(4) = h(2)x(2)$$

After calculating all these values; write down the result of convolution as,

$$y(n) = \{y(0), y(1), y(2), y(3), y(4)\}$$
$$\uparrow$$

Prob. 9 : Compute the convolution $y(n) = x(n) * h(n)$

where $x(n) = \{1,1, 0, 1, 1\}$ and $h(n) = \{1, -2, -3, 4\}$

$\uparrow$ $\uparrow$

Soln. : Here x (n) contains '5' samples and h (n) has '4' samples. To make the length of x (n) and h (n) same; rewrite the sequence h (n) as follows :

$$h(n) = \{1, -2, -3, 4, 0\}$$
$$\uparrow$$

We can add zeros at the end or at the beginning of sequence. This will not affect the given sequence. This method of adding zeros, to adjust the length of sequence is called as zero-padding.

Range of 'n' : The range of 'n' for y (n) is calculated as follows :

Lowest index of y (n) $\Rightarrow y_l = x_l + h_l = -2 + (-3) = -5$

and Highest index of y (n) $\Rightarrow y_h = x_h + h_h = 2 + 1 = 3$

Thus y (n) varies from y (– 5) to y (3). Now using tabulation method the convolution is obtained as shown in Fig. A-60.

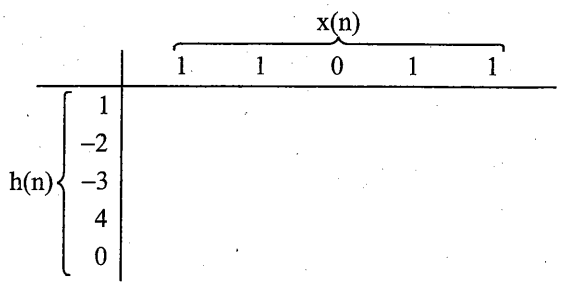


Fig. A-60(a) : Matrix of x (n) and h (n)

	1	1	0	1	1
1	1×1	1×1	1×0	1×1	1×1
−2	−2×1	−2×1	−2×0	−2×1	−2×1
−3	−3×1	−3×1	−3×0	−3×1	−3×1
4	4×1	4×1	4×0	4×1	4×1
0	0×1	0×1	0×0	0×1	0×1

Fig. A-60(b) : Convolution of x (n) and h (n)

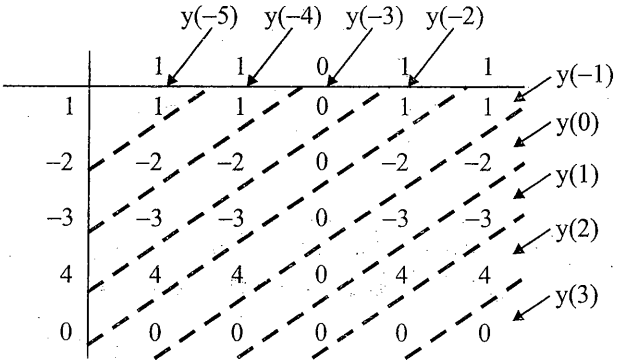


Fig. A-60(c) : Diagonally separation of elements

Different values of y (n) are calculated by adding corresponding elements as follows :

$y(-5) = 1$

$y(-4) = -2+1 = -1$

$y(-3) = -3-2+0 = -5$

$y(-2) = 4-3+0+1 = 2$

$y(-1) = 0+4+0-2+1 = 3$

$y(0) = 0+0-3-2 = -5$

$y(1) = 0+4-3 = 1$

$y(2) = 0+4 = 4$

$y(3) = 0$

Thus result of convolution y (n) is,

$$y(n) = \{y(-5), y(-4), y(-3), y(-2), \underset{\uparrow}{y(-1)}, y(0), y(1), y(2), y(3)\}$$

$$\therefore \quad y(n) = \{1, -1, -5, 2, 3, -5, 1, 4, 0\}$$
$$\uparrow$$

Neglecting the last '0' term we have,

$$y(n) = \{1, -1, -5, 2, 3, -5, 1, 4\}$$
$$\uparrow$$

Prob. 10 : The impulse response of linear time invarient is

$$h(n) = \{1, 2, 1, -1\}$$
$$\uparrow$$

Determine the response of system to the input

$$x(n) = \{1, 2, 3, 1\}$$
$$\uparrow$$

Soln. : The response of the system is,

$$y(n) = x(n) * h(n)$$

Range of n :

Lowest of $y(n) \Rightarrow y_l = x_l + h_l = 0 + 0 = 0$

Highest range of $y(n) \Rightarrow y_h = x_h + h_h = 3 + 3 = 6$

Using tabulation method; the convolution is obtained as shown in Fig. A-61.

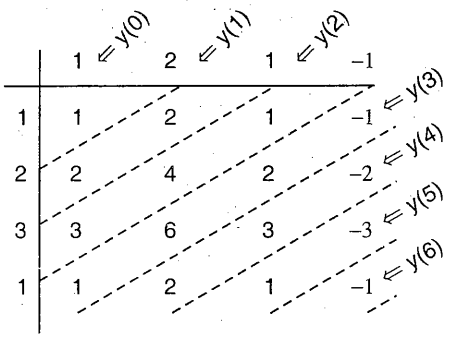


Fig. A-61 : Linear convolution of x (n) and h (n)

$$\therefore \quad y(n) = \{1, 4, 8, 8, 3, -2, -1\}$$
$$\uparrow$$

Prob. 11 : Use discrete convolution to find the response to the input x (n) of the LTI system with impulse response h (n) for the corresponding x (n) shown in Figs. A-62(a) and (b).

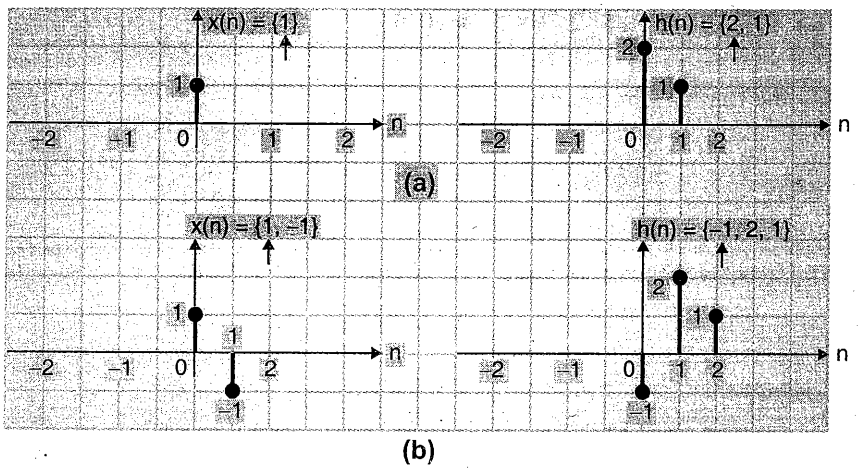


Fig. A-62 : Given sequence x (n) and h (n)

Soln. : (i) From Fig. A-62(a) the sequences x (n) and h (n) can be written as,

$$x(n) = \underset{\uparrow}{\{1\}} \text{ and } h(n) = \underset{\uparrow}{\{2, 1\}}$$

Making lengths equal we have,

$$x(n) = \underset{\uparrow}{\{1, 0\}} \text{ and } h(n) = \underset{\uparrow}{\{2, 1\}}$$

$\therefore$ $x(0) = 1$ $\quad\quad$ $\therefore$ $h(0) = 2$

$x(1) = 1$ $\quad\quad$ $h(1) = 1$

Range of 'n' for y (n) :

Lowest index of $n \Rightarrow y_l = x_l + h_l = 0 + 0 = 0$

Highest index of $n \Rightarrow y_h = x_h + h_h = 1 + 1 = 2$

Thus y (n) varies from y (0) to y (2)

By using tabulation method, convolution of x (n) and h (n) is obtained as shown in Fig. A-63.

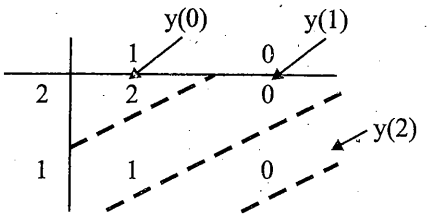
	x(n)	
	1	0
h(n) 2		
h(n) 1		

(a)

	1	0
2	2×1	2×0
1	1×1	1×0

(b)

Fig. A-63



(c)

Fig. A-63 : Convolution of x (n) = {1, 0} and h (n) = {0, 1}
↑ ↑

Different values of y (n) are obtained by adding corresponding terms.

$\therefore \quad y(0) = 2, \qquad y(1) = 1 + 0 = 1, \qquad y(2) = 0$

Thus result of convolution is, $y(n) = \{y(0), y(1), y(2)\}$
↑

$\therefore \quad y(n) = \{2, 1, 0\}$
↑

Neglecting last term we have,

$y(n) = \{2, 1\}$
↑

(ii) From Fig. A-62(b) we can write sequences x (n) and h (n) as,

$x(n) = \{1, -1\}$ and $h(n) = \{-1, 2, 1\}$
↑ ↑

Making lengths equal we have,

$x(n) = \{1, -1, 0\}$ and $h(n) = \{-1, 2, 1\}$
↑ ↑

$\therefore \; x(0) = 1$ $\qquad$ $\therefore \; h(0) = -1$

$x(1) = -1$ $\qquad$ $h(1) = 2$

and $x(2) = 0$ $\qquad$ and $\qquad h(2) = 1$

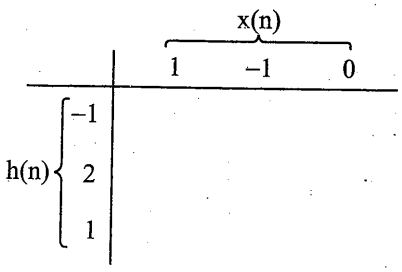
Range of 'n' for y (n) :

Lowest index $\Rightarrow y_l = x_l + h_l = 0 + 0 = 0$

Highest index $\Rightarrow y_h = x_h + h_h = 2 + 2 = 4$

Thus y (n) varies from y (0) to y (4).

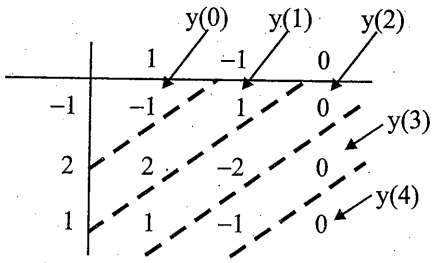
By using tabulation method; convolution of x (n) and h (n) is obtained as shown in Fig. A-64.



(a)

	1	−1	0
−1	(−1×1)	(−1×−1)	(−1×0)
2	(2×1)	(2×−1)	(2×0)
1	(1×1)	(1×−1)	(1×0)

(b)



(c)

Fig. A-64 : Convolution of x (n) = {1, – 1, 0} and h (n) = {– 1, 2, 1}
↑ ↑

Different values of y (n) are obtained by adding corresponding terms.

$\therefore$ $y(0) = -1$ $\quad y(1) = 2 + 1 = 3$

$y(2) = 1 - 2 = -1$ $\quad y(3) = -1 + 0 = -1$

$y(4) = 0$

Thus result of convolution y (n) is,

$$y(n) = \underset{\uparrow}{\{y(0), y(1), y(2), y(3), y(4)\}}$$

$$\therefore \quad y(n) = \underset{\uparrow}{\{-1, 3, -1, -1, 0\}}$$

Neglecting last '0' term we can write,

$$y(n) = \underset{\uparrow}{\{-1, 3, -1, -1\}}$$

4) Multiplication method of linear convolution :

This is another easy method to obtain convolution of two sequences. This method is similar to multiplication of multidigit numbers. Write down the two sequences, x (n) and h (n) and obtain the multiplication by usual method. The result of multiplication will be equal to the convolution of two sequences. We will solve one example by this method.

Prob. 12 : Obtain linear convolution of following sequences :

$$x(n) = \{1, 2, 1, 2\} \text{ and } h(n) = \{2, 2, -1, 1\}$$
$$\uparrow \qquad\qquad \uparrow$$

Soln. : First we will decide the range of 'n' for y (n).

Lowest index $\Rightarrow y_l = x_l + h_l = -1 + (-2) = -3$

Highest index $\Rightarrow y_h = x_h + h_h = 2 + 1 = 3$

Thus y (n) varies from y (– 3) to y (3)

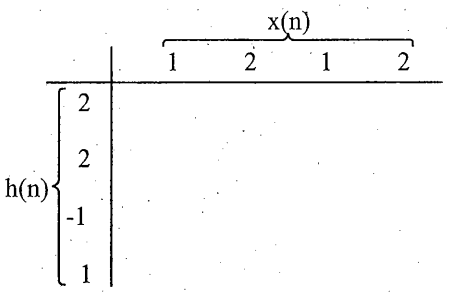
We will solve this problem using multiplication method.

	x(n) :-	1	2	1	2	
Multiplication →	h(n) :- ⊗ 2	2	−1	1		
	1	2	1	2		⇐ x(n) is multiplied by (1)
Addition ® ⊕	−1	−2	−1	−2	0	⇐ x(n) is multiplied by (−1)
						← Zero digit similar to normal multiplication
Addition ® ⊕	2	4	2	4	0	0 ⇐ x(n) is multiplied by (2)
						← Two zero digits similar to normal multiplication
Addition ® ⊕	2	4	2	4	0	0 0 ⇐ x(n) is multiplied by (2)
						← Three zero digits similar to normal multiplication
	2	6	5	5	5	−1 2 ← These values are obtained by adding all digits in corresponding columns.

This result of multiplication is same as linear convolution. We know that range of y (n) is from y (– 3) to y (3). So the first term from L.H.S. represents the value of y (– 3). Now mark the arrow at the position of y (0). Thus output y (n) can be written as,

$$y(n) = \{2, 6, 5, 5, 5, -1, 2\}$$
$$\uparrow$$

Let us verify this result using tabulation method shown in Fig. A-65.

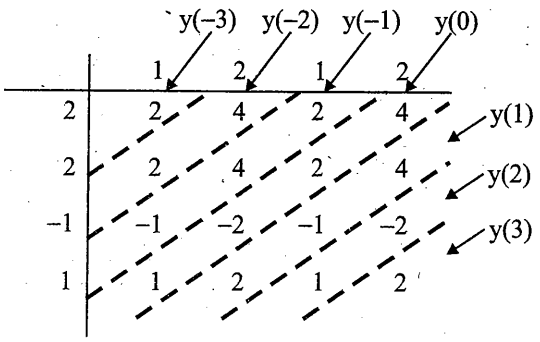


(a)

	1	2	1	2
2	(2×1)	(2×2)	(2×1)	(2×2)
2	(2×1)	(2×2)	(2×1)	(2×2)
−1	(−1×1)	(−1×2)	(−1×1)	(−1×2)
1	(1×1)	(1×2)	(1×1)	(1×2)

(b)

Fig. A-65



(c)

Fig. A-65 : Convolution of x (n) = {1, 2, 1, 2} and h (n) = {2, 2, – 1, 1}
↑

Thus using tabulation method we get,

$y(-3) = 2$ $\quad$ $y(-2) = 2+4 = 6$

$y(-1) = -1+4+2 = 5$ $\quad$ $y(0) = 1-2+2+4 = 5$

$y(1) = 2-1+4 = 5$ $\quad$ $y(2) = 1-2 = -1$

$y(3) = 2$

$\therefore$ $y(n) = \{2, 6, 5, 5, 5, -1, 2\}$
↑

Prob. 13 : Find output of the system if input x (n) and impulse response h (n) are given by,

$x(n) = 1, \quad n = -2, 0, 1$

$= 2, \quad n = -1$

$= 0, \quad$ otherwise

and $h(n) = \delta(n) - \delta(n-1) + \delta(n-2) - \delta(n-3)$.

Soln. : First we will obtain the sequences x (n) and h (n).

Input sequence x (n) is,

at $n = 0 \Rightarrow x(0) = 1$

at $n = 1 \Rightarrow x(1) = 1$

at $n = -1 \Rightarrow x(-1) = 2$

at $n = -2 \Rightarrow x(-2) = 1$

$\therefore$ $x(n) = \{1, 2, 1, 1\}$
↑

Now we will calculate the sequence h (n).

It is $h(n) = \delta(n) - \delta(n-1) + \delta(n-2) - \delta(n-3)$.

It is calculated as shown in Fig. A-66.

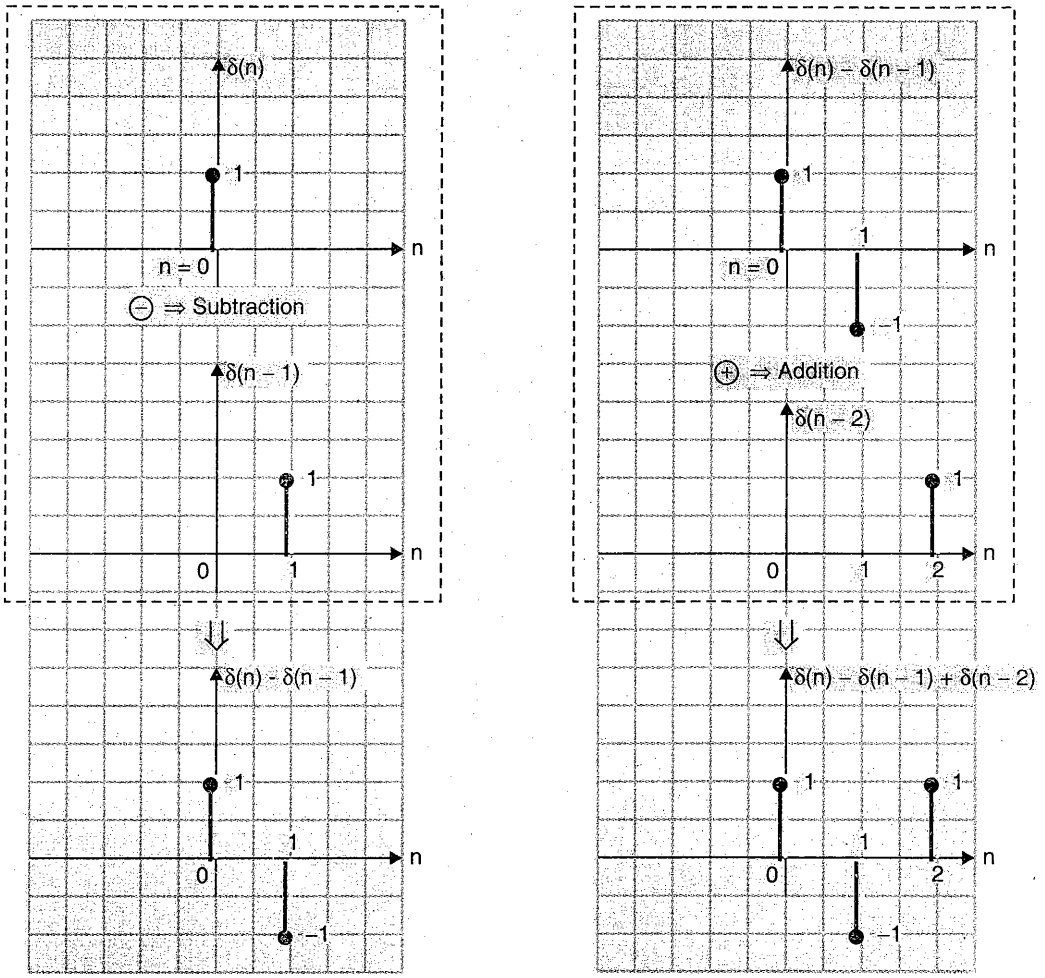
$$\therefore \quad h(n) = \{1, -1, 1, -1\}$$
$$\uparrow$$

Now we will use tabular method of linear convolution. The range of output y (n) is,

$$y_l = x_l + h_l = -2 + 0 = -2$$

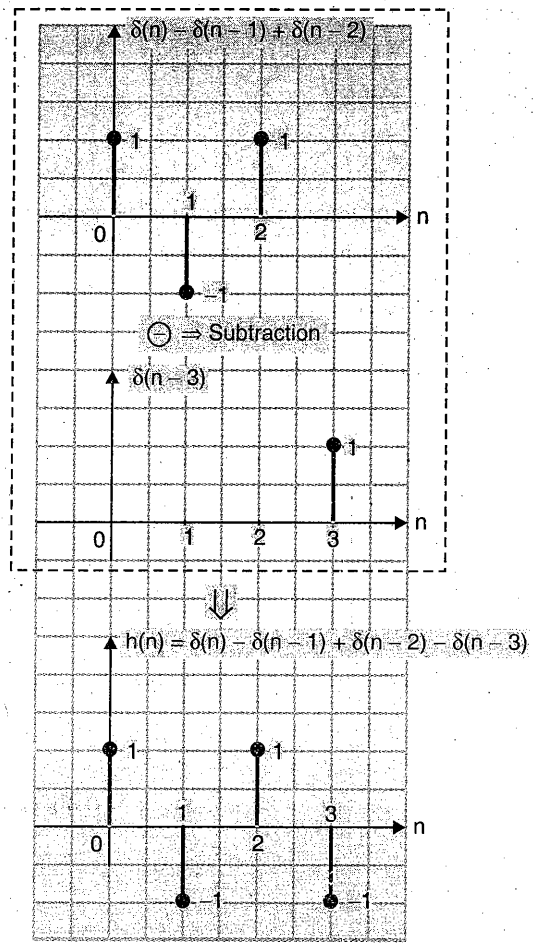
and $$y_h = x_h + h_h = 1 + 3 = 4$$

Thus the range of y (n) is from n = – 2 to n = 4.



(a) **(b)**

Fig. A-66



(c)

Fig. A-66

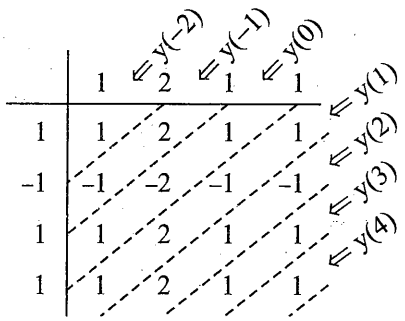


Fig. A-67

The different values of $y(n)$ are as follows :

$$y(-2) = 1,\ y(-1) = 2-1 = 1$$

$$y(0) = 1-2+1 = 0,\ y(1) = 1+2-1+1 = 3$$

$$y(2) = 2+1-1 = 2,\ y(3) = 1+1 = 2$$

and $y(4) = 1$

$$\therefore\ y(n) = \{1, 1, 0, 3, 2, 2, 1\}$$
$$\uparrow$$

1.10.2 Properties of Linear Convolution :

We have discussed how to obtain the convolution of two sequences. In this sub-section we will study some important properties of linear convolution. These properties are :

1. Commutative property
2. Associative property
3. Distributive property.

1. Commutative property :

Statement :

It states that linear convolution is commutative operation. That means

$$x(n) * h(n) = h(n) * x(n).$$

Proof :

The linear convolution of $x(n)$ and $h(n)$ is given by,

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) \qquad ...(1)$$

In this equation $h(n-k)$ is folded delayed sequence; while $x(k)$ is unchanged.

Put $m = n-k$ as new index of summation in Equation (1).

$$\therefore\ k = n-m$$

Limits will change as follows :

When $k = -\infty \Rightarrow n-m = -\infty \quad \therefore -m = -\infty - n \quad \therefore m = \infty$

When $k = +\infty \Rightarrow n-m = \infty \quad \therefore -m = \infty - n \quad \therefore m = -\infty$

Putting these values in Equation (1) we get,

$$y(n) = \sum_{m=+\infty}^{-\infty} x(n-m)\,h(m) \qquad ...(2)$$

In case of summation operation limits from $+\infty$ to $-\infty$ are same as $-\infty$ to $+\infty$. Thus changing limits of summation we get,

$$y(n) = \sum_{m=-\infty}^{\infty} x(n-m)h(m) \quad ...(3)$$

In Equation (3), m is dummy variable. It can be replaced by any other variable. So replacing 'm' by 'k' in Equation (3) we get,

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \quad ...(4)$$

In this equation $x(n-k)$ is folded and delayed sequence; while sequence $h(k)$ is unchanged. As per the definition of convolution Equation (4) can be written as,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k) \cdot x(n-k) = h(n) * x(n) \quad ...(5)$$

Thus from Equations (1) and (5) we get,

$$y(n) = x(n) * h(n) = h(n) * x(n) \quad ...(6)$$

Hence proved that linear convolution is commutative.

Graphical Presentation :

Graphically it is presented as shown in Fig. A-68(a).

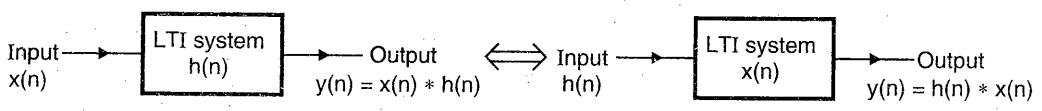


Fig. A-68(a) : $x(n) * h(n) = h(n) * x(n)$

2. Associative property :

Statement : It states that $[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$

Explanation : We know that '*' indicates convolution operation. Every convolution operation indicates that certain input is applied to the LTI system and output is the convolution of input signal and the impulse response of system.

Now consider L.H.S. term. That is the term, $[x(n) * h_1(n)] * h_2(n)$.

Let this output is $y(n)$

$$\therefore \quad y(n) = [x(n) * h_1(n)] * h_2(n) \quad ...(1)$$

There are two convolution operations. The bracket term indicates one convolution of $x(n)$ and $h_1(n)$. Let this output be denoted by $y_1(n)$. Thus Equation (1) becomes,

$$y(n) = y_1(n) * h_2(n) \quad ...(2)$$

The convolution $y_1(n) = x(n) * h_1(n)$ indicates that, we are applying input to LTI system whose impulse response is $h_1(n)$. The output of system is $y_1(n)$. Such system is represented in Fig. A-68(b).

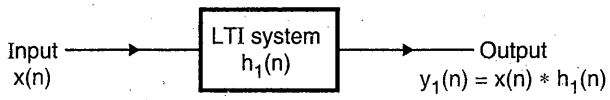


Fig. A-68(b) : $y_1(n) = x(n) * h_1(n)$

Now according to Equation (2); $y(n)$ is obtained by computing convolution of $y_1(n)$ and $h_2(n)$. Here $y_1(n)$ acts as input signal to the LTI system whose impulse response is $h_2(n)$. The output of this system is $y(n)$. Diagrammatically it is represented as shown in Fig. A-68(c).

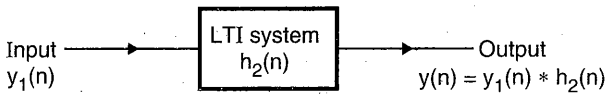


Fig. A-68(c) : $y(n) = y_1(n) * h_2(n)$

Thus to implement Equation (1) we have to connect (cascade) two systems shown in Figs. A-68(b) and A-68(c). This cascade connection is shown in Fig. A-68(d).

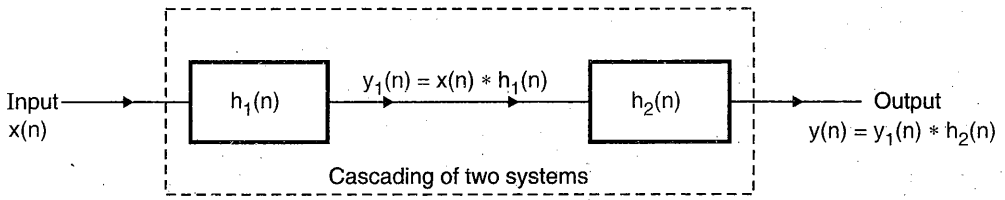


Fig. A-68(d) : $y(n) = [x(n) * h_1(n)] * h_2(n)$

Now consider the R.H.S. term $x(n) * [h_1(n) * h_2(n)]$. We can design one LTI system whose impulse response $h(n)$ is the convolution of $h_1(n)$ and $h_2(n)$. That means convolution of two impulse responses. Thus $h(n) = h_1(n) * h_2(n)$.

$$\therefore \quad x(n) * [h_1(n) * h_2(n)] = x(n) * h(n) \quad ...(3)$$

The diagrammatic representation of Equation (3) is shown in Fig. A-68(e).

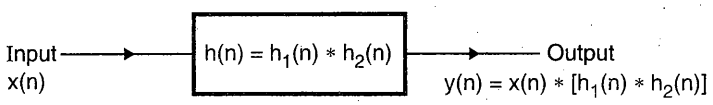


Fig. A-68(e)

Meaning of associative property :

This property indicates that we can replace cascade combination of LTI systems by a single system whose impulse response is convolution of individual impulse responses. It is represented in Fig. A-68(f).

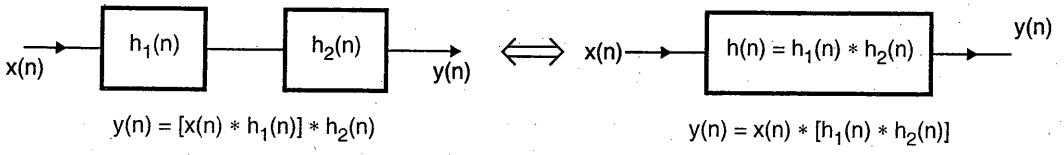


Fig. A-68(f) : Meaning of associative property

3. Distributive property :

Statement : It states that linear convolution is distributive. That means

$$x(n) * [h_1(n) + h_2(n)] = [x(n) * h_1(n)] + [x(n) * h_2(n)] \quad ...(1)$$

Proof : Consider the R.H.S. term of Equation (1). We will denote it by $y(n)$.

$$\therefore \quad y(n) = [x(n) * h_1(n)] + [x(n) * h_2(n)] \quad ...(2)$$

Every bracket term in Equation (2), indicates the convolution operation. Thus we will denote output of each convolution as follows :

$$y_1(n) = x(n) * h_1(n) \quad ...(3)$$

$$\text{and} \quad y_2(n) = x(n) * h_2(n) \quad ...(4)$$

$$\therefore \quad y(n) = y_1(n) + y_2(n) \quad ...(5)$$

Observe Equations (3) and (4). It indicates that there are two LTI systems whose impulse responses are $h_1(n)$ and $h_2(n)$. Important thing is that, the same input signal $x(n)$ is applied to both systems. While Equation (5) shows that we have to add the outputs of these two LTI system. This process is represented diagrammatically as shown in Fig. A-69(a).

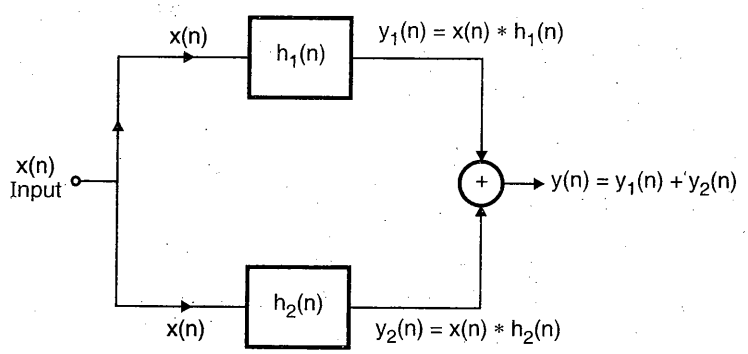


Fig. A-69(a) : $y(n) = y_1(n) + y_2(n) = [x(n) * h_1(n)] + [x(n) * h_2(n)]$

As shown in Fig. A-69(a), the same input is applied to both systems. So this is a parallel connection of two LTI systems. Now as shown in Fig. A-69(a); two impulse responses are added.

$$\therefore \quad h(n) = h_1(n) + h_2(n) \qquad ...(6)$$

Thus we can design a single LTI system having impulse response $h(n) = h_1(n) + h_2(n)$. This will perform the same operation performed by the system shown in Fig. A-69(a). This system is represented in Fig. A-69(b).

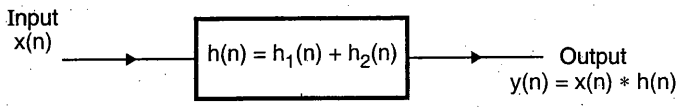


Fig. A-69(b) : $y(n) = x(n) * h(n) = x(n) * [h_1(n) + h_2(n)]$

Meaning of distributive property :

This property indicates that, we can replace a parallel combination of LTI systems by a single LTI system whose impulse response is the sum of individual impulse responses. It is represented as shown in Fig. A-69(c).

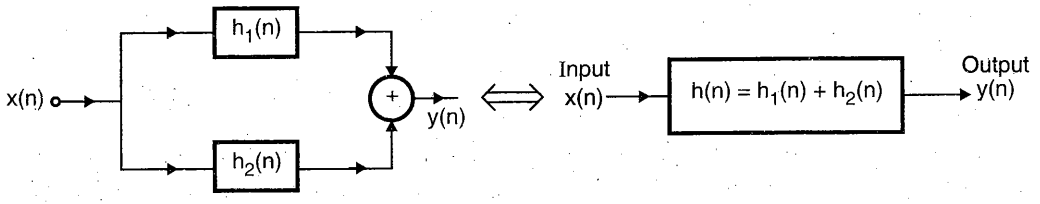


Fig. A-69(c) : Meaning of distributive property

Summary of properties :

Table A-5 shows the summary of properties of linear convolution.

Table A-5

Sr. No.	Name of property	Mathematical equation
1.	Commutative	$x(n) * h(n) = h(n) * x(n)$
2.	Associative	$[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$
3.	Distributive	$x(n) * [h_1(n) + h_2(n)] = [x(n) * h_1(n)] + [x(n) * h_2(n)]$

1.11 Properties of LTI Systems :

We have studied properties of discrete time systems. In this section, we will discuss some important properties of LTI systems like stability, causality etc.

1.11.1 Stability of LSI System :

We have discussed the stability condition for discrete time system. It states that a discrete time system is stable if bounded input produces bounded output. Here the meaning of word "bounded" is; input and output signals should have some finite magnitude. The magnitude of these signals should not be infinity. Now in this section we will derive the stability condition applicable to LTI systems; in terms of unit sample response.

Proof of stability criteria for LTI (or LSI) systems in terms of unit impulse response :

Statement : LTI system is stable if its impulse response is absolutely summable.

Proof : Consider a linear time invarient system having impulse response $h(n)$. Let $x(n)$ be the input applied to this system. Now according to the definition of convolution; the output of such system is expressed as,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad ...(1)$$

Equation (1) can also be written as,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) \quad ...(2)$$

Now first we will define bounded sequence. A discrete time sequence is said to be bounded if the absolute value of every element is less than some finite number. Thus input $x(n)$ will be bounded if $|x(n)|$ is less than some finite number. Let us denote this finite number by M_x. Thus for input signal to be bounded.

$$|x(n)| \leq M_x \quad ...(3)$$

Here M_x is a finite number; so definately its value should be less than infinity. Thus Equation (3) can be written as,

$$|x(n)| \leq M_x < \infty \quad ...(4)$$

Now taking absolute value of both sides of Equation (2) we get,

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} h(k)x(n-k) \right| \quad ...(5)$$

We will read R.H.S. of Equation (5) as "absolute value of summation of terms''. If we take '$\sum$' sign outside then the term becomes $\sum_{k=-\infty}^{\infty} |h(k)||x(n-k)|$. This is summation of absolute values of terms. Always absolute value of sum of terms is less than or equal to the sum of absolute values of terms.

Thus $\left| \sum_{k=-\infty}^{\infty} h(k)\,x(n-k) \right| \leq \sum_{k=-\infty}^{\infty} |h(k)|\,|x(n-k)|$ ...(6)

But from Equation (5); the L.H.S. term is $|y(n)|$.

$$\therefore \quad |y(n)| \leq \sum_{k=-\infty}^{\infty} |h(k)|\,|x(n-k)| \qquad ...(7)$$

Here $x(n-k)$ is delayed input signal. If input is bounded then its delayed version is also bounded. This is because delay or folding is related to time shifting operations. By performing these operations; the magnitude is not changed. Now for bounded input we have,

$$|x(n)| \leq M_x$$

$$\therefore \quad |x(n-k)| \leq M_x \qquad ...(8)$$

Putting this value in Equation (7) we get,

$$|y(n)| \leq \sum_{k=-\infty}^{\infty} |h(k)| \cdot M_x$$

$$\therefore \quad |y(n)| \leq M_x \sum_{k=-\infty}^{\infty} |h(k)| \qquad ...(9)$$

We know that M_x is a finite number. We want output $y(n)$ to be bounded. That means $|y(n)|$ should be finite. So from Equation (9), to obtain finite output we have,

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty \qquad ...(10)$$

Here $h(k) = h(n)$ is the impulse response of LTI system. Thus Equation (10) gives the condition of stability in terms of impulse response of the system.

Now the stability factor is denoted by 's'.

$$\therefore S = \sum_{k=-\infty}^{\infty} |h(k)| < \infty \qquad ...(11)$$

Thus LTI system is stable if its impulse response is absolutely summable.

1.11.2 Causality of LTI System :

We have studied that a discrete time system is causal if output of system depends only on present and past inputs and not on the present input. In this section we will prove that LTI system is causal if its impulse response $h(n) = 0$ for $n < 0$.

Statement : A LTI system is causal if and only if $h(n) = 0$ for $n < 0$.

This is the necessary and sufficient condition for causality of LTI system.

In simple words we can write this statement as "A LTI system is causal if its impulse response, $h(n) = 0$ for negative values of n".

Proof : According to the equation of linear convolution we have,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \qquad ...(1)$$

This equation can also be written as,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) \qquad ...(2)$$

Here limits of summation are $k = -\infty$ to $+\infty$. This summation can be expressed into two parts as follows :

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) + \sum_{k=-\infty}^{-1} h(k)x(n-k) \qquad ...(3)$$

Note that total range of k is from $-\infty$ to $+\infty$. In the first term we have taken the range of k from 0 to ∞. Thus for second term the range of k is from -1 to $-\infty$. Because we have covered $k = 0$ in the first term. So for the second summation the range of k should not be from $k = 0$ to $-\infty$ but it should be from $k = -1$ to $-\infty$. Now expanding the terms of summation we get,

$$y(n) = [h(0)x(0)+h(1)x(n-1)+h(2)x(n-2)+....]$$
$$+[h(-1)x(n+1)+h(-2)x(n+2)+h(-3)x(n+3)+....] \qquad ...(4)$$

Let us consider the 'present' condition is at $n = 0$. Thus,

$x(0) \Rightarrow$ Present input

$x(n-1), x(n-2), ... \Rightarrow$ Delayed inputs that means past inputs.

and $x(n+1), x(n+2), ... \Rightarrow$ Advanced inputs that means future inputs.

We discussed that for the system to be causal we want only present and past inputs. We do not want future inputs. Thus second bracket in Equation (4) should be zero to make the system causal.

Here $x(n+1), x(n+2)$ ... are future inputs. We do not want to make input to the system zero. So the second bracket is zero if we adjust $h(-1), h(-2), h(-3)$ ... to be equal to zero. Then in this case we will have only first bracket terms. That means we will have only present and past input terms. And then the system will be causal.

Thus for causality,

$$h(-1) = h(-2) = h(-3) = \ldots = 0 \quad \ldots(5)$$

That means,

$$h(n) = 0 \quad \text{for } n < 0 \quad \ldots(6)$$

Modification of convolution equation :

We have the equation of linear convolution,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad \text{OR} \quad y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

We have derived that, the LTI system is causal then $h(n) = h(k) = 0$ for n (or k) < 0. Thus if LTI system is causal then the limits of summation will be from $k = 0$ to $+\infty$; because for negative values of 'k', $h(k)$ is zero. Thus for causal LTI system the equation of linear convolution is,

$$y(n) = \sum_{k=0}^{\infty} x(k)h(n-k) \quad \text{OR} \quad y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$

Prob. 1 : Determine range of values of parameter 'a' for which the linear time invariant system with impulse response.

$$h(n) = a^n \quad n \geq 0 \text{ and } n \text{ even}$$
$$= 0 \quad \text{otherwise}$$

is stable

Soln. : We know that LTI system is BIBO stable when,

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty \quad \ldots(1)$$

Given equation is $h(n) = a^n$ for $n \geq 0$ and n even and $h(n) = 0$ otherwise. That means sequence $h(n)$ is present from $n = 0$ to $n = \infty$ and for even values of n only. So limits of summation will change from $n = 0$ to $n = \infty$. We will also replace $h(k)$ by $h(n)$. Thus Equation (1) becomes,

$$\sum_{k=-\infty}^{\infty} |h(k)| = \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} |a^n| \quad \ldots(2)$$

Since 'n' is even, to satisfy this condition we will put n = 2 p. Here value of p is 0, 1, 2 .... ∞. So 'n' is always even for any value of 'p'. Thus Equation (2) becomes,

$$\sum_{k=-\infty}^{\infty} |h(k)| = \sum_{p=0}^{\infty} |a^p|$$

$$= \sum_{p=0}^{\infty} |a|^p$$

Expanding the summation we get,

$$\sum_{k=-\infty}^{\infty} |h(k)| = |a|^0 + |a|^1 + |a|^2 + |a|^3 + ...$$

$$= 1 + |a| + |a|^2 + |a|^3 .... \quad ...(3)$$

This is geometric series. For geometric series we have,

$$\sum_{n=0}^{\infty} a^n = 1 + a + a^2 + a^3 + ... = \frac{1}{1-a} \quad \text{if } |a| < 1$$

Thus Equation (3) becomes,

$$\sum_{k=-\infty}^{\infty} |h(k)| = \frac{1}{1-|a|} \quad \text{if } |a| < 1$$

This condition (if $|a| < 1$) indicates that the given series converges to $\frac{1}{1-|a|}$ if and only if $|a| < 1$. Thus for the system to be stable $\mathbf{|a| < 1}$ is the condition. Otherwise the system will be unstable.

1.12 Parallel and Cascade Connection of LTI Systems :

In case of complicated operations; it is necessary to interconnect different subsystems. There are three types of interconnections as follows :

(1) Cascade or series interconnection
(2) Parallel interconnection
(3) Feedback interconnection

(1) Cascade or series interconnection :

In this case two systems are connected in series. That means output of one system is connected to the input of other system. This is shown in Fig. A-70(a).

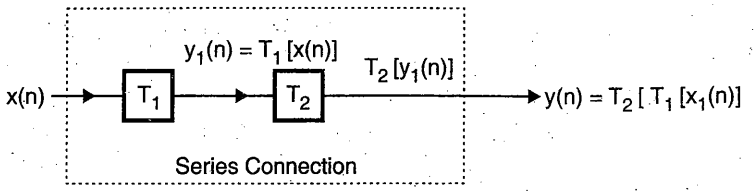


Fig. A-70(a) : Cascade or series interconnection

Here T_1 and T_2 represents impulse responses of systems 1 and 2 respectively. Thus output of system-1 is,

$$y_1(n) = T_1[x(n)]$$

This is applied as input to system 2. Thus total output is,

$$y(n) = T_2\left[y_1(n)\right]$$

$$\therefore \quad y(n) = T_2 T_1\left[x_1(n)\right]$$

(2) Parallel connection : Two systems can be connected in parallel as shown in Fig. A-70(b). Here same input, x (n) is applied to two systems.

Thus output of system-1 is,

$$y_1(n) = T_1[x(n)]$$

and output of system-2 is,

$$y_2(n) = T_2[x(n)].$$

Finally these two outputs are added by using adder block. Thus the total output is,

$$y(n) = y_1(n) + y_2(n)$$

$$\therefore \quad y(n) = T_1[x(n)] + T_2[x(n)]$$

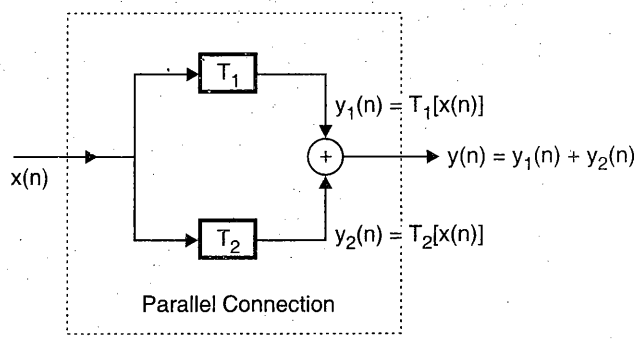


Fig. A-70(b) : Parallel interconnection

(3) Feedback interconnection : As the name indicates, a feedback connection is done from output to the input. This interconnection is as shown in Fig. A-70(c).

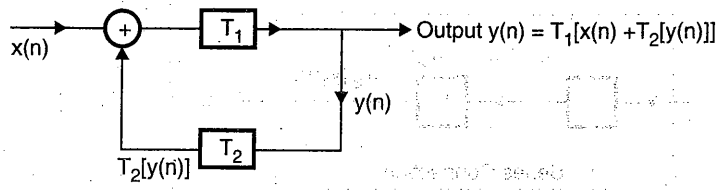


Fig. A-70(c) : Feedback interconnection

Here output $y(n)$ is applied to system-2 (T_2). Its output is $T_2[y(n)]$. This signal is added with input $x(n)$. So the output of adder is $x(n)+T_2[y(n)]$. This signal is applied to system-1 (T_1). Thus the total output is,

$$y(n) = T_1\left[x(n)+T_2[y(n)]\right]$$

Solution of Linear Constant Coefficient Difference Equation :

In this subsection we will discuss how to obtain exploit solution of a given linear constant coefficient difference equation. Basically there are two methods to obtain this solution.

These methods are as follows :

(1) Direct method

(2) Indirect method

Indirect method is based on z transformation. So will study that method in the later chapter. Presently we will discuss direct method for obtaining the solution.

In the direct method; it is assumed that the solution consists of two parts :

(1) Homogeneous or complementary solution. It is denoted by $y_h(n)$.

(2) Particular solution. It is denoted by $y_p(n)$.

Thus the total solution is expressed as,

$$y(n) = y_h(n)+y_p(n) \qquad ...(1)$$

(1) Homogeneous solution of difference equation :

The general form of difference equation. It is,

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k)+\sum_{k=0}^{M} b_k x(n-k) \qquad ...(2)$$

In the first term the limits of summation are from k = 1 to N. We can change these limits from k = 0 to N as follows. Take the limits from k = 0 to N then,

$$\sum_{k=0}^{N} a_k y(n-k) = a_0 y(n)+\sum_{k=1}^{N} a_k y(n-k)$$

$$\therefore \quad -\sum_{k=1}^{N} a_k y(n-k) = a_0 y(n)-\sum_{k=0}^{N} a_k y(n-k) \qquad ...(3)$$

Putting this value in Equation (2) we get,

$$y(n) = a_0 y(n) - \sum_{k=0}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

Assume $a_0 = 1$

$$\therefore \quad y(n) = y(n) - \sum_{k=0}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

$$\therefore \quad \sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \qquad ...(4)$$

We will obtain homogeneous solution by assuming input $x(n) = 0$.

Since $x(n) = 0 \Rightarrow x(n-k) = 0$. Thus Equation (4) becomes,

$$\sum_{k=0}^{N} a_k y(n-k) = 0 \qquad ...(5)$$

We will assume that the solution is in the form of exponential.

Thus let, homogeneous solution,

$$[y_h(n)] = \lambda^n \qquad ...(6)$$

Replace n by $n-k$

$$\therefore \quad y_h(n-k) = \lambda^{n-k}$$

Putting this value in Equation (5) we get,

$$\sum_{k=0}^{N} a_k \lambda^{n-k} = 0 \qquad ...(7)$$

Expanding the summation we get,

$$a_0 \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + ... + a_N \lambda^{n-N} = 0$$

We have assumed that $a_0 = 1$

$$\therefore \quad \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \ldots + a_N \lambda^{n-N} = 0$$

Taking λ^{n-N} common we get,

$$\lambda^{n-N}\left[\lambda^n \cdot \lambda^{-n+N} + a_1 \lambda^{n-1} \cdot \lambda^{-n+N} + a_2 \lambda^{n-2} \cdot \lambda^{-n+N} + \ldots + a_N\right] = 0$$

$$\therefore \quad \lambda^{n-N}\left[\lambda^N + a_1 \lambda^{N-1} + a_2 \lambda^{N-2} + \ldots + a_N\right] = 0 \qquad ...(8)$$

The term inside the bracket is called as characteristic polynomial of the system. This

polynomial has 'N' roots denoted by $\lambda_1, \lambda_2 \ldots \lambda_N$. These roots can be real or complex valued.

Case I : Roots are distinct :

When the roots are distinct then we can write the solution as,

$$y_h(n) = c_1 \lambda_1^n + c_2 \lambda_2^n + \ldots + c_N \lambda_N^n \quad \ldots(9)$$

Here $c_1, c_2, \ldots c_N$ are constants.

Case II : Multiple roots :

In case of multiple roots; the solution can be written as,

$$y_h(n) = c_1 \lambda_1^n + c_2 n \lambda_1^n + c_3 n^2 \lambda_1^n + \ldots + c_N \lambda_N$$

Solved Problems :

Prob. 1 : Determine the homogeneous solution of the system described by $y(n) - 3y(n-1) - 4y(n-2) = x(n)$.

Soln. :

Step I : Assume input $x(n) = 0$

$$\therefore \quad y(n) - 3y(n-1) - 4y(n-2) = 0 \quad \ldots(1)$$

Step II : Assume the the homogeneous solution is in the form

$$y_h(n) = \lambda^n \quad \ldots(2)$$

Thus we get,

$$y(n-1) = \lambda^{n-1} \text{ and } y(n-2) = \lambda^{n-2}$$

Step III : Putting these values in Equation (1) we get,

$$\lambda^n - 3\lambda^{n-1} - 4\lambda^{n-2} = 0 \quad \ldots(3)$$

Taking λ^{n-2} common we get,

$$\lambda^{n-2}\left[\lambda^n \cdot \lambda^{-n+2} - 3\lambda^{n-1} \cdot \lambda^{-n+2} - 4\right] = 0$$

$$\therefore \quad \lambda^{n-2}\left[\lambda^2 - 3\lambda - 4\right] = 0$$

$$\therefore \quad \lambda^2 - 3\lambda - 4 = 0 \quad \ldots(4)$$

Now obtain roots of Equation (4) using the formula,

$$\text{roots} = \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here $a = 1$, $b = -3$ and $c = -4$

$$\therefore \quad \lambda = \frac{3 \pm \sqrt{9 - 4(1)(-4)}}{2 \times 1} = \frac{3 \pm \sqrt{25}}{2} = \frac{3 \pm 5}{2}$$

$$\therefore \quad \lambda_1 = \frac{3-5}{2} \text{ and } \lambda_2 = \frac{3+5}{2}$$

$$\therefore \quad \lambda_1 = -1 \text{ and } \lambda_2 = 4$$

Step IV : The homogeneous solution is given by,

$$y_h(n) = c_1 \lambda_1^n + c_2 \lambda_2^n$$

$$\therefore y_h(n) = c_1 (-1)^n + c_2 (4)^n \quad \ldots(5)$$

Now we will obtain values of c_1 and c_2.

Step V : From Equation (5) we get,

$$\text{For } n = 0 \Rightarrow y(0) = c_1 + c_2 \quad \ldots(6)$$

$$\text{and for } n = 1 \Rightarrow y(1) = -c_1 + 4c_2 \quad \ldots(7)$$

Step VI : Now take original equation,

$$y(n) - 3y(n-1) - 4y(n-2) = 0$$

$$\therefore \quad y(n) = 3y(n-1) + 4y(n-2) \quad \ldots(8)$$

$$\therefore \quad \text{For } n = 0 \Rightarrow y(0) = 3y(-1) + 4y(-2) \quad \ldots(9)$$

$$\text{For } n = 1 \Rightarrow y(1) = 3y(0) + 4y(-1) \quad \ldots(10)$$

Putting Equation (9) in Equation (10) we get,

$$y(1) = 3[3y(-1) + 4y(-2)] + 4y(-1)$$

$$\therefore \quad y(1) = 9y(-1) + 12y(-2) + 4y(-1)$$

$$\therefore \quad y(1) = 13y(-1) + 12y(-2) \quad \ldots(11)$$

Step VII : Equate equations in step V and VI. That means equating Equations (6) and (9) we get,

$$c_1 + c_2 = 3y(-1) + 4y(-2) \quad \ldots(12)$$

And equating Equations (7) and (11) we get,

$$-c_1 + 4c_2 = 13y(-1) + 12y(-1) \quad \ldots(13)$$

Now we will solve Equations (12) and (13) to obtain c_1 and c_2.

Adding Equations (12) and (13) we get,

$$5c_2 = 16y(-1) + 16y(-2)$$

$$\therefore c_2 = \frac{16}{5} y(-1) + \frac{16}{5} y(-2) \quad \ldots(14)$$

Putting this value in Equation (12) we get,

$$c_1 + \frac{16}{5} y(-1) + \frac{16}{5} y(-2) = 3y(-1) + 4y(-2)$$

$$\therefore \quad c_1 = 3y(-1) - \frac{16}{5}y(-1) + 4y(-2) - \frac{16}{5}y(-2)$$

$$\therefore \quad c_1 = -\frac{1}{5}y(-1) + \frac{4}{5}y(-2) \qquad ...(15)$$

Step VIII : Putting these values in Equation (5) we get,

$$y_h(n) = \left[-\frac{1}{5}y(-1) + \frac{4}{5}y(-2)\right](-1)^n + \left[\frac{16}{5}y(-1) + \frac{16}{5}y(-2)\right](4)^n \qquad ..(16)$$

If we assume initial conditions zero that means,

$y(-1) = y(-2) = 0$ then Equation (16) becomes,

$$y_h(n) = (-1)^n + (4)^n \qquad ...(17)$$

Note : To obtain this solution we have assumed input $x(n) = 0$. So this output is also called as zero input response.

Prob. 2 : Find the impulse response of the system shown in Fig. A-71.

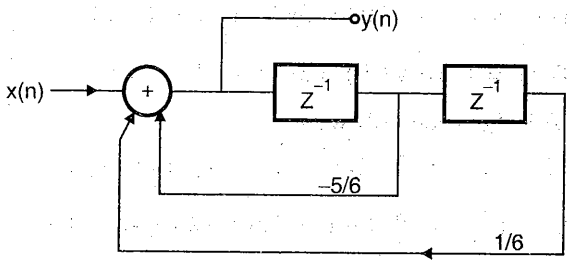


Fig. A-71 : Given discrete time system

Soln. : Impulse response means, the response (output) of system when input $x(n) = \delta(n)$. Using homogeneous solution we can find the impulse response, assuming that the system is initially relaxed.

First we will write the difference equation from given Fig. A-71. At the output of first delay we get $y(n-1)$ and it is multiplied by $\frac{-5}{6}$. While at the output of second delay we get $y(n-2)$ and it is multiplied by $\frac{1}{6}$. These two outputs are added with input signal $x(n)$.

$$\therefore \quad y(n) = x(n) - \frac{5}{6}y(n-1) + \frac{1}{6}y(n-2) \qquad ...(1)$$

Now we will find out the homogeneous solution.

Step I : Assume input $x(n) = 0$. Thus Equation (1) becomes,

$$y(n) + \frac{5}{6}y(n-1) - \frac{1}{6}y(n-2) = 0 \qquad ...(2)$$

Step II : Assume that the homogeneous solution is in the form

$$y_h(n) = \lambda^n \qquad ...(3)$$

Thus we get, $y(n-1) = \lambda^{n-1}$ and $y(n-2) = \lambda^{n-2}$

Step III : Putting these values in Equation (2) we get,

$$\lambda^n + \frac{5}{6}\lambda^{n-1} - \frac{1}{6}\lambda^{n-2} = 0$$

Taking λ^{n-2} common we get,

$$\lambda^{n-2}\left[\lambda^n \cdot \lambda^{-n+2} + \frac{5}{6}\lambda^{n-1} \cdot \lambda^{-n+2} - \frac{1}{6}\right] = 0$$

$$\therefore \quad \lambda^{n-2}\left[\lambda^2 + \frac{5}{6}\lambda - \frac{1}{6}\right] = 0$$

$$\therefore \quad \lambda^2 + \frac{5}{6}\lambda - \frac{1}{6} = 0 \qquad ...(4)$$

Now we will obtain roots of Equation (4)

$$\therefore \quad \lambda = \frac{-\frac{5}{6} \pm \sqrt{\frac{25}{36} - \left[4 \times 1 \times \left(-\frac{1}{6}\right)\right]}}{2}$$

$$\therefore \quad \lambda = \frac{-\frac{5}{6} \pm \sqrt{\frac{25}{36} + \frac{4}{6}}}{2} = \frac{-\frac{5}{6} \pm \sqrt{\frac{25+24}{36}}}{2} = \frac{\frac{-5}{6} \pm \frac{7}{6}}{2}$$

$$\therefore \quad \lambda_1 = \frac{1}{6} \text{ and } \lambda_2 = -1$$

Step IV : The homogeneous solution is given by,

$$y_h(n) = c_1 \lambda_1^n + c_2 \lambda_2^n$$

$$\therefore \quad y_h(n) = c_1\left(\frac{1}{6}\right)^n + c_2(-1)^n$$

We will denote $y_h(n)$ by $h(n)$

$$\therefore \quad h(n) = c_1\left(\frac{1}{6}\right)^n + c_2(-1)^n \qquad ...(5)$$

Now we will obtain values of c_1 and c_2.

Step V : We will apply initial conditions. In this case $x(n) = \delta(n)$ and $y(n) = h(n)$. Putting these values in Equation (1) we get,

$$h(n) = \delta(n) - \frac{5}{6}h(n-1) + \frac{1}{6}h(n-2)$$

For $n = 0 \Rightarrow h(0) = \delta(0) - \frac{5}{6}h(-1) + \frac{1}{6}h(-2)$

Here $\delta(0) = 1$ and since initial conditions are zero thus $h(-1) = 0$ and $h(-2) = 0$

$$\therefore \quad h(0) = 1 - \frac{5}{6} \times 0 + \frac{1}{6} \times 0$$

$$\therefore \quad h(0) = 1$$

For $n = 1 \Rightarrow h(1) = \delta(1) - \frac{5}{6}h(0) + \frac{1}{6}h(-1)$

Here $\delta(1) = 0$, $h(-1) = 0$ and $h(0) = 1$

$$\therefore \quad h(1) = 0 - \frac{5}{6} \times 1 + \frac{1}{6} \times 0$$

$$\therefore \quad h(1) = -\frac{5}{6}$$

Step VI : Putting $n = 0$ and $n = 1$ in Equation (5) we get,

For $n = 0 \Rightarrow c_1\left(\frac{1}{6}\right)^0 + c_2(-1)^0 = h(0)$

$$\therefore \quad c_1 + c_2 = 1 \qquad \ldots(6)$$

For $n = 1 \Rightarrow c_1\left(\frac{1}{6}\right)^1 + c_2(-1)^1 = h(1)$

$$\therefore \quad c_1 \cdot \frac{1}{6} - c_2 = -\frac{5}{6}$$

$$\therefore \quad \frac{1}{6}c_1 - c_2 = -\frac{5}{6} \qquad \ldots(7)$$

We will obtain values of c_1 and c_2

From Equation (6)

$$c_2 = 1 - c_1$$

Putting this value in Equation (7) we get,

$$\frac{1}{6}c_1 - 1 + c_1 = -\frac{5}{6}$$

$$\therefore \quad +\frac{7}{6}c_1 = -\frac{5}{6} + 1 = \frac{1}{6} \qquad \therefore \quad 7c_1 = 1$$

$$\therefore \quad c_1 = \frac{1}{7}$$

$$\text{and} \quad c_2 = 1 - c_1$$

$$\therefore \quad c_2 = 1 - \frac{1}{7} = \frac{6}{7}$$

$$\therefore c_2 = \frac{6}{7}$$

Putting c_1 and c_2 in Equation (5) we get,

$$\therefore h(n) = \frac{1}{7}\left(\frac{1}{6}\right)^n + \frac{6}{7}(-1)^n$$

This is impulse response of given system.

(2) Particular solution of difference equation :

The particular solution is denoted by $y_p(n)$. It is calculated by assuming that the output has same form at that of input. For example if input $x(n) = u(n)$ then the output is assumed to be $y_p(n) = k\,u(n)$; where 'k' is arbitrary constant.

Important rules to assume the solution :

(i) If any term in a particular solution (y_p) is also present in homogeneous solution then multiply 'y_p' by n.

(ii) If such term appears 'r' times in a homogeneous solution then multiply 'y_p' by n^r.

Some commonly used input signals and the particular solution that we have to assume is shown in following Table A-6.

Table A-6

Sr. No.	Input x (n)	Particular solution
1.	Constant term (A)	k
2.	u (n)	k u (n)
3.	$A \cdot c^n$	$k\,c^n$
4.	$A^n \cdot n^c$	$k_0 n^c + k_1 n^{M-1} + \ldots + k_c$
5.	$A \cos \omega_0 n$	$k_1 \cos \omega_0 n + k_2 \sin \omega_0 n$

A particular solution is basically used to calculate the zero state response (y_{zs}) of the system. The following steps are used to calculate the forced response of system :

Step I : Find the nature of zero input response from the roots of characteristics polynomials. This is $y_h(n)$ or y_{zi}.

Step II : Assume the particular solution $y_p(n)$

Step III : Determine the coefficients in the zero input response. Then the forced response $y_{zs} = y_h(n) + y_p(n)$.

Solved Problems :

Prob. 1 : Find out the particular solution for the following differential equation

$$y(n) + 3y(n-1) = x(n)$$

Assume $x(n) = u(n)$

Soln. : Here input $x(n) = u(n)$ = unit step

Now corresponding to this input, we will assume the particular solution as,

$$y_p(n) = k\,u(n) \quad ..(1)$$

Here k is some constant.

Now the given equation is,

$$y(n) + 3y(n-1) = x(n) \quad ...(2)$$

From Equation (1) we can write,

$$y(n-1) = k\,u(n-1) \quad ...(3)$$

Putting Equations (1) and (3) in Equation (2) we get,

$$k\,u(n) + 3ku(n-1) = u(n) \quad ..(4)$$

Now we will calculate the value of k. We know that input is unit step, $u(n)$. Its value is 1 for $n \geq 0$. Putting $n = 1$ in Equation (4) we get,

$$ku(1) + 3ku(0) = u(1)$$

Remember that while putting value of 'n' any term should not vanish.

But $u(0) = u(1) = 1$

$$\therefore \quad k + 3k = 1$$

$$\therefore \quad 4k = 1 \Rightarrow k = \frac{1}{4}$$

Putting this value in Equation (1) we get,

$$y_p(n) = \frac{1}{4}u(n)$$

Prob. 2 : Determine the particular solution of the difference equation,

$$y(n) = 3y(n-1) - 4y(n-2) + x(n)$$

When the forcing function $x(n) = 2^n$, $n \geq 0$ and zero everywhere.

Soln. : Refer the Table A-6(a); for input $x(n) = 3^n$ we will assume the particular solution,

$$y_p(n) = k\,2^n \quad ...(1)$$

Now the given input $x(n) = 2^n$ is present only for positive value of 'n' ($n \geq 0$). Thus we can write,

$$x(n) = 2^n u(n) \quad ...(2)$$

Accordingly Equation (1) is modified as,

$$y_p(n) = k\,2^n u(n) \quad ..(3)$$

From Equation (3) we can write,

$y(n-1) = k\,2^{n-1} u(n-1)$ and $y(n-2) = k\,2^{n-2} u(n-2)$

Putting these values in the given equation of $y(n)$ we get,

$$k\,2^n u(n) = 3k\,2^{n-1} u(n-1) - 4k\,2^{n-2} u(n-2) + 2^n u(n) \quad ..(4)$$

Now we will calculate value of k. Given input is present for $n \geq 0$. But while putting value of 'n' any term in equation should not vanish. For example if we will put $n = 1$ then the second term of R.H.S. becomes,

$$-4k\,2^{n-2} u(n-2) \Rightarrow -4k\,(2)^{-1} u(-1)$$

But $u(-1) = 0$; so this value of 'n' is not allowed. Thus we will put $n = 2$ in Equation (4) so that any term will not vanish.

$$\therefore \quad k\,2^2 u(2) = 3k\,(2)^1 u(1) - 4k\,(2)^0 u(0) + 2^2 u(2)$$

Here $u(0) = u(1) = u(2) = 1$

$$\therefore \quad 4k = 6k - 4k + 4$$

$$\therefore \quad 2k = 4 \Rightarrow \boxed{k = 2}$$

Putting this value in Equation (3) we get,

$$\boxed{y_p(n) = 2.2^n u(n)}$$

Note : If total solution is asked to calculate, then first obtain $y_h(n)$ and $y_p(n)$ separately. The total solution is $y(n) = y_h(n) + y_p(n)$.

Prob. 3 : Determine zero input response for a second order difference equation.

$y(n) - 3y(n-1) - 4y(n-2) = x(n)$.

Soln. : Given difference equation is,

$$y(n) - 3y(n-1) - 4y(n-2) = x(n)$$

Zero input means $x(n) = 0$

$$\therefore \quad y(n) - 3y(n-1) - 4y(n-2) = 0$$

$$\therefore \quad y(n) = 3y(n-1) + 4y(n-2)$$

For $n = 0 \Rightarrow \quad y(0) = 3y(-1) + 4y(-2)$

For $n = 1 \Rightarrow \quad y(1) = 3y(0) + 4y(-1)$

For $n = 2 \Rightarrow \quad y(2) = 3y(1) + 4y(0)$

For $n = 3 \Rightarrow \quad y(3) = 3y(2) + 4y(1)$

From these equations, we can write the zero input response as,

$$y(k) = 3y(k-1) + 4y(k-2)$$

1.13 Eigen Functions for LTI Systems and Frequency Response :

Because of property of LTI system; it is possible to represent the signals in terms of sinusoids or complex exponentials. Consider input signal $x(n) = e^{j\omega n}$.

According to the equation of linear convolution we have,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k) \qquad ...(1)$$

$$\text{Since } x(n) = e^{j\omega n} \Rightarrow x(n-k) = e^{j\omega(n-k)}$$

$$\therefore \quad y(n) = \sum_{k=-\infty}^{\infty} h(k)\,e^{j\omega(n-k)} \qquad ...(2)$$

$$= \sum_{k=-\infty}^{\infty} h(k)\,e^{j\omega n} \cdot e^{-j\omega k}$$

$$\therefore \quad y(n) = e^{j\omega n} \sum_{k=-\infty}^{\infty} h(k)\,e^{-j\omega k} \qquad ...(3)$$

In Equation (3), the summation term is denoted by $H(e^{j\omega})$ or $H(\omega)$ and the eigen value $H(\omega)$ represents the frequency response of a system.

$$\therefore \quad H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)\,e^{-j\omega k} \qquad ...(4)$$

$$\text{and} \quad y(n) = e^{j\omega n} \cdot H(e^{j\omega}) \qquad ...(5)$$

Here $H(e^{j\omega})$ is a complex quantity and it can be expressed in terms of real and imaginary parts.

$$\therefore \quad H(e^{j\omega}) = H_R(e^{j\omega}) + j\,H_I(e^{j\omega}) \qquad ...(6)$$

In terms of magnitude and phase we can write,

$$H(e^{j\omega}) = |H(e^{j\omega})|\,e^{j\arg[H(e^{j\omega})]} \qquad ...(7)$$

Let input signal be,

$$x(n) = A\cos(\omega_o n + \theta)$$

Using Euler's identity we can write,

$$x(n) = A\left[\frac{e^{j(\omega_o n+\theta)} + e^{-j(\omega_o n+\theta)}}{2}\right]$$

$$\therefore \quad x(n) = \frac{A}{2}e^{j\omega_o n}\cdot e^{j\theta} + \frac{A}{2}e^{-j\omega_o n}\cdot e^{-j\theta} \qquad ...(8)$$

Let $x_1(n) = \frac{A}{2}e^{j\omega_o n}\cdot e^{j\theta}$ ...(9)

and $x_2(n) = \frac{A}{2}e^{-j\omega_o}\cdot e^{-j\theta}$ ...(10)

Now response of a system to the input $x_1(n)$ is,

$$y_1(n) = H(e^{j\omega_o})\cdot\frac{A}{2}e^{j\omega_o n}\cdot e^{j\theta} \qquad ...(11)$$

Similarly response of a system to the input $x_2(n)$ is,

$$y_2(n) = H(e^{j\omega_o})\cdot\frac{A}{2}e^{-j\omega_o n}\cdot e^{-j\theta} \qquad ...(12)$$

Now total output is, $y(n) = y_1(n) + y_2(n)$

$$\therefore \quad y(n) = \frac{A}{2}\left[H(e^{j\omega_o})e^{j\omega_o n}\cdot e^{j\theta} + H(e^{j\omega_o})\cdot e^{-j\omega_o n}\cdot e^{j\theta}\right]$$

$$\therefore \quad y(n) = A\left[|H(e^{j\omega_o})|\cdot\cos(\omega_o n+\theta)\right] \qquad ...(13)$$

As $\cos\phi = \frac{e^{j\phi}+e^{-j\phi}}{2}$

Magnitude and phase of H (ω) :

The magnitude of $H(\omega)$ is given by,

$$|H(\omega)| = \sqrt{H_R^2(\omega) + H_I^2(\omega)}$$

Here $H_R(\omega)$ = Real component of $H(\omega)$

and $H_I(\omega)$ = Imaginary component of $H(\omega)$.

And the phase of $H(\omega)$ is given by,

$$\theta(\omega) = \tan^{-1}\frac{H_I(\omega)}{H_R(\omega)}$$

Solved Problems :

Prob. 1 : If impulse response of a system is,

$$h(n) = \left(\frac{1}{4}\right)^n u(n)$$

then determine the output sequence when input is complex exponential sequence,

$$x(n) = 5\,e^{j\pi n/2} \quad \ldots \quad -\infty < n < \infty$$

Soln. : The response of a system to the complex exponential input $A\,e^{j\omega n}$ is given by,

$$y(n) = A\,e^{j\omega n} \cdot H(\omega) \qquad \ldots(1)$$

Here $H(\omega)$ = Fourier transform of $h(n)$

Given $h(n) = \left(\frac{1}{4}\right)^n u(n)$

We have standard fourier transform pair,

$$a^n u(n) \xleftrightarrow{\text{F.T.}} \frac{1}{1 - a\,e^{-j\omega}}$$

$$\therefore \quad H(\omega) = \left(\frac{1}{4}\right)^n u(n) \xleftrightarrow{\text{F.T.}} \frac{1}{1 - \frac{1}{4}e^{-j\omega}} \qquad \ldots(2)$$

The given input signal is,

$$x(n) = 5\,e^{j\pi n/2} \qquad \ldots(3)$$

We have standard exponential input signal,

$$x(n) = A\,e^{j\omega n} \qquad \ldots(4)$$

Comparing equations (3) and (4),

$$\omega = \frac{\pi}{2}$$

Putting $\omega = \frac{\pi}{2}$ in Equation (2) we get,

$$H\left(\frac{\pi}{2}\right) = \frac{1}{1 - \frac{1}{4}e^{\frac{-j\pi}{2}}} = \frac{1}{1 - \frac{1}{4}\left[\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right]} = \frac{1}{1 + j\frac{1}{4}}$$

$$\therefore \quad H\left(\frac{\pi}{2}\right) = \frac{1 + j0}{1 + j\frac{1}{4}} = \frac{1\angle 0}{1.03\angle 14.03^\circ} = 0.97\ \angle -14.03^\circ$$

$$\therefore \quad H\left(\frac{\pi}{2}\right) = 0.97\,e^{-j\,14.03^\circ} \qquad \ldots(5)$$

Putting this value in Equation (1) we get,

$$y(n) = 5e^{\frac{j\pi n}{2}} \cdot 0.97 e^{-j14.03}$$

$$\therefore \quad y(n) = 4.85 e^{j\left(\frac{\pi n}{2} - 14.03\right)}$$

Prob. 2 : If impulse response of a system is,

$$h(n) = \left(\frac{1}{4}\right)^n u(n)$$

then determine the response of system to the input signal.

$$x(n) = 20 - 10 \sin\frac{\pi}{2} n + 10 \cos \pi n; \quad -\infty < n < \infty$$

Soln. :

We have already obtained the frequency response $H(\omega)$ for $h(n) = \left(\frac{1}{4}\right)^n u(n)$.

$$\text{It is} \quad H(\omega) = \frac{1}{1 - \frac{1}{4} e^{-j\omega}} \qquad ...(1)$$

Given input signal is,

$$x(n) = 20 - 10 \sin\frac{\pi}{2} n + 10 \cos \pi n \qquad ...(2)$$

Let $x(n) = x_1(n) + x_2(n) + x_3(n)$

Consider first term of input signal. It is $x_1(n) = 20$. That means for this term $\omega = 0$. Thus putting $\omega = 0$ in Equation (1) we get,

$$H(0) = \frac{1}{1 - \frac{1}{4} e^0} = \frac{4}{3} \qquad ...(3)$$

Now consider second term of input signal. It is $x_2(n) = -10 \sin\frac{\pi}{2} n$. It indicates $\omega = \frac{\pi}{2}$. Thus from Equation (1) we get,

$$H\left(\frac{\pi}{2}\right) = \frac{1}{1 - \frac{1}{4} e^{-j\frac{\pi}{2}}} = \frac{1}{1 - \frac{1}{4}\left[\cos\frac{\pi}{2} - j \sin\frac{\pi}{2}\right]} = \frac{1}{1 + j\frac{1}{4}}$$

$$\therefore \quad H\left(\frac{\pi}{2}\right) = 0.97 \angle -14.03 = 0.97 e^{-j\,14.03^\circ} \qquad ...(4)$$

The last term of input signal is $x_3(n) = 10 \cos \pi n$. It represents $\omega = \pi$. Thus from

Equation (1) we get,

$$H(\pi) = \frac{1}{1-\frac{1}{4}e^{-j\pi}} = \frac{1}{1-\frac{1}{4}[\cos\pi - j\sin\pi]}$$

$$\therefore \quad H(n) = \frac{1}{1-\frac{1}{4}(-1)} = \frac{1}{1+\frac{1}{4}} = \frac{4}{5}$$

If input to the system is $x(n) = A\cos\omega_n$ then the system response is given by,

$$y(n) = A|H(\omega)|\cos[\omega_n + \theta(\omega)]$$

And if input to the system is $x(n) = A\sin\omega_n$ then the system response is given by,

$$y(n) = A|H(\omega)|\sin[\omega_n + \theta(\omega)]$$

Thus for the given input; the response of system is,

$$y(n) = 20\left(\frac{4}{3}\right) - 0.97(10)\sin\left(\frac{\pi}{2} - 14.03^\circ\right) + 10\left(\frac{4}{5}\right)\cos\pi$$

$$\therefore y(n) = 26.67 - 9.7\sin\left(\frac{\pi}{2} - 14.03^\circ\right) + 8\cos\pi$$

1.14 Correlation of Discrete Time Systems :

Basics of correlation :

The word co-relation means comparison. Basically it is comparison between two signals or sequences. A good application of correlation is the radar system.

In case of radar system the signal is transmitted from radar to the target. Let signal is $x(n)$. Now if target is present then the signal is reflected back from the target. We will denote this reflected signal by $y(n)$. This signal will be attenuated because of different losses taking place in the path of signal. We will denote this attenuation by factor β. Similarly this signal $y(n)$ will be delayed in time, compared to transmitted signal $x(n)$. Thus we can write,

$$y(n) = \beta x(n - n_0) + N \qquad ...(1)$$

Here β = Attenuation factor

n_0 = Amount by which signal is delayed.

N = Noise levels added in the signal.

Now correlation means compare this output with the transmitted signal $x(n)$. That means check whether $y(n) = x(n)$ or not.

Applications of correlation :

(1) Detection of difference between two signals.

(2) Identification of signals in the presence of noise.

(3) Observing effects of inputs and outputs in control engineering.

(4) Determination of time delays between transmitted and received signals.
(5) Used in image processing to compare data from different images.
(6) Identification of specific pattern within the data stream.
(7) Computation of average power in the waveforms.
(8) For diagnosis of medical disorders.

Types of correlation :

There are two types of correlation as follows :

(1) Auto-correlation
(2) Cross-correlation.

(1) Auto-correlation :

Many times it is required to correlate the signal with itself then this correlation is called as auto-correlation. Thus if the two signals are generated from the same source then it is called as auto-correlation. Auto-correlation is denoted by $r_{xx}(l)$. Here 'l' is a variable and r_{xx} indicates the comparison between $x(n)$ and $x(n)$.

(2) Cross-correlation :

The correlation between $x(n)$ and $y(n)$ is called as cross-correlation. That if the two signals are generated from different sources then it is called as cross-correlation. It is denoted by $r_{xy}(l)$. Here 'l' is a variable and r_{xy} indicates the comparison between $x(n)$ and $y(n)$.

Mathematical Equations :

(1) Auto-correlation : It is defined as,

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)\,x(n-l) \qquad ...(1)$$

Here first sequence $x(n)$ is kept as it is; while the sequence $x(n)$ is delayed by 'l'. Equation (1) can also be written as,

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n+l)\,x(n) \qquad ...(2)$$

(2) Cross-correlation :

The cross-correlation between $x(n)$ and $y(n)$ is defined as,

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\,y(n-l) \qquad ...(3)$$

Here sequence $x(n)$ is kept as it is; while $y(n)$ is delayed by 'l'. Equation (3) can also be written as,

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n+l)\,y(n) \qquad ...(4)$$

Here sequence $y(n)$ is kept as it is; while $x(n)$ is advanced by 'l'.

If we change the role of $x(n)$ and $y(n)$ then the cross-correlation is denoted by $r_{yx}(l)$.

. It is given by,

$$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n)\,x(n-l) \qquad ...(5)$$

Alternately Equation (5) can be written as,

$$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n+l)\,x(n) \qquad ...(6)$$

Solved Problems :

Prob. 1 : Compute the cross-correlation between.

$x(n) = \{1, 1, 0, 1\}$ and
$\uparrow$

$y(n) = \{4, -3, -2, 1\}$
$\uparrow$

Soln. : The cross-correlation of $x(n)$ and $y(n)$ is given by,

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\,y(n-l) \qquad ...(1)$$

Range of 'n' :

In Equation (1) sequence $x(n)$ is not changed but sequence $y(n)$ is delayed by 'l'. Thus range of summation ('n') will be same as $x(n)$.

Thus range of 'n' is from -2 to 1.

Putting these values in Equation (1) we get,

$$r_{xy}(l) = \sum_{n=-2}^{1} x(n)\,y(n-l) \qquad ...(2)$$

Now we will decide range of 'l' as follows :

Range of 'l' :

Observe the given sequences. Here maximum value of $y(n)$ is $y(1)$. So in Equation (2) $y(n-l)$ should have maximum value equals to $y(1)$. If we get $y(n-l) = y(2)$ then, result of summation will be zero.

Now starting value of 'n' in summation is $n = -2$.

Thus $y(n-l) = y(1)$ maximum $\Rightarrow n - l = 1$ maximum that means,

$$-2 - l = 1 \Rightarrow l = -2 - 1 = -3.$$

So we will start from $l = -3$

Now from the given sequence; lowest value of $y(n)$ is $y(-2)$. Thus from Equation (2), we have $y(n-l) = y(-2)$. Now according to Equation (2) we should stop the summation at n = 1.

Thus $y(n-l) = y(-2) \Rightarrow 1-l = -2 \Rightarrow l = 1+2 = 3$

Thus we will stop at $l = +3$.

So range of l is from -3 to $+3$.

Calculation of $r_{xy}(-3)$:

Putting $l = -3$ in Equation (2) we get,

$$r_{xy}(-3) = \sum_{n=-2}^{1} x(n)\,y(n+3) \qquad ...(3)$$

Expanding the summation,

$$r_{xy}(-3) = x(-2)y(-2+3)+x(-1)y(-1+3)+x(0)y(0+3)+x(1)y(1+3)$$

$$\therefore\ r_{xy}(-3) = x(-2)y(1)+x(-1)y(2)+x(0)y(3)+x(1)y(4) \qquad ...(4)$$

Now we have given sequences,

$$x(n) = \{1,\ 1,\ 0,\ 1\} \quad \text{and} \quad y(n) = \{4,\ -3,\ -2,\ 1\}$$

($\uparrow$ at $x(-2)$; $\uparrow$ at $y(-2)$)

$x(-2) = 1$ and $y(-2) = 4$

$x(-1) = 1$ and $y(-1) = -3$

$x(0) = 0$ and $y(0) = -2$

$x(1) = 1$ and $y(1) = 1$

Putting these values in Equation (4) we get,

$$r_{xy}(-3) = (1\times1)+(1\times0)+(0\times0)+(1\times0) = 1+0+0+0$$

$$\therefore\ r_{xy}(-3) = 1$$

Calculation of $r_{xy}(-2)$:

Putting $l = -2$ in Equation (2) we get,

$$r_{xy}(-2) = \sum_{n=-2}^{1} x(n)\,y(n+2) \qquad ...(5)$$

Expanding the summation,

$$r_{xy}(-2) = x(-2)y(-2+2)+x(-1)y(-1+2)+x(0)y(2)+x(1)y(1+2)$$

$$\therefore\ r_{xy}(-2) = x(-2)y(0)+x(-1)y(1)+x(0)y(2)+x(1)y(3)$$

$$\therefore\ r_{xy}(-2) = (1\times(-2))+(1\times1)+(0\times0)+(1\times0) = -2+1$$

$$\therefore\ r_{xy}(-2) = -1$$

Calculation of $r_{xy}(-1)$:

Putting $l = -1$ in Equation (2) we get,

$$r_{xy}(-1) = \sum_{n=-2}^{1} x(n)\,y(n+1) \qquad ...(6)$$

Expanding the summation,

$$r_{xy}(-1) = x(-2)\,y(-2+1)+x(-1)\,y(-1+1)+x(0)\,y(0+1)+x(1)\,y(1+1)$$

$$\therefore\ r_{xy}(-1) = x(-2)\,y(-1)+x(-1)\,y(0)+x(0)\,y(1)+x(1)\,y(2)$$

$$\therefore\ r_{xy}(-1) = [1\times(-3)]+[1\times(-2)]+[0\times 1]+[1\times 0] = -3-2$$

$$\therefore\ r_{xy}(-1) = -5$$

Calculation of $r_{xy}(0)$:

Putting $l = 0$ in Equation (2) we get,

$$r_{xy}(0) = \sum_{n=-2}^{1} x(n)\,y(n) \qquad ...(7)$$

Expanding the summation,

$$\therefore\ r_{xy}(0) = x(-2)\,y(-2)+x(-1)\,y(-1)+x(0)\,y(0)+x(1)\,y(1)$$

$$= [1\times 4]+[1\times(-3)]+[0\times(-2)]+[1\times 1] = 4-3+0+1$$

$$\therefore\ r_{xy}(0) = 2$$

Calculation of $r_{xy}(1)$:

Putting $l = 1$ in Equation (2) we get,

$$r_{xy}(1) = \sum_{n=-2}^{1} x(n)\,y(n-1) \qquad ...(8)$$

Expanding the summation,

$$r_{xy}(1) = x(-2)\,y(-2-1)+x(-1)\,y(-1-1)+x(0)\,y(0-1)+x(1)\,y(1-1)$$

$$= x(-2)\,y(-3)+x(-1)\,y(-2)+x(0)\,y(-1)+x(1)\,y(0)$$

$$= [1\times 0]+[1\times 4]+[0\times(-3)]+[1\times(-2)] = 0+4+0-2$$

$$\therefore\ r_{xy}(1) = 2$$

Calculation of $r_{xy}(2)$:

Putting $l = 2$ in Equation (2) we get,

$$r_{xy}(2) = \sum_{n=-2}^{1} x(n)\,y(n-2) \qquad ...(9)$$

Expanding the summation,

$$r_{xy}(2) = x(-2)y(-2-2)+x(-1)y(-1-2)+x(0)y(0-2)+x(1)y(1-2)$$

$$= x(-2)y(-4)+x(-1)y(-3)+x(0)y(-2)+x(1)y(-1)$$

$$= [1\times0]+[1\times0]+[0\times4]+[1\times(-3)] = 0+0+0-3$$

$$\therefore\ r_{xy}(2) = -3$$

Calculation of $r_{xy}(3)$:

Putting $l = 3$ in Equation (2) we get,

$$r_{xy}(3) = \sum_{n=-2}^{1} x(n)y(n-3) \qquad ...(10)$$

Expanding the summation,

$$r_{xy}(3) = x(-2)y(-2-3)+x(-1)y(-1-3)+x(0)y(0-3)+x(1)y(1-3)$$

$$= x(-2)y(-5)+x(-1)y(-4)+x(0)y(-3)+x(1)y(-2)$$

$$= [1\times0]+[1\times0]+[0\times0]+[1\times4] = 0+0+0+4$$

$$\therefore\ r_{xy}(3) = 4$$

Now the final sequence is written as,

$$r_{xy}(l) = \{ r_{xy}(-3), r_{xy}(-2), r_{xy}(-1), r_{xy}(0), r_{xy}(1), r_{xy}(2), r_{xy}(3) \}$$

$$\therefore\ r_{xy}(l) = \{ 1, -1, -5, \underset{\uparrow}{2}, 2, -3, 4 \}$$

Note that arrow is marked at the value of $r_{xy}(0)$.

This is a tedious method. We will solve some examples using simple method.

Simple method to calculate correlation :

We will use one of the important properties of correlation. It is as follows :

$$r_{xy}(l) = x(n) * y(-n) \qquad ...(11)$$

Observe Equation (11). The L.H.S. indicates cross correlation of $x(n)$ and $y(n)$. R.H.S. term indicates the convolution ('*') operation. It is the convolution of $x(n)$ and $y(-n)$. Here $y(-n)$ is the folded version of $y(n)$.

So this equation implies :

(i) Take signal $x(n)$ as it is.

(ii) Fold sequence $y(n)$ to obtain $y(-n)$

(iii) Obtain the convolution of $x(n)$ and $y(-n)$

(iv) The result of convolution will be same as the correlation.

Let us solve the same example using the method. Given sequences are,

$x(n) = \{1, 1, 0, 1\}$ and $y(n) = \{4, -3, -2, 1\}$

↑ ↑

Step I : To obtain $y(-n)$; write down sequence $y(n)$ in the reverse order. Here arrow is marked at sample (-2). Do not change the arrow.

Thus $y(-n) = \{1, -2, -3, 4\}$

↑

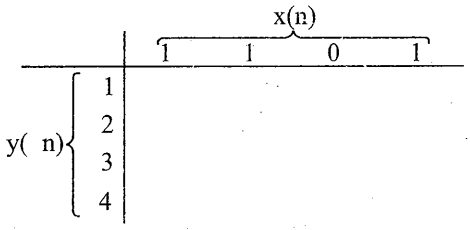
Step II : We will decide the range of n for convolution of $x(n)$ and $y(-n)$

Lowest index 'l' $\Rightarrow -2+(-1) = -3$

Highest index 'l' $\Rightarrow 1 + 2 = 3$

Thus output sequence will be from -3 to $+3$.

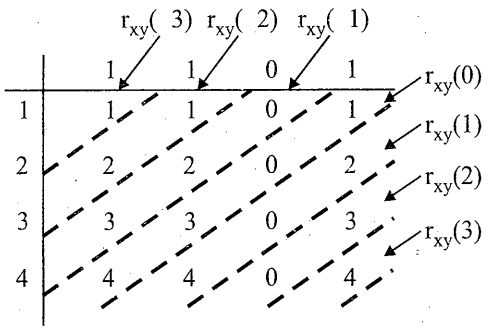
Step III : The convolution of $x(n)$ and $y(-n)$ using tabulation method is shown in Fig. A-72.



(a)

	1	1	0	1
1	(1×1)	(1×1)	(1×0)	(1×1)
−2	(−2×1)	(−2×1)	(−2×0)	(−2×1)
−3	(−3×1)	(−3×1)	(−3×0)	(−3×1)
4	(4×1)	(4×1)	(4×0)	(4×1)

(b)



(c)

Fig. A-72 : Convolution of $x(n)$ and $y(-n)$

$\therefore \quad r_{xy}(-3) = 1 \qquad r_{xy}(-2) = -2+1 = -1$

$r_{xy}(-1) = -3-2+0 = -5 \qquad r_{xy}(0) = 4-3+0+1 = 2$

$r_{xy}(1) = 4+0-2 = 2 \qquad r_{xy}(2) = 0-3 = -3$

and $r_{xy}(3) = 4$

Thus we get $r_{xy}(l) = x(n) * y(-n) = \{1, -1, -5, 2, 2, -3, 4\}$
$\uparrow$

This is the same sequence, we obtained earlier using definition of correlation.

Prob. 2 : Determine the auto-correlation of the following signals :

(i) $x(n) = \{1, 2, 1, 1\}$ (ii) $y(n) = \{1, 1, 2, 1\}$

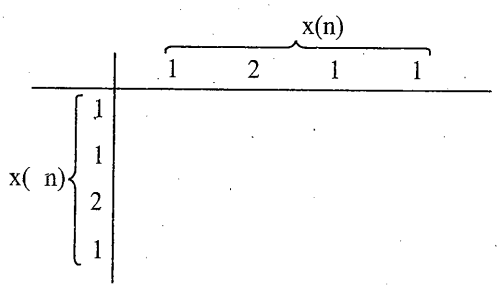
What is your conclusion ?

Soln. : Auto-correlation means correlation of sequence with itself.

(i) Thus $r_{xx}(l) = x(n) * x(-n)$

Here $x(n) = \{1, 2, 1, 1\}$ and $x(-n) = \{1, 1, 2, 1\}$

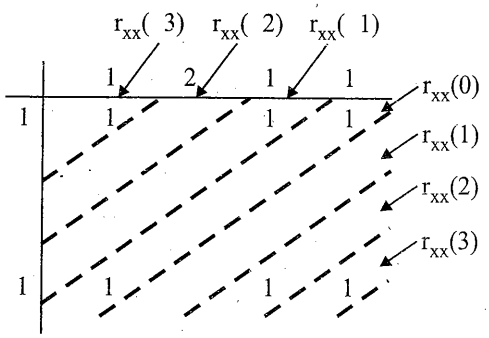
The convolution $x(n) * x(-n)$ is shown in Fig. A-73.



(a)

	1	2	1	1
1	(1×1)	(1×2)	(1×1)	(1×1)
1	(1×1)	(1×2)	(1×1)	(1×1)
2	(2×1)	(2×2)	(2×1)	(2×1)
1	(1×1)	(1×2)	(1×1)	(1×1)

(b)



(c)

Fig. A-73 : Convolution of $x(n) * x(-n)$

Here the range of output sequence is,

Lowest value = $0 + (-3) = -3$

Highest value = $3 + 0 = 3$

Thus output sequence will be from -3 to $+3$

$$\therefore\ r_{xx}(-3) = 1 \qquad r_{xx}(-2) = 1+2 = 3$$

$$r_{xx}(-1) = 2+2+1 = 5 \qquad r_{xx}(0) = 1+4+1+1 = 7$$

$$r_{xx}(1) = 2+2+1 = 5 \qquad r_{xx}(2) = 2+1 = 3$$

$$r_{xx}(3) = 1$$

$$\therefore\ r_{xx}(l) = \{ r_{xx}(-3), r_{xx}(-2), r_{xx}(-1), r_{xx}(0), r_{xx}(1), r_{xx}(2), r_{xx}(3) \}$$

$$\therefore\ r_{xx}(l) = 1, 3, 5, 7, 5, 3, 1$$

(ii) Here $y(n) = \{1, 1, 2, 1\}$ Thus $y(-n) = \{1, 2, 1, 1\}$

$$\therefore\ r_{yy}(l) = y(n) * y(-n)$$

Observe the given sequences. In part (i) we have obtained the convolution of same sequences. Thus the result of this convolution will be same as the previous part.

But the position of arrow in the sequences is changed. We will check the range of output sequence. It is,

$$\text{Lowest value} = 0+(-3) = -3$$

$$\text{Highest value} = 0+3 = 3.$$

Thus the position of arrow in the output sequence will be same.

$$\therefore\ r_{yy}(l) = \{1, 3, 5, 7, 5, 3, 1\}$$

Conclusion :

The auto-correlation sequences are same. That means $r_{xx}(l) = r_{yy}(l)$.

This is because in both convolutions the range of output sequence is same. And the magnitude of two sequences in both cases is same.

Properties of correlation :

Some important properties of auto-correlation and cross-correlation are as follows :

(1) The result of auto-correlation is maximum when signal matches with itself and there is no phase shifting.

(2) Auto-correlation is an even function.

$$\therefore\ r_{xx}(l) = r_{xx}(-l)$$

(3) The cross-correlation is not commutative

That means $r_{xy}(l) = r_{yx}(-l)$

(4) The correlation is equation to convolution of one sequence with the folded version of other sequence.

That means $r_{xx}(l) = x(n) * x(-n)$

and $r_{xy}(l) = x(n) * y(-n)$

(5) The normalized auto-correlation is,

$$\rho_{xx}(l) = \frac{r_{xx}(l)}{r_{xx}(0)}$$

and normalized cross-correlation is,

$$\rho_{xy}(l) = \frac{r_{xy}(l)}{\sqrt{r_{xx}(0)\, r_{yy}(0)}}$$

Theory Questions

Q. 1 Define a linear shift-invariant discrete time system. Show how this system can be completely characterised by its unit sample response.

Q. 2 Define a discrete time system. What are the properties of a discrete time system ?

Q. 3 Define causality of a system. Show that for LSI system, causality implies that the unit sample response h (n) is zero for $n < 0$.

Q. 4 Explain clearly : Linear convolution.

Q. 5 Write note on : Quantization error.

Q. 6 Prove that LTI system is stable if and only if it's impulse response is absolutely summable.

Q. 7 Show that the output of an LTI system can be expressed in terms of its unit step response $\delta(n)$.

Q. 8 For a causal LTI system prove that impulse response $h(n) = 0$ for $n < 0$.

Q. 9 Explain the following properties of DT-LTI system with suitable examples :

(i) Time invariance

(ii) Stability

(iii) Linearity

Q. 10 Prove that a DT-LTI system is casual if $h(n) = 0$ for $n < 0$.

Unsolved Numericals

Q. 1 With suitable examples define the periodic and non-periodic signals.

Let $x_1(t) = \sin(2\pi f_1 t)$ and $x_2(t) = \cos(8\pi f_2 t)$ be the two periodic signals, where f_1 and f_2 are two arbitrary constants. Determine the condition for which the signal $x(t) = x_1(t) + x_2(t)$ should also be a periodic signal.'

Ans. : $\frac{T_1}{T_2} = \frac{4 f_2}{f_1}$

Q. 2 Find whether the following signal is periodic, if yes find periodicity.

$x(t) = 2\cos t + 3\cos\frac{t}{3}$

Ans. : Periodic, $T = 2T_1$

Q. 3 Consider a sequence $x(n)$

$x(n) = \cos(\omega_0 n)$

Show that it is periodic if $\omega_0 = 2\pi\frac{N_1}{N_2}$ where N_1 and N_2 are integers.

Q. 4 Determine whether or not the signals below are periodic and for each signal that is periodic, determine the fundamental period :

(i) $x(n) = R_e\left\{e^{jn\pi/12}\right\} + I_m\left\{e^{jn\pi/18}\right\}$

(ii) $x(n) = \sin(\pi + 0.2n)$ (iii) $x(n) = e^{\frac{j\pi}{16}n}\cos(n\pi/17)$

Ans. : (i) Periodic, N = 12

(ii) Nonperiodic

(iii) Periodic, N = 34

Q. 5 Consider a discrete-time system with input $x(n)$ and output $y(n)$. The input-output relationship for this system is :

$y(n) = x(n)x(n-2)$

(i) Is the system memory less ?

(ii) Determine the output of the system when the input is $A\delta[n]$, where A is any real or complex number.

Ans. : (i) Memoryless

(ii) $y[n] = 0$ if $\delta[n]$ and $\delta[n-2]$ are not 1 simultaneously.

Q. 6 Determine if the systems described by the following input-output equations are causal or noncausal.

(i) $y(n) = x(n) - x(n-1)$

(ii) $y(n) = x(n)^2$

(iii) $y(n) = x(-n)$

Ans. : (i) Causal

(ii) Noncausal

(iii) Causal

Q. 7 Determine if the following system described by,

$y(t) = \sin[x(t+2)]$;

is memoryless, causal, linear, time invariant, stable.

Ans. : memoryless, noncausal, time invarient, nonlinear, stable.

Q. 8 From the given signal $x(t)$ as shown in Fig. A-74(a) plot $x(1 - t/2)$.

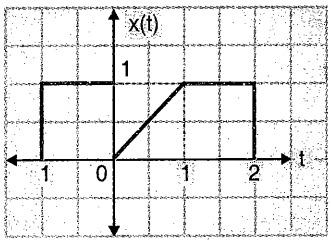


Fig. A-74(a)

Ans. :

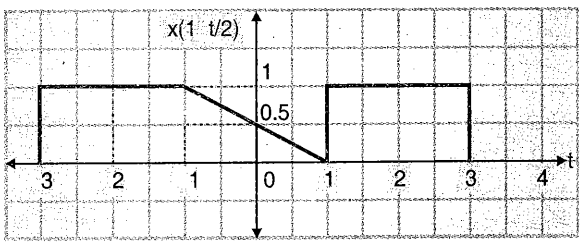


Fig. A-74(b)

Q. 9 Show that a system with excitation $x(t)$ and response $y(t)$ described by $y(t) = x(t/2)$ is linear, time varient, non-causal.

Q. 10 Find whether the following signals are energy or power signals and find the value :

(i) $e^{-at}u(t)$

(ii) $u[n]$

Ans. : (i) $E = \frac{1}{2a}$

(ii) $P = \frac{1}{2}$

Q. 11 If $x_1(n) = \delta(n+1) + 2\delta(n) + \delta(n-1)$ and $x_2(n) = 3\delta(n+1) + 5\delta(n) + 3\delta(n-1)$.

Determine $x(n)$, where $x(n)$ is the convolution of $x_1(n)$ and $x_2(n)$.

Ans. : $x(n) = \{3, 11, \underset{\uparrow}{16}, 11, 3\}$

Q. 12 Let $x_1(n) = \cos\left(2\pi\frac{n}{3}\right), 0 \le n \le 10$

$= 0$ otherwise

$x_2(n) = 1, 0 \le n \le 10$

$= 0$ otherwise

Obtain $x_1(n) * x_2(n)$

Ans. : $\{0, 0.5, -0.5, 0, 0.5, -0.5, 0, 0.5, -0.5\}$

❑❑❑

Z-Transform

Section No.	Name of the Topic
1.1	Introduction
1.2	Z Transform
1.3	Properties of Z-transform

• Code : DSP2

1.1 Introduction :

Fourier transform provides a valuable technique for frequency domain analysis and design of continuous time signals and LTI systems. While the Z-transform provides a valuable technique for analysis and design of discrete time signals and discrete time LTI systems.

The Z-transform has real and imaginary parts like fourier transform. A plot of imaginary part versus real part is called as Z-plane or complex Z-plane. The poles and zeros of discrete time system are plotted in the complex Z-plane. The pole-zero plot is main characteristic of discrete time LTI systems. We can also check the stability of system using pole-zero plot.

1.1.1 Advantages of Z-transform :

The importance of Z-transform is as follows :

1. Discrete time signals and LTI systems can be completely characterized using Z-transform.
2. The stability of LTI system can be determined using Z-transform.
3. Mathematical calculations are reduced using Z-transform. For example convolution operation is transformed into simple multiplication operation.
4. By calculating Z-transform of given signal, DFT and FT can be determined.
5. Entire family of digital filters can be obtained from one proto-type design using Z-transform.
6. The solution of differential equations can be simplified using Z-transform.

1.2 Z-Transform :

There are two types of Z-transform :

1. Single sided Z-transform
2. Double sided Z-transform.

1. Single sided Z-transform :

Definition : A single sided Z-transform of discrete time signal $x(n)$ is defined as,

$$X(Z) = \sum_{n=0}^{\infty} x(n) Z^{-n} \qquad \ldots(1)$$

Here "Z" is a complex variable. In Equation (1), limits of summation are from 0 to ∞. So while expanding the summation we will put only positive values of n (from $n = 0$ to $n = \infty$). So this is single sided or one-sided Z-transform.

2. Double sided Z-transform :

Definition : A double sided Z-transform of discrete time signal $x(n)$ is defined as,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \qquad \ldots(2)$$

While expanding the summation, we will put both positive and negative values of 'n'. Thus this is double-sided Z- transform.

What is Z-domain ?

Observe Equations (1) and (2). Here we are transforming discrete time sequence $x(n)$ into $X(Z)$. We know that discrete time sequence $x(n)$ is drawn by plotting amplitude versus 'n'. This is called as discrete domain. In case of Z-transform we are plotting real part of Z on X-axis and imaginary part of Z on Y-axis as shown in Fig. U-1. This is called as Z-domain.

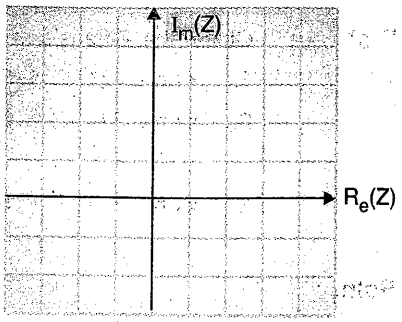


Fig. U-1 : Z-domain

How to denote Z-transform ?

The relationship between $x(n)$ and $X(Z)$ is indicated as follows :

$$x(n) \overset{Z}{\longleftrightarrow} X(Z) \qquad ...(3)$$

Note that $X(Z)$ is 'Z' transform of $x(n)$. Always, when Z-transform of sequence is obtained then it is denoted by capital letter, 'X'(Z). Here the arrow is bidirectional. This is because we can also obtain $x(n)$ from $X(Z)$ using inverse Z-transform.

The 'Z' transform of $x(n)$ is also denoted as,

$$X(Z) = \mathbf{Z}\{x(n)\} \qquad ...(4)$$

Here $x(n)$ and $X(Z)$ are called as Z-transform pairs.

1.2.1 Region of Convergence (ROC) :

In case of Z-transform; the limits of summation are from $n = -\infty$ to $n = \infty$. So if we will expand this summation then we will get an infinite power series. This infinite power series (that means Z-transform) will exist only for those values of Z for which the series attains a finite value. (That means the series converges).

Definition of ROC :

The region of convergence (ROC) of $X(Z)$ is set for all the values of Z for which $X(Z)$ attains a finite value. Everytime when we find the Z-transform, we must indicate its ROC.

Significance of ROC :

1. ROC will decide whether a system (filter) is stable or unstable.
2. ROC also determines the type of sequence that means.
   (i) Causal or non-causal
   (ii) Finite or infinite.

Solved Problems :

Prob. 1 : Obtain the Z-transform of following finite duration sequences.

1. $x(n) = \{1, 2, 4, 5, 0, 7\}$
2. $x(n) = \{\underset{\uparrow}{1}, 2, 4, 5, 0, 7\}$
3. $x(n) = \{1, 2, 4, \underset{\uparrow}{5}, 0, 7\}$

Soln. :

1. Here arrow is not mentioned. So by default it is at first position.

$$\therefore \quad x(n) = \{\underset{\uparrow}{\mathbf{1}}, \mathbf{2}, \mathbf{4}, \mathbf{5}, \mathbf{0}, \mathbf{7}\} \qquad \ldots(1)$$

According to definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \qquad \ldots(2)$$

But $x(n)$ is present from $n = 0$ to $n = 5$. The different values are as follows :

$x(0) = 1$ $\qquad$ $x(3) = 5$

$x(1) = 2$ $\qquad$ $x(4) = 0$

$x(2) = 4$ $\qquad$ $x(5) = 7$

Thus we will change the limits of summation from $n = 0$ to $n = 5$.

$$\therefore \quad X(Z) = \sum_{n=0}^{5} x(n) Z^{-n} \qquad \ldots(3)$$

Expanding the summation we get,

$$\therefore \quad X(Z) = x(0)Z^0 + x(1)Z^{-1} + x(2)Z^{-2} + x(3)Z^{-3} + x(4)Z^{-4} + x(5)Z^{-5}$$

$$\therefore \quad X(Z) = 1.Z^0 + 2Z^{-1} + 4Z^{-2} + 5Z^{-3} + 0.Z^{-4} + 7Z^{-5}$$

But $Z^0 = 1$

$$\therefore X(Z) = 1 + \frac{2}{Z} + \frac{4}{Z^2} + \frac{5}{Z^3} + \frac{7}{Z^5} \qquad \ldots(4)$$

ROC : This is finite duration sequence. For this sequence we will check for which values of 'Z'; the value of $X(Z)$ becomes infinity. The simple method to decide the ROC is, put $Z = 0$ and $Z = \infty$ in the equation of $X(Z)$.

(i) Putting Z = 0 in Equation (4) we get,

$$X(Z) = 1+\frac{2}{0}+\frac{4}{0}+\frac{5}{0}+\frac{7}{0} = 1+\infty = \infty$$

We know that $\frac{2}{0}, \frac{4}{0}$ etc. are equal to ∞. And $1+\infty$ is again ∞. Thus Z = 0 is not valid as it results $X(Z) = \infty$.

(ii) Putting $Z = \infty$ in Equation (4) we get,

$$X(Z) = 1+\frac{2}{\infty}+\frac{4}{\infty}+\frac{5}{\infty}+\frac{7}{\infty} = 1+0 = 1$$

Here $\frac{2}{\infty}, \frac{4}{\infty} = 0$. Thus putting $Z = \infty$ we get finite value. So $Z = \infty$ is allowed.

Now ROC of given sequence is written as follows :

ROC : Entire Z-plane except $|Z| = 0$

This ROC is shown in Fig. U-2.

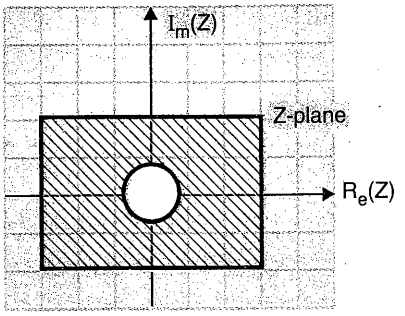


Fig. U-2 : ROC

Note : This sequence is causal, since x(n) is present only for positive values of n. Thus for a causal finite duration sequence ROC is entire Z-plane except $|Z| = 0$.

2. Given sequence is,

$$x(n) = \{1, 2, 4, 5, 0, 7\} \quad \text{...(1)}$$
$$\uparrow$$

According to definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \quad \text{...(2)}$$

Note that the range of given sequence is from n = −5 to n = 0. The different values of x(n) are :

$x(-5) = 1$ $\qquad$ $x(-2) = 5$

$x(-4) = 2$ $\qquad$ $x(-1) = 0$

$x(-3) = 4$ $\qquad$ $x(0) = 7$

We will change the limits of summation from n = −5 to n = 0.

$$\therefore \quad X(Z) = \sum_{n=-5}^{0} x(n) Z^{-n} \quad \text{...(3)}$$

Expanding the summation we get,

$$X(Z) = x(-5)Z^{+5} + x(-4)Z^{+4} + x(-3)Z^{+3} + x(-2)Z^{+2} + x(-1)Z^{+1} + x(0)Z^{0}$$

$$\therefore \quad X(Z) = 1Z^{5} + 2Z^{4} + 4Z^{3} + 5Z^{2} + 0Z^{1} + 7Z^{0}$$

$$X(Z) = Z^5 + 2Z^4 + 4Z^3 + 5Z^2 + 7 \qquad ...(4)$$

ROC : We will determine ROC by putting Z = 0 and Z = ∞.

(i) Putting Z = 0 in Equation (4) we get,

$$X(Z) = 0 + 0 + 0 + 0 + 7 = 7$$

This is a finite value. So Z = 0 is allowed.

(ii) Putting Z = ∞ in Equation (4) we get,

$$X(Z) = \infty + \infty + \infty + \infty + 7 = \infty + 7 = \infty$$

This is because ∞ + 7 etc. = ∞. So Z = ∞ is not allowed.

Thus ROC is entire Z-plane except |Z| = ∞. This ROC is shown in Fig. U-3.

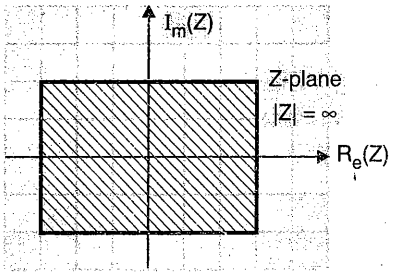


Fig. U-3 : ROC

Note : The given sequence x(n) is anticausal. This is because x(n) is present only for negative values of 'n'. Thus for anticausal finite duration sequence, ROC is entire Z-plane except |Z| = ∞

3. Given sequence is,

$$x(n) = \{1, 2, 4, 5, 0, 7\} \qquad ...(1)$$
$$\uparrow$$

According to definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \qquad ...(2)$$

But x(n) is present from n = −2 to n = 3. The different values of x(n) are as follows :

$$x(-2) = 1 \qquad x(1) = 5$$
$$x(-1) = 2 \qquad x(2) = 0$$
$$x(0) = 4 \qquad x(3) = 7$$

We will change the limits of summation from n = −2 to n = 3.

$$\therefore \quad X(Z) = \sum_{n=-2}^{n=3} x(n) Z^{-n} \qquad ...(3)$$

Expanding the summation we get,

$$X(Z) = x(-2)Z^{+2} + x(-1)Z^{+1} + x(0)Z^{0} + x(1)Z^{-1} + x(2)Z^{-2} + x(3)Z^{-3}$$

$$\therefore \quad X(Z) = 1Z^2 + 2Z^1 + 4Z^0 + 5Z^{-1} + 2Z^{-2} + 7Z^{-3}$$

$$\therefore \quad X(Z) = Z^2 + Z + 4 + \frac{5}{Z} + \frac{2}{Z^2} + \frac{7}{Z^3} \qquad \ldots(4)$$

ROC : We will determine ROC by putting Z = 0 and Z = ∞ in Equation (4).

(i) Putting Z = 0 in Equation (4) we get,

$$X(Z) = 0 + 0 + 4 + \frac{5}{0} + \frac{2}{0} + \frac{7}{0} = 4 + \infty = \infty$$

This is because $\frac{5}{0}, \frac{2}{0}$ and $\frac{7}{0} = \infty$

Thus Z = 0 is not allowed.

(ii) Putting Z = ∞ in Equation (4) we get,

$$X(Z) = \infty + \infty + 4 + \frac{5}{\infty} + \frac{2}{\infty} + \frac{7}{\infty} = \infty$$

Thus Z = ∞ is not allowed.

ROC : ROC is entire Z-plane except | Z | = 0 and | Z | = ∞.

This ROC is shown in Fig. U-4.

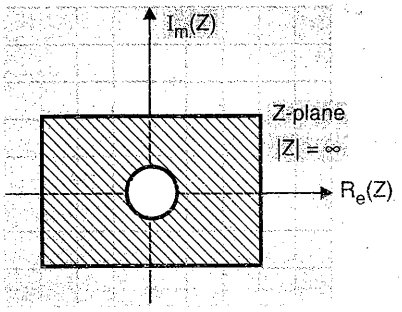


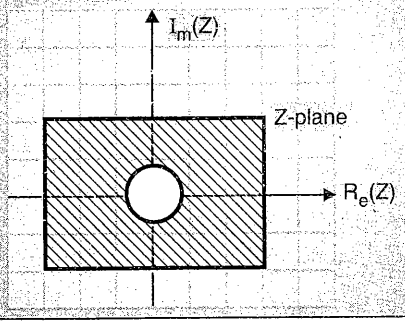
Fig. U-4 : ROC

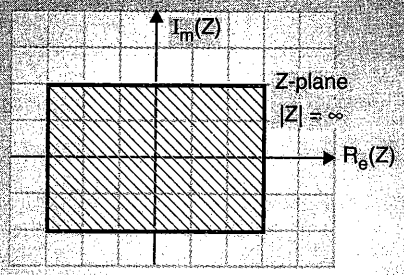
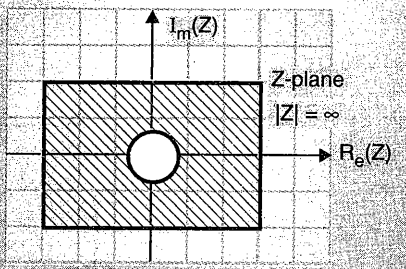
Note : The given sequence is a bothsided sequence. This is because x (n) is present for both positive and negative values of 'n'. Thus for bothsided finite duration sequence the ROC is entire Z-plane except | Z | = 0 and | Z | = ∞.

Summary of ROC :

Table U-1 shows the ROC for different finite duration sequences.

Table U-1

Sr. No.	Finite duration sequence	ROC (Diagram)	ROC (Statement)
1)	Causal sequence		Entire Z-plane except | Z | = 0

Sr. No.	Finite duration sequence	ROC (Diagram)	ROC (Statement)
2)	Anticausal sequence		Entire Z-plane except |Z| = ∞
3)	Two sided sequence		Entire Z-plane except |Z| = 0 and |Z| = ∞

1.2.2 Z-transform of Standard Sequences :

In this section we will obtain Z-transform of some standard sequences. Most of the standard sequences are having infinite duration. But there are some exceptions like unit impulse, $\delta(n)$.

1. Z-transform of unit impulse $\delta(n)$:

Unit impulse $\delta(n)$ is shown in Fig. U-5. It is given by,

$$\delta(n) = 1 \quad \text{only at } n = 0$$
$$= 0 \quad \text{otherwise}$$

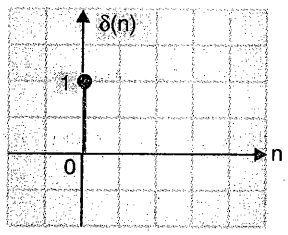


Fig. U-5 : Unit impulse $\delta(n)$

According to definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \qquad ...(1)$$

Here $x(n) = \delta(n)$

$$\therefore X(Z) = \sum_{n=-\infty}^{\infty} \delta(n) Z^{-n} \qquad ...(2)$$

Since $\delta(n)$ is present only at $n = 0$ we can directly write,

$$X(Z) = \delta(0) Z^0$$

But $\delta(0) = 1$ and $Z^0 = 1$

$$\therefore \quad X(Z) = 1 \qquad \ldots(3)$$

ROC : In Equation (3), there is no 'Z' term. So *ROC is entire Z-plane.* That means Z can have any value.

Thus the Z-transform pair is

$$\delta(n) \overset{Z}{\longleftrightarrow} 1$$

2. Z-transform of delayed unit impulse, $\delta(n-k)$:

Here $\delta(n-k)$ is a delayed unit impulse. It indicates that $\delta(n)$ is delayed by 'k' samples. It is shown in Fig. U-6.

It is given by,

$$\delta(n-k) = 1 \quad \text{only at } n = k \text{ and } k > 0$$
$$= 0 \quad \text{otherwise}$$

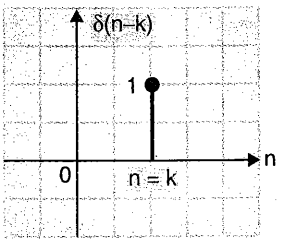


Fig. U-6 : Delayed unit impulse

According to definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \qquad \ldots(1)$$

Here $x(n) = \delta(n-k), \quad k > 0$

$$\therefore \quad X(Z) = \sum_{n=-\infty}^{\infty} \delta(n-k) Z^{-n}$$

But $\delta(n-k) = 1$ only at $n = k$. Thus we get,

$$X(Z) = 1 \cdot Z^{-k}$$

$$\therefore \quad X(Z) = Z^{-k}$$

ROC :

Here $X(Z) = Z^{-k} = \frac{1}{Z^k}$. Since k is positive, $(k > 0)$ for any value of 'Z' (except Z = 0) we will get finite value of $X(Z)$. Thus *ROC is entire Z-plane except Z = 0.* This is because if we put Z = 0 then $X(Z) = \infty$. Thus the Z-transform pair is,

$$\delta(n-k) \overset{Z}{\longleftrightarrow} Z^{-k}$$

3. Z-transform of advanced unit impulse , $\delta(n+k)$, $k > 0$:

Here $\delta(n+k)$, $k > 0$ is an advanced unit impulse. It indicates that $\delta(n)$ is advanced by '+ k' samples. It is shown in Fig. U-7.

It is given by,

$$\delta(n+k) = 1 \quad \text{only at } n = -k,\ k > 0$$
$$= 0 \quad \text{otherwise}$$

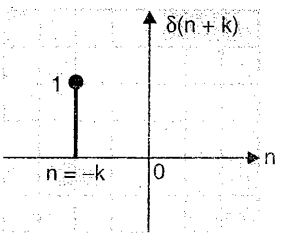


Fig. U-7 : Advanced unit impulse

According to definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

Here $x(n) = \delta(n+k)$

$$\therefore \quad X(Z) = \sum_{n=-\infty}^{\infty} \delta(n+k) Z^{-n}$$

But $\delta(n+k) = 1$ only at n = k. Thus we get,

$$X(Z) = 1 \cdot Z^k$$

$$\therefore X(Z) = Z^k$$

ROC :

Here $k > 0$. So we will get finite values of $X(Z)$ for all 'Z' except $Z = \infty$. Because for $Z = \infty$ we get, $X(Z) = (\infty)^k = \infty$. Thus *ROC is entire Z-plane except Z = ∞.*

Thus Z-transform pair is,

$$\delta(n+k) \overset{Z}{\longleftrightarrow} Z^{+k}$$

4. Z-transform of unit step, u (n) :

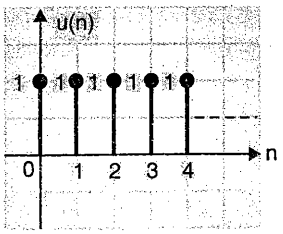


Fig. U-8 : Unit step

We know that u (n) is unit step as shown in Fig. U-8.

It is given by,

$$u(n) = 1 \quad \text{for } n \geq 0$$
$$= 0 \quad \text{otherwise}$$

According to the definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \qquad ...(1)$$

Here x (n) = u (n). Since u (n) is present from n = 0 to n = ∞ we will change the limits of summation.

$$\therefore \quad X(Z) = \sum_{n=0}^{\infty} u(n) Z^{-n} \qquad ...(2)$$

But the magnitude of u (n) is always 1.

$$\therefore \quad X(Z) = \sum_{n=0}^{\infty} 1 \cdot Z^{-n}$$

$$\therefore \quad X(Z) = \sum_{n=0}^{\infty} (Z^{-1})^n \qquad ...(3)$$

The standard equation of geometric series is,

$$\sum_{n=0}^{\infty} A^n = A^0 + A + A^2 + A^3 + ... = 1 + A + A^2 + A^3 + ... = \frac{1}{1-A} \text{ if } |A| < 1 \qquad ...(4)$$

Let $A = Z^{-1}$. Thus Equation (3) becomes,

$$X(Z) = \frac{1}{1-A} \quad \text{if} \quad |A| < 1 = \frac{1}{1-Z^{-1}} \text{ if } |Z^{-1}| < 1$$

∴ Multiplying numerator and denominator by Z we get,

$$X(Z) = \frac{Z}{Z - Z^{-1} Z} \quad \text{if} \quad |Z^{-1}| < 1$$

$$\therefore \quad X(Z) = \frac{Z}{Z-1} \quad \textbf{if } |Z^{-1}| < 1$$

In Equation (4), the condition "if $|A| < 1$" indicates that if this condition is satisfied then only the geometric series is convergent otherwise not. In the given example we get this condition as, $|Z^{-1}| < 1$. That means this is the condition of ROC.

Thus ROC : $|Z^{-1}| < 1$

that means $\left|\frac{1}{Z}\right| < 1$

$\therefore |1| < |Z| \cdot 1$

$\therefore$ ROC is $|Z| > 1$

This ROC is plotted as shown in Fig. U-9.

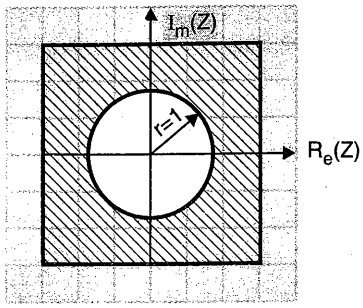


Fig. U-9 : ROC of unit step

Thus ROC is exterior part of circle having radius 1.

Thus the Z-transform pair is,

$$u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-1} \text{ if } |Z| > 1$$

Why to draw circle only ?

The given ROC is $|Z| > 1$. In case of circle having radius 1, every point outside the periphery of circle satisfies this condition ($|Z| > 1$). For other diagrams like square or triangle this condition cannot be satisfied.

Note : The unit step is an infinite sequence and as shown in Fig. U-9, the range of $u(n)$ is from $n = 0$ to $n = \infty$. That means this sequence is present only for positive values of n. So it is a causal sequence.

5. Z-transform of unit ramp :

We know that unit ramp sequence is denoted by $r(n)$. It is as shown in Fig. U-10.

Its magnitude is as follows :

at $n = 0$, $r(n) = 0$
at $n = 1$, $r(n) = 1$
at $n = 2$, $r(n) = 2$
at $n = 3$, $r(n) = 3$ ...

Thus it is expressed as,

$$r(n) = n \quad \text{for } n \geq 0$$
$$= 0 \quad \text{otherwise} \qquad ...(1)$$

Since this sequence is again causal sequence we can write

$$r(n) = n\,u(n) \qquad ...(2)$$

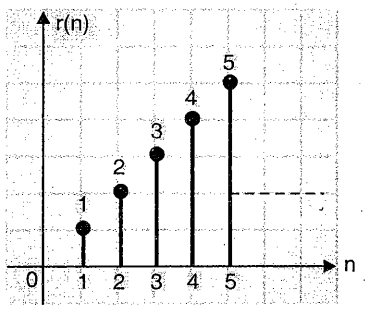


Fig. U-10 : Unit ramp

Note : Whenever any sequence is multiplied by u (n) then the magnitude of that sequence is not changed. But since u (n) is present, only for positive values of n (n ≥ 0); the result of multiplication is a causal sequence. Thus multiplication of any sequence with u (n) always results in causal sequence.

According to the definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \qquad ...(3)$$

But here $x(n) = r(n) = n\,u(n)$

$$\therefore \quad X(Z) = \sum_{n=-\infty}^{\infty} n\,u(n) Z^{-n} \qquad ...(4)$$

Just now we discussed that the multiplication by u (n) results in a causal sequence. So the limits of summation will be from n = 0 to n = ∞. Remember that since the magnitude of u (n) is 1; it will not affect the calculation. It is just used to change the limits. So now onwards we will not write u (n) in the calculations.

$$\therefore \quad X(Z) = \sum_{n=0}^{\infty} n Z^{-n} = \sum_{n=0}^{\infty} n (Z^{-1})^{n} \qquad ...(5)$$

Now use standard summation formula,

$$\sum_{n=0}^{\infty} n A^{n} = \frac{A}{(1-A)^2} \quad \text{if } |A| < 1$$

Let $A = Z^{-1}$. Thus Equation (5) becomes,

$$X(Z) = \frac{Z^{-1}}{(1-Z^{-1})^2} \quad \text{if } |Z^{-1}| < 1$$

To convert this equation into positive powers of 'Z' we will multiply numerator and denominator by Z^2.

$$\therefore \quad X(Z) = \frac{Z^2 \cdot Z^{-1}}{Z^2 (1-Z^{-1})^2} \quad \text{if } |Z^{-1}| < 1$$

$$= \frac{Z}{Z^2 (1-2Z^{-1}+Z^{-2})} \quad \text{if } |Z^{-1}| < 1$$

$$= \frac{Z}{Z^2-2Z+1} \quad \text{if } |Z^{-1}| < 1$$

$$\therefore \quad X(Z) = \frac{Z}{(Z-1)^2} \qquad \text{if } |Z^{-1}| < 1$$

ROC : Here the condition "$|Z^{-1}| < 1$" indicates the ROC.

$$\therefore \quad \text{ROC is } \left|\frac{1}{Z}\right| < 1 \Rightarrow 1 < |Z|$$

$$\therefore \quad \text{ROC is } |Z| > 1$$

Thus ROC is exterior part of circle having radius 1. This ROC is same as shown in Fig. U-9.

Thus the Z-transform pair is

$$nu(n) \overset{Z}{\longleftrightarrow} \frac{Z}{(Z-1)^2} \qquad \text{if } |Z| > 1$$

6. Z-transform of right hand exponential sequence :

As the name indicates, it is an exponential sequence present at the right hand that means only for positive values of 'n'. So it is a causal exponential sequence. Thus it is expressed as,

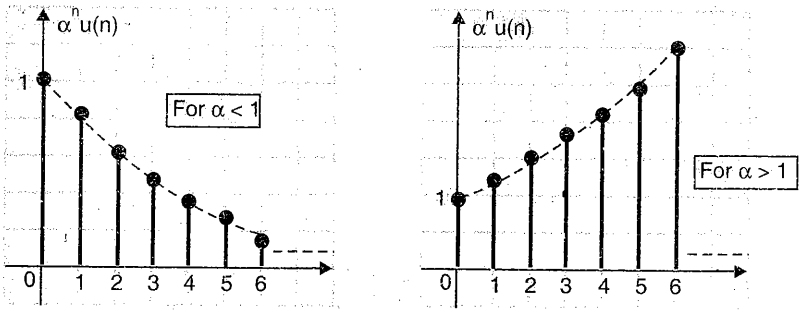
$$x(n) = \alpha^n u(n) = \alpha^n \quad \text{for } n \geq 0$$
$$= 0 \quad \text{for } n < 0 \qquad \ldots(1)$$

This exponential sequence is shown in Fig. U-11.

According to definition of Z-transform,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \qquad \ldots(2)$$

Here $x(n) = \alpha^n u(n)$. Since the sequence is multiplied by $u(n)$, the limits of summation will change. It will be $n = 0$ to $n = \infty$.



(a) (b)

Fig. U-11 : Exponential sequence $\alpha^n u(n)$

$$\therefore \quad X(Z) = \sum_{n=0}^{\infty} \alpha^n Z^{-n} \qquad \ldots(3)$$

Let us bring it in the form of A^n.

$$\therefore \quad X(Z) = \sum_{n=0}^{\infty} (\alpha Z^{-1})^n \qquad \ldots(4)$$

Now we have the equation of geometric series,

$$\sum_{n=0}^{\infty} A^n = \frac{1}{1-A} \quad \text{if } |A| < 1$$

Let $A = \alpha Z^{-1}$. Thus Equation (4) becomes,

$$X(Z) = \frac{1}{1-\alpha Z^{-1}} \quad \textbf{if } |\alpha Z^{-1}| < 1 \qquad \ldots(5)$$

To convert it into positive powers of Z; multiply numerator and denominator by Z.

$$\therefore \quad X(Z) = \frac{Z}{Z(1-\alpha Z^{-1})} \quad \text{if } |\alpha Z^{-1}| < 1$$

$$\therefore \quad X(Z) = \frac{Z}{Z-\alpha} \quad \text{if } |\alpha Z^{-1}| < 1$$

ROC :

This condition $|\alpha Z^{-1}| < 1$ indicates the ROC.

$$\therefore \text{ ROC is } |\alpha Z^{-1}| < 1 \Rightarrow \left|\frac{\alpha}{Z}\right| < 1 \Rightarrow |\alpha| < |Z|$$

Thus ROC is $|Z| > |\alpha|$.

It indicates that ROC is exterior part of circle having radius α.

This ROC is shown in Fig. U-12.

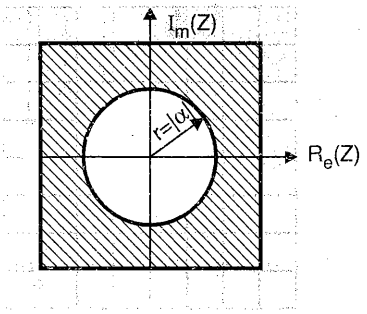


Fig. U-12 : ROC of $\alpha^n u(n)$

Thus we can write Z-transform pair as,

$$\alpha^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha} \quad \text{if } |Z| > |\alpha|$$

7. Z-transform of left handed exponential sequence :

As the name indicates; it is an exponential sequence present on left hand side; that means only at negative values of 'n'. So it is anticausal or non-causal exponential sequence. It is expressed as,

$$x(n) = -\alpha^n u(-n-1) \quad ...(1)$$

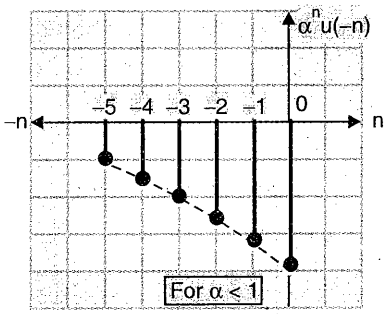
Here $u(-n)$ is folded unit step as shown in Fig. U-13(a).



(a) Folded unit step, $u(-n)$ **(b) Sequence $u(-n-1)$**

Fig. U-13

While $u(-n-1)$ indicates the advancing of $u(-n)$ by '1' sample. It is obtained by shifting Fig. U-13(a) towards left by 1 sample. This sequence is shown in Fig. U-13(b).



Note that this sequence is having magnitude equals to 1. From $n = -1$ to $n = -\infty$. So we can express it as,

$$u(-n-1) = 1 \quad \text{for } n \le -1$$
$$= 0 \quad \text{for } n \ge 0 \quad ...(2)$$

Clearly this is an anticausal sequence. So *multiplication of any sequence with $u(-n-1)$ gives anticausal sequence.* And since its magnitude is unity; it will not change the magnitude of sequence after multiplication. Now anticausal exponential sequence is shown in Fig. U-14.

Fig. U-14 : Anticausal exponential sequence

According to definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \quad ...(3)$$

But $x(n) = -\alpha^n u(-n-1)$

$$\therefore X(Z) = \sum_{n=-\infty}^{\infty} -\alpha^n u(-n-1) Z^{-n} \quad ...(4)$$

Here multiplication by $u(-n-1)$ gives anticausal sequence present from $n = -1$ to $n = -\infty$. So we will change the limits of summation. And since the magnitude of $(-\alpha^n)$ is not changed with this multiplication; we will not consider the term $u(-n-1)$ in the calculations.

$$\therefore \quad X(Z) = \sum_{n=-\infty}^{-1} (-\alpha^n) Z^{-n}$$

Here also we will have to bring it in the form A^n.

$$\therefore \quad X(Z) = \sum_{n=-\infty}^{-1} -(\alpha^{+1} Z^{-1})^n \qquad ...(5)$$

The limits of summation are negative, so to make it positive we will substitute

$$l = -n$$

$$\therefore \quad n = -l$$

$\therefore$ when $n = -\infty$, we get $-\infty = -l$; That means $l = \infty$

and when $n = -1$, we get $-1 = -l$; That means $l = 1$.

$$\therefore \quad X(Z) = \sum_{l=1}^{\infty} -(\alpha^1 Z^{-1})^{-l} \qquad ...(6)$$

Take negative sign outside and change the term inside the bracket.

$$\therefore \quad X(Z) = -\sum_{l=1}^{\infty} (\alpha^{-1} Z)^l \qquad ...(7)$$

Now put $A = \alpha^{-1} Z$

$$\text{Thus} \quad \sum_{l=1}^{\infty} A^l = A + A^2 + A^3 + A^4 + ... = A(1 + A + A^2 + A^3 + ...)$$

The term inside the bracket is standard geometric series which converges to $\frac{1}{1-A}$ if $|A| < 1$. Thus we get,

$$\sum_{l=1}^{\infty} A^l = A \times \frac{1}{1-A} \qquad \text{if} \quad |A| < 1 \qquad ...(8)$$

Thus Equation (7) becomes,

$$X(Z) = -(\alpha^{-1} Z) \times \frac{1}{1-\alpha^{-1} Z} \qquad \text{if } |\alpha^{-1} Z| < 1$$

$$= -\frac{(Z/\alpha)}{1-(Z/\alpha)} \qquad \text{if } \left|\frac{Z}{\alpha}\right| < 1$$

$$\therefore \quad X(Z) = \frac{Z/\alpha}{Z/\alpha - 1} \qquad \text{if } |Z| < \alpha$$

Multiplying numerator and denominator by α we get,

$$X(Z) = \frac{Z}{Z-\alpha} \qquad \text{if} \quad |Z| < |\alpha|$$

ROC :

Here ROC is $|Z| < |\alpha|$. That means ROC is interior part of circle having radius α. This ROC is shown in Fig. U-15.

Thus Z-transform pair is

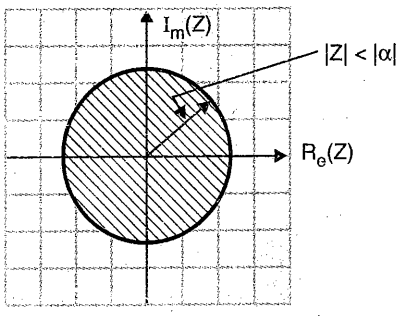


Fig. U-15 : ROC of $-\alpha^n u(-n-1)$

$$-\alpha^n u(-n-1) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha} \qquad \text{if } |Z| < |\alpha|$$

Note : This sequence is an infinite exponential sequence and it is anticausal. Thus ROC of infinite anticausal sequence is interior part of circle having radius $|\alpha|$.

Comment : Z-transforms of $\alpha^n u(n)$ and $-\alpha^n u(-n-1)$ are same but their ROC's are different.

(8) Z-transform of two sided exponential :

Consider two sided exponential sequence, which is addition of causal and non-causal exponential sequences. Thus,

$$x(n) = \alpha^n u(n) + \beta^n u(-n-1) \qquad \ldots(1)$$

This two sided exponential sequence is shown in Fig. U-16(a).

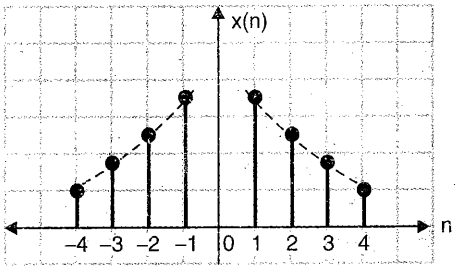


Fig. U-16(a) : Two sided exponential

Let $x_1(n) = \alpha^n u(n)$ and $x_2(n) = \beta^n u(-n-1)$.

Now $X(Z) = X_1(Z) + X_2(Z)$ ...(2)

We have obtained the Z-transform for such sequences separately.

Z-transform of $\alpha^n u(n)$:

$$\text{We have} \quad \alpha^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha} \qquad \text{if } |Z| > |\alpha|$$

$$\therefore \quad X_1(Z) = \frac{Z}{Z-\alpha} \qquad \text{if } |Z| > |\alpha| \qquad \ldots(3)$$

Z-transform of $\beta^n u(-n-1)$:

$$\text{We have} \quad -\alpha^n u(-n-1) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha} \qquad \text{if } |Z| < |\alpha|.$$

From this we can write,

$$\beta^n u(-n-1) = -\frac{Z}{Z-\beta} \qquad \text{if } |Z| < |\beta|$$

$$\therefore \quad X_2(Z) = \frac{-Z}{Z-\beta} \qquad \text{if } |Z| < |\beta| \qquad \ldots(4)$$

Putting Equations (3) and (4) in Equation (2) we get,

$$X(Z) = \frac{Z}{Z-\alpha} - \frac{Z}{Z-\beta}$$

ROC : $|Z| > |\alpha|$ and $|Z| < |\beta|$

Deciding combined ROC :

We have obtained ROC separately. Now for the total sequence we will find ROC for which $X(Z) = X_1(Z) + X_2(Z)$ will convergent. In other words there should be some overlap of two ROC's. Now this will depend on the values of α and β.

We will consider two cases as follows :

Case I : If $|\beta| < |\alpha|$

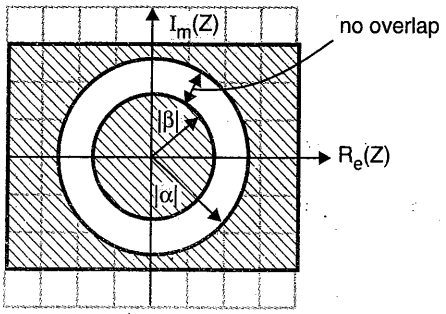
This ROC is plotted as shown in Fig. U-16(b).

We know that for $X_1(Z)$, ROC is $|Z| > |\alpha|$; which is exterior part of circle having radius α, that means exterior part of outer circle. And for $X_2(Z)$, ROC is $|Z| < |\beta|$; which is interior part of circle having radius β. That means interior part of inner circle.

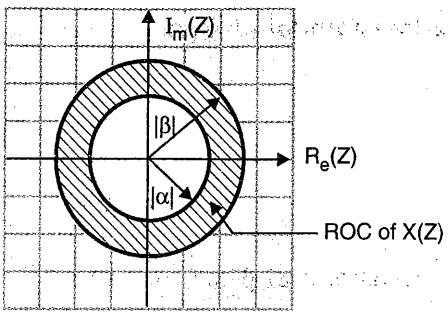
Now as shown in Fig. U-16(a), the two ROC's will not overlap. There is a gap between two ROC's. For this portion, $X(Z)$ will not convergent. That $X(Z)$ will not exist.

Case II : If $|\beta| > |\alpha|$

This ROC is plotted as shown in Fig. U-16(c).



(b) ROC for $|\beta| < |\alpha|$



(c) ROC for $|\beta| > |\alpha|$

Fig. U-16

In this case there is a ring in the Z-plane. In this case both power series converge simultaneously. That means $X(Z)$ exist.

$$\therefore \quad X(Z) = X_1(Z) + X_2(Z) = \frac{Z}{Z-\alpha} - \frac{Z}{Z-\beta}$$

$$\therefore \quad X(Z) = \frac{Z}{Z-\alpha} + \frac{Z}{\beta-Z} \qquad \text{for } |\alpha| < |Z| < |\beta|$$

ROC :

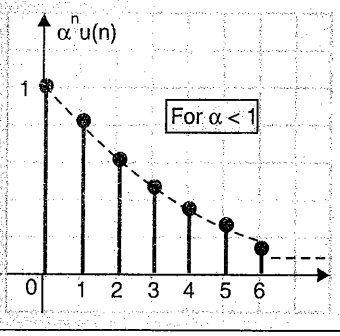
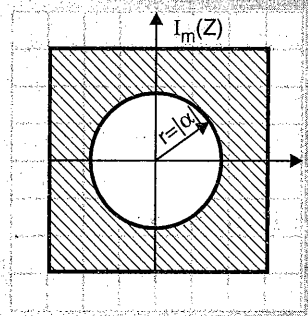
Thus combined ROC is $|\alpha| < |Z| < |\beta|$

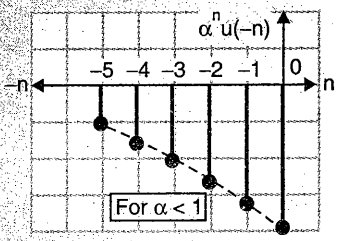
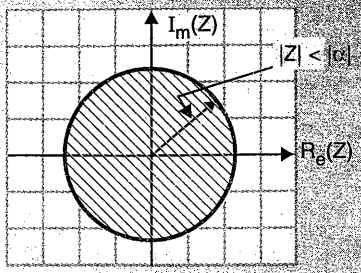
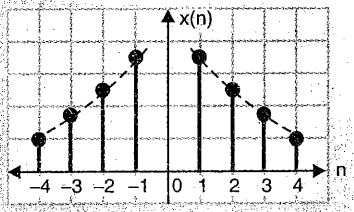
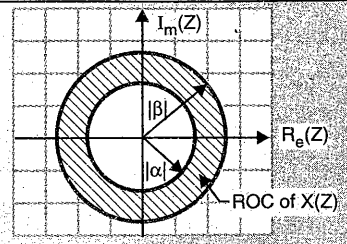
Note : When we combine many Z-transforms the ROC of the overall Z-transform is at least the intersection of the individual Z-transforms.

Summary of ROC for infinite duration sequences :

Table U-2 shows the ROC for infinite duration sequence.

Table U-2

Sr. No.	Infinite duration sequence	ROC
(1)	Causal e.g. $x(n) = \alpha^n u(n)$, 	

Sr. No.	Infinite duration sequence	ROC
(2)	Anticausal e.g. $x(n) = \alpha^n u(-n-1)$, 	
(3)	Two sided e.g. $x(n) = \alpha^n u(n) + \beta^n u(-n-1)$ 	

1.2.3 Properties of ROC :

We have solved some examples using Z transform and we have discussed ROC in each case. Based on this; the properties of ROC are summarized as follows :

(1) The ROC is a ring, whose center is at origin.
(2) ROC cannot contain any pole.
(3) If ROC of X(Z) includes unit circle then and then only the fourier transform of D.T. sequence x(n) converges.
(4) The ROC must be a connected region.
(5) For a finite duration sequence, x(n); the ROC is entire Z plane except Z = 0 and Z = ∞.
(6) If x(n) is causal then ROC is exterior part of circle of radius say 'α'.
(7) If x(n) is anticausal then ROC is interior part of circle of radius say 'α'.
(8) If x(n) is two sided sequence then ROC is intersection of two circles of radii 'α' and 'β'.

1.3 Properties of Z-transform :

In this section we will discuss different properties of Z-transform. These properties are applicable to both double sided as well as single sided Z-transform.

1.3.1 Linearity :

Statement : It states that if $x(n) = a_1 x_1(n) + a_2 x_2(n)$

and if $x_1(n) \overset{Z}{\longleftrightarrow} X_1(Z)$ and $x_2(n) \overset{Z}{\longleftrightarrow} X_2(Z)$ then,

$$x(n) \overset{Z}{\longleftrightarrow} X(Z) = a_1 X_1(Z) + a_2 (X_2(Z)$$

Where a_1 and a_2 are constants.

Proof : According to definition of Z-transform,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \quad ...(1)$$

$$\text{Here } x(n) = a_1 x_1(n) + a_2 x_2(n)$$

$$\therefore X(Z) = \sum_{n=-\infty}^{\infty} [a_1 x_1(n) + a_2 x_2(n)] Z^{-n} \quad ...(2)$$

Writing two terms separately we get,

$$X(Z) = \sum_{n=-\infty}^{\infty} a_1 x_1(n) Z^{-n} + \sum_{n=-\infty}^{\infty} a_2 x_2(n) Z^{-n}$$

Here a_1 and a_2 are constants. So we can take it outside the summation sign.

$$\therefore X(Z) = a_1 \sum_{n=-\infty}^{\infty} x_1(n) Z^{-n} + a_2 \sum_{n=-\infty}^{\infty} x_2(n) Z^{-n} \quad ...(3)$$

Comparing Equation (3) with Equation (1) we get,

$$X(Z) = a_1 X_1(Z) + a_2 X_2(Z)$$

Comment on ROC :

The combined ROC is the overlap or intersection of the individual ROC's of $X_1(Z)$ and $X_2(Z)$.

Use of this property :

We can express the given signal in the form of sum of two or more elementary signals. Then find their individual Z- transforms and add them. This will simplify the solution.

Note : Linearity property states that Z- transform of linear combination of signals is same as linear combination of Z- transform. While the ROC of X(Z) is intersection or overlap of individual region of convergence of $X_1(Z)$ and $X_2(Z)$.

Solved Problems on Linearity Property :

Prob. 1 : Determine Z transform of

$x(n) = (n+1)u(n)$

Soln. : The given function is,

$$x(n) = (n+1)u(n)$$

$$\therefore x(n) = nu(n) + u(n) \quad ...(1)$$

$$\text{Let } x_1(n) = nu(n) \text{ and } x_2(n) = u(n)$$

$$\therefore x(n) = x_1(n) + x_2(n) \quad ...(2)$$

Here $n\,u(n)$ is ramp sequence. We have standard Z transform pair,

$$n\,u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{(Z-1)^2} \qquad \text{ROC} : |Z| > 1$$

$$\therefore \quad X_1(Z) = Z\{n\,u(n)\} = \frac{Z}{(Z-1)^2}, \; |Z| > 1 \qquad \ldots(3)$$

Now $u(n)$ is a unit step and we have,

$$u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-1} \qquad \text{ROC} : |Z| > 1$$

$$\therefore \quad X_2(Z) = Z\{u(n)\} = \frac{Z}{Z-1} \qquad \text{ROC} : |Z| > 1 \qquad \ldots(4)$$

From Equation (2) we can write,

$$X(Z) = X_1(Z) + X_2(Z) \qquad \ldots(5)$$

Putting Equations (3) and (4) in Equation (5) we get,

$$X(Z) = \frac{Z}{(Z-1)^2} + \frac{Z}{Z-1}, \; \text{ROC} : |Z| > 1$$

Thus ROC is exterior part of unit circle as shown in Fig. U-17.

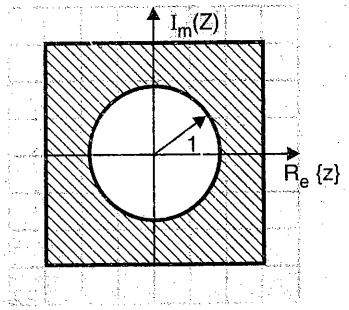


Fig. U-17

Prob. 2 : Determine Z-transform and ROC of signal :

$$x(n) = [3(4^n) - 5(3^n)]\,u(n)$$

Soln. : Let us bring it into the known form that is,

$$x(n) = a_1 x_1(n) + a_2 x_2(n) \qquad \ldots(1)$$

$$\therefore \quad x(n) = 3(4^n)\,u(n) - 5(3^n)\,u(n) \qquad \ldots(2)$$

Comparing Equations (1) and (2) we can write,

$$x_1(n) = 4^n u(n) \quad \text{and} \quad x_2(n) = 3^n u(n)$$

Thus using linearity property,

$$X(Z) = 3X_1(Z) - 5X_2(Z) \qquad \ldots(3)$$

Now recall the standard Z-transform pair,

$$\alpha^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha} \quad \text{and ROC is } |Z| > |\alpha|$$

Here $x_1(n) = 4^n u(n)$ $\therefore \alpha = 4$

$$\therefore X_1(Z) = \frac{Z}{Z-4}, \quad \text{ROC } |Z| > 4 \qquad ...(4)$$

And $x_2(n) = 3^n u(n)$ $\therefore \alpha = 3$

$$\therefore X_2(Z) = \frac{Z}{Z-3}, \quad \text{ROC } |Z| > 3 \qquad ...(5)$$

Putting Equations (4) and (5) in Equation (3) we get,

$$X(Z) = 3 \times \frac{Z}{Z-4} - 5 \times \frac{Z}{Z-3}$$

ROC :

For $X_1(Z)$, ROC is $|Z| > 4$ and for $X_2(Z)$, ROC is $|Z| > 3$. Now we have to decide the common ROC such that both $X_1(Z)$ and $X_2(Z)$ will converge. We have discussed that the combined ROC is overlap or intersection of two ROC's. The common ROC is $|Z| > 4$. It is as shown in Fig. U-18.

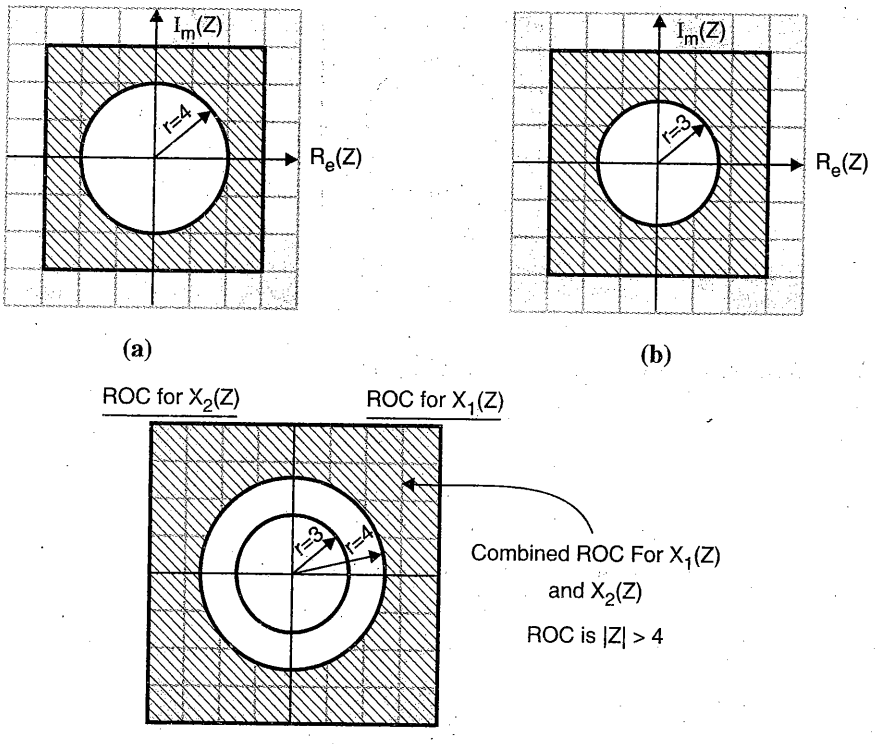


Fig. U-18 : ROC of given signal

Note : We cannot take combined ROC as $|Z| > 3$. Because for $X_2(Z)$ ROC is $|Z| > 4$. So in this case $X_2(Z)$ will not converge. But if we take ROC $|Z| > 4$ then by default it is > 3 so both sequences will converge. That means in this case $|Z| > 4$ is the common ROC for both sequences.

Prob. 3 : Determine the Z-transform and sketch the ROC of :

$$x(n) = \begin{cases} \left(\frac{1}{3}\right)^n, & n > 0, \\ \left(\frac{1}{2}\right)^{-n}, & n < 0, \end{cases}$$

Soln. : For the range $n > 0$, we will get a causal sequence and for the range $n < 0$, we will get anticausal sequence. Thus the given expression can be written as,

$$x(n) = \left(\frac{1}{3}\right)^n u(n) + \left(\frac{1}{2}\right)^{-n} u(-n-1) \quad ...(1)$$

The known form of sequence $x(n)$ is,

$$x(n) = a_1 x_1(n) + a_2 x_2(n) \quad ...(2)$$

Comparing Equations (1) and (2) we get,

$$x_1(n) = \left(\frac{1}{3}\right)^n u(n) \quad \text{and} \quad x_2(n) = \left(\frac{1}{2}\right)^{-n} u(-n-1)$$

Using linearity property,

$$X(Z) = X_1(Z) + X_2(Z) \quad ...(3)$$

We have standard Z-transform pair for causal sequence.

$$\alpha^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha} \qquad \text{ROC is } |Z| > |\alpha|$$

$$\text{Here } x_1(n) = \left(\frac{1}{3}\right)^n u(n) \qquad \therefore \alpha = \frac{1}{3}$$

$$\therefore X_1(Z) = \frac{Z}{Z-\frac{1}{3}} \qquad \text{ROC is } |Z| > \left|\frac{1}{3}\right| \quad ...(4)$$

$$\text{Now } x_2(n) = \left(\frac{1}{2}\right)^{-n} u(-n-1) \quad ...(5)$$

Recall the standard Z-transform pair for anticausal sequence,

$$-\alpha^n u(-n-1) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha} \qquad \text{ROC is } |Z| < |\alpha|$$

From this we can write,

$$+\alpha^n u(-n-1) \overset{Z}{\longleftrightarrow} -\frac{Z}{Z-\alpha} \qquad \text{ROC is } |Z| < |\alpha|$$

Now Equation (5) can be written as,

$$x_2(n) = \left[\left(\frac{1}{2}\right)^{-1}\right]^n u(-n-1) = \left[\left(\frac{1}{1/2}\right)^n\right] u(-n-1)$$

$$\therefore \quad x_2(n) = 2^n u(-n-1)$$

Thus, here $\alpha = 2$,

$$\therefore \quad X_2(Z) = \frac{Z}{Z-2} \qquad \text{ROC is } |Z| < |2| \qquad ...(6)$$

Putting Equations (4) and (5) in Equation (3) we get,

$$X(Z) = \frac{Z}{Z-\frac{1}{3}} + \frac{Z}{Z-2}$$

ROC :

For $X_1(Z)$, ROC is $|Z| > \frac{1}{3}$ and for $X_2(Z)$ ROC is $|Z| < 2$. We know that the combined ROC is overlap or intersection of two ROC's. This ROC is shown in Fig. U-19. Thus combined ROC is $\left|\frac{1}{3}\right| < |Z| < |2|$.

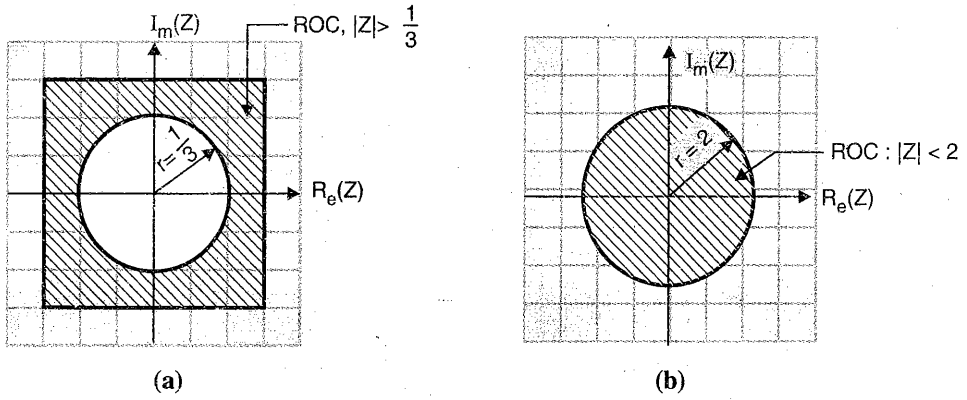
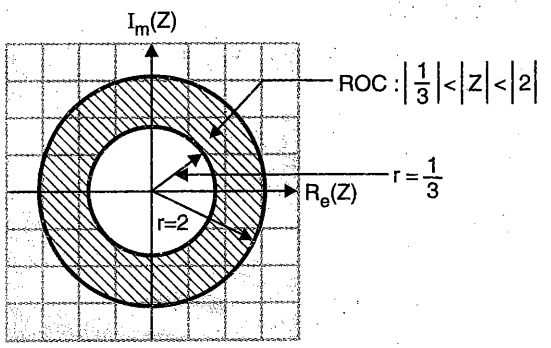


Fig. U-19



(c)

Fig. U-19 : ROC of given sequence

Prob. 4 : Find Z transform of following function along with ROC, $x(n) = a^n u(n) + \delta(n-5)$.

Soln. : The given function is,

$$x(n) = a^n u(n) + \delta(n-5) \quad ...(1)$$

$$\text{Let} \quad x_1(n) = a^n u(n) \quad ...(2)$$

$$\text{and} \quad x_2(n) = \delta(n-5) \quad ...(3)$$

$$\therefore \quad x(n) = x_1(n) + x_2(n) \quad ...(4)$$

First we will calculate Z transform of $x_1(n)$.

We have, $a^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-a}$; ROC : $|Z| > |a|$

$$\therefore \quad Z\{a^n u(n)\} = X_1(Z) = \frac{Z}{Z-a}; \quad \text{ROC } |Z| > |a|$$

Now we will calculate Z transform of $x_2(n)$.

We have standard Z transform pair,

$$\delta(n-k) \overset{Z}{\longleftrightarrow} Z^{-k}; \quad \text{ROC : Entire Z plane except Z = 0.}$$

$$\therefore \quad Z\{\delta(n-5)\} = X_2(Z) = Z^{-5}; \quad \text{ROC : Entire Z plane except Z = 0.}$$

From Equation (4) we can write,

$$X(Z) = X_1(Z) + X_2(Z)$$

$$\therefore \quad X(Z) = \frac{Z}{Z-a} + Z^{-5}$$

The combined ROC is $|Z| > |a|$.

Prob. 5 : Determine the Z-transform of :

$$x(n) = (\cos \omega_0 n)\, u(n)$$

Soln. : According to Euler's identity we have,

$$\cos\theta = \frac{1}{2} e^{j\theta} + \frac{1}{2} e^{-j\theta}$$

$$\therefore \quad \cos \omega_0 n = \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}$$

$$\therefore \quad x(n) = \frac{1}{2} e^{j\omega_0 n}\, u(n) + \frac{1}{2} e^{-j\omega_0 n}\, u(n) \qquad \ldots(1)$$

Now we have the known form,

$$x(n) = a_1 x_1(n) + a_2 x_2(n) \qquad \ldots(2)$$

Comparing Equations (1) and (2) we get,

$$x_1(n) = e^{j\omega_0 n}, \qquad \text{here} \quad a_1 = \frac{1}{2}$$

$$\text{and} \quad x_2(n) = e^{-j\omega_0 n}, \qquad \text{here} \quad a_2 = \frac{1}{2}$$

Using Linearity property we get,

$$X(Z) = a_1 X_1(Z) + a_2 X_2(Z) \qquad \ldots(3)$$

$$\text{Here} \quad x_1(n) = e^{j\omega_0 n}\, u(n) = \left(e^{j\omega_0}\right)^n u(n)$$

Recall the standard Z-transform pair,

$$\alpha^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha} \qquad \text{ROC is} \quad |Z| > |\alpha|$$

Here $\alpha = e^{j\omega_0}$ Thus we get,

$$X_1(Z) = \frac{Z}{Z - e^{j\omega_0}} \qquad \text{ROC is} \quad |Z| > |e^{j\omega_0}| \qquad \ldots(4)$$

$$\text{Now} \quad x_2(n) = e^{-j\omega_0 n}\, u(n) = \left(e^{-j\omega_0}\right)^n u(n)$$

Here $\alpha = e^{-j\omega_0}$

$$\therefore \quad X_2(Z) = \frac{Z}{Z - e^{-j\omega_0}} \qquad \text{ROC is} \quad |Z| > |e^{-j\omega_0}| \qquad \ldots(5)$$

Putting Equations (4) and (5) in Equation (3) we get,

$$X(Z) = a_1 \frac{Z}{Z - e^{j\omega_0}} + a_2 \frac{Z}{Z - e^{-j\omega_0}}$$

But $a_1 = a_2 = \frac{1}{2}$

$$X(Z) = \frac{1}{2}\left[\frac{Z}{Z - e^{j\omega_0}} + \frac{Z}{Z - e^{-j\omega_0}}\right] \quad ...(6)$$

ROC :

For $X_1(Z)$, ROC is $|Z| > |e^{j\omega_0}|$

We have $e^{j\theta} = \cos\theta + j\sin\theta$

$\therefore\ e^{j\omega_0} = \cos\omega_0 + j\sin\omega_0$

Now $|e^{j\omega_0}| = \sqrt{\cos^2\omega_0 + \sin^2\omega_0} = 1$

Thus ROC is $|Z| > 1$

Similarly for $X_2(Z)$, ROC is $|Z| > |e^{-j\omega_0 n}|$

We have, $e^{-j\omega_0} = \cos\omega_0 - j\sin\omega_0$

Thus $\left|e^{-j\omega_0}\right| = \sqrt{\cos^2\omega_0 + \sin^2\omega_0} = 1$

Thus ROC is $|Z| > 1$

So the combined ROC is $|Z| > 1$.
This is shown in Fig. U-20.

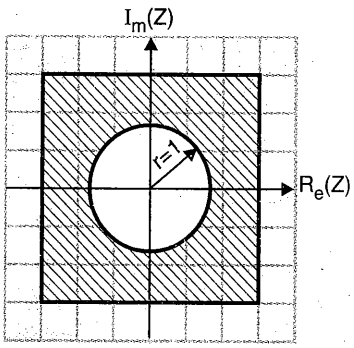


Fig. U-20 : ROC of $x(n) = (\cos\omega_0 n)\, u(n)$

This is a standard Z-transform pair.

We can further simplify Equation (6) as follows.

We have $e^{j\omega_0} = \cos\omega_0 + j\sin\omega_0$ and $e^{-j\omega_0} = \cos\omega_0 - j\sin\omega_0$

Putting these values in Equation (6) we get,

$$X(Z) = \frac{1}{2}\left[\frac{Z}{Z - (\cos\omega_0 + j\sin\omega_0)} + \frac{Z}{Z - (\cos\omega_0 - j\sin\omega_0)}\right] \quad ...(7)$$

Consider the first term inside the bracket,

$$\frac{Z}{Z - (\cos\omega_0 + j\sin\omega_0)} = \frac{Z}{Z - \cos\omega_0 - j\sin\omega_0}$$

Rationalizing the equation,

$$= \frac{Z(Z - \cos\omega_0 + j\sin\omega_0)}{(Z - \cos\omega_0 - j\sin\omega_0)(Z - \cos\omega_0 + j\sin\omega_0)}$$

$$= \frac{Z(Z - \cos\omega_0 + j\sin\omega_0)}{Z^2 - Z\cos\omega_0 + jZ\sin\omega_0 - Z\cos\omega_0 + \cos^2\omega_0 - j\sin\omega_0\cos\omega_0 - jZ\sin\omega_0 + j\sin\omega_0\cos\omega_0 - j^2\sin^2\omega_0}$$

$$= \frac{Z(Z - \cos\omega_0 + j\sin\omega_0)}{Z^2 - 2Z\cos\omega_0 + 1} \quad ...(8)$$

.... [Here $j^2 = -1$ and $\sin^2\omega_0 + \cos^2\omega_0 = 1$]

Now consider second term,

$$\frac{Z}{Z - (\cos\omega_0 - j\sin\omega_0)} = \frac{Z}{Z - \cos\omega_0 + j\sin\omega_0}$$

Rationalizing the equation.

$$= \frac{Z(Z - \cos\omega_0 - j\sin\omega_0)}{(Z - \cos\omega_0 + j\sin\omega_0)(Z - \cos\omega_0 - j\sin\omega_0)}$$

$$= \frac{Z(Z - \cos\omega_0 - j\sin\omega_0)}{Z^2 - 2Z\cos\omega_0 + 1} \quad ...(9)$$

Putting Equations (8) and (9) in Equation (7) we get,

$$X(Z) = \frac{1}{2}\left[\frac{Z(Z - \cos\omega_0 + j\sin\omega_0)}{Z^2 - 2Z\cos\omega_0 + 1} + \frac{Z(Z - \cos\omega_0 - j\sin\omega_0)}{Z^2 - 2Z\cos\omega_0 + 1}\right]$$

$$\therefore \quad X(Z) = \frac{1}{2}\left[\frac{Z^2 - Z\cos\omega_0 + jZ\sin\omega_0 + Z^2 - Z\cos\omega_0 - jZ\sin\omega_0}{Z^2 - 2Z\cos\omega_0 + 1}\right]$$

$$\therefore \quad X(Z) = \frac{1}{2}\left[\frac{2Z^2 - 2Z\cos\omega_0}{Z^2 - 2Z\cos\omega_0 + 1}\right]$$

$$\therefore \quad X(Z) = \frac{Z^2 - Z\cos\omega_0}{Z^2 - 2Z\cos\omega_0 + 1}$$

Therefore the standard Z-transform identity is,

$$(\cos \omega_0 n)\, u(n) \xleftrightarrow{Z} \frac{Z^2 - Z\cos\omega_0}{Z^2 - 2Z\cos\omega_0 + 1}, \quad \text{ROC is } |Z| > 1$$

Prob. 6 : Determine Z-transform of :

$$x(n) = \sin \omega_0 n\, u(n)$$

Soln. : We can obtain Z-transform of $\sin \omega_0 n\, u(n)$ similar to the last problem using Euler's identify. This problem can also be solved using another method, which is more simple.

We have standard Z-transform pair,

$$\alpha^n u(n) \xleftrightarrow{Z} \frac{Z}{Z-\alpha} \qquad \text{ROC is } |Z| > |\alpha| \qquad ...(1)$$

Let $\alpha = e^{j\omega n}$. Thus we can write,

$$Z\left\{e^{j\omega n} u(n)\right\} = \frac{Z}{Z - e^{j\omega}} \qquad \text{ROC : } |Z| > |e^{j\omega_0}| \qquad ...(2)$$

But we have, $e^{j\omega_0 n} = \cos \omega_0 n + j \sin \omega_0 n$

Putting this value in Equation (3) we get,

$$Z\left\{\cos \omega_0 n + j \sin \omega_0 n\right\} = \frac{Z}{Z - (\cos\omega_0 + j\sin\omega_0)} = \frac{Z}{Z - \cos\omega_0 - j\sin\omega_0}$$

$$= \frac{Z(Z - \cos\omega_0 + j\sin\omega_0)}{(Z - \cos\omega_0 - j\sin\omega_0)(Z - \cos\omega_0 + j\sin\omega_0)}$$

$$\therefore \quad Z\left\{\cos \omega_0 n + j \sin \omega_0 n\right\} = \frac{Z^2 - Z\cos\omega_0 + jZ\sin\omega_0}{Z^2 - 2Z\cos\omega_0 + 1}$$

$$\therefore \quad Z\left\{\cos \omega_0 n\right\} + Z\left\{j \sin \omega_0 n\right\} = \frac{Z^2 - Z\cos\omega_0}{Z^2 - 2Z\cos\omega_0 + 1} + j\frac{Z\sin\omega_0}{Z^2 - 2Z\cos\omega_0 + 1} \qquad ...(3)$$

Comparing imaginary part we get,

$$Z\left\{j \sin \omega_0 n\right\} = \frac{jZ\sin\omega_0}{Z^2 - 2Z\cos\omega_0 + 1}$$

$$\therefore \quad Z\left\{\sin \omega_0 n\right\} = \frac{Z\sin\omega_0}{Z^2 - 2Z\cos\omega_0 + 1} \qquad \text{ROC is } |Z| > |e^{j\omega_0}|$$

Similarly comparing real part we can obtain Z-transform of $\cos \omega_0 n$. Now ROC is $|Z| > |e^{j\omega_0}|$. But we know that $|e^{j\omega_0}| = 1$. Thus ROC is $|Z| > 1$. This is a standard Z-transform pair.

$$\therefore \quad \sin \omega_0 n\, u(n) \xleftrightarrow{Z} \frac{Z \sin \omega_0}{Z^2 - 2Z \cos \omega_0 + 1} \qquad ROC : |Z| > 1$$

1.3.2 Time Shifting :

Statement :

$$\text{If} \quad x(n) \xleftrightarrow{Z} X(Z)$$

$$\text{then} \quad x(n-k) \xleftrightarrow{Z} Z^{-k} X(Z)$$

The ROC of $Z^{-k} X(Z)$ is same as that of $X(Z)$ except for $Z = 0$ if $k > 0$ and $Z = \infty$ if $k < \infty$

Proof : According to definition of Z-transform,

$$Z\{x(n)\} = X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \qquad ...(1)$$

Then $Z\{x(n-k)\}$ can be written as,

$$\therefore \quad Z\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-k) Z^{-n} \qquad ...(2)$$

Now Z^{-n} can be written as $Z^{-n} = Z^{-(n-k)} \cdot Z^{-k}$. Thus Equation (2) becomes,

$$Z\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-k) Z^{-(n-k)} \cdot Z^{-k}$$

Since the limits of summation are in terms of 'n' we can take Z^{-k} outside the summation sign.

$$\therefore \quad Z\{x(n-k)\} = Z^{-k} \sum_{n=-\infty}^{\infty} x(n-k) Z^{-(n-k)} \qquad ...(3)$$

Now put $n - k = m$ on R.H.S. The limits will change as follows.

at $n = -\infty$, $-\infty - k = m \Rightarrow m = -\infty$

at $n = +\infty$, $\infty - k = m \Rightarrow m = \infty$

$$\therefore \quad Z\{x(n-k)\} = Z^{-k} \sum_{m=-\infty}^{\infty} x(m) Z^{-m} \qquad ...(4)$$

Compare Equation (4) with Equation (1),

$$\therefore \quad Z\{x(n-k)\} = Z^{-k} X(Z)$$

$$\therefore \; x(n-k) \overset{Z}{\longleftrightarrow} Z^{-k} X(Z)$$

Similarly it can be shown that,

$$x(n+k) \overset{Z}{\longleftrightarrow} Z^{+k} X(Z)$$

Note : Here $x(n-k)$ indicates that the sequence is shifted in time domain (delayed). Thus shifting the sequence in time domain by $(-k)$ samples corresponds to multiplication by Z^{-k} in the frequency domain.

Solved Problems on Time Shifting Property :

Prob. 1 : Find the Z-transform of :

$$x(n) = \delta(n-k)$$

Soln. :

We know that $\delta(n)$ is unit impulse and $Z\{\delta(n)\} = 1$ that means $\delta(n) \overset{Z}{\longleftrightarrow} 1$

According to time shifting property we have,

$$x(n-k) \overset{Z}{\longleftrightarrow} Z^{-k} X(Z)$$

$$\therefore \; \delta(n-k) \overset{Z}{\longleftrightarrow} Z^{-k} \cdot 1$$

ROC : Entire Z-plane except Z = 0.

Prob. 2 : Find the Z-transform of :

$$x(n) = \delta(n+2)$$

Soln. : We have,

$$\delta(n) \overset{Z}{\longleftrightarrow} 1$$

Using time shifting property we can write,

$$x(n+k) \overset{Z}{\longleftrightarrow} Z^{+k} X(Z)$$

$$\therefore \delta(n+2) \overset{Z}{\longleftrightarrow} Z^2$$

ROC : Entire Z-plane except $Z = \infty$

Prob. 3 : It is given that :

$$x_1(n) = \{1, 2, 3, 4, 0, 1\}$$
$$\uparrow$$

Using time shifting property find Z-transform of $x_2(n)$ where :

$$x_2(n) = \{1, 2, 3, 4, 0, 1\}$$
$$\uparrow$$

Soln. : Comparing the given sequences, we can conclude that $x_2(n)$ is advanced version of $x_1(n)$ and the advance is by two units.

$$\therefore \quad x_2(n) = x_1(n+2) \qquad ...(1)$$

According to time shifting property

$$x(n+k) \overset{Z}{\longleftrightarrow} Z^k X(Z)$$

$$\therefore \quad x(n+2) \overset{Z}{\longleftrightarrow} Z^2 X(Z) \qquad ...(2)$$

Now we will obtain $X_1(Z)$

Here $x_1(n) = \{1, 2, 3, 4, 0, 1\}$
$$\uparrow$$

According to definition of Z-transform.

$$X_1(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

Here range of sequence x (n) is from n = 0 to 5

$$\therefore \quad X_1(Z) = \sum_{n=0}^{5} x(n) Z^{-n}$$

Expanding the summation and putting the values of x (n) we get,

$$X_1(Z) = 1Z^0 + 2Z^{-1} + 3Z^{-2} + 4Z^{-3} + 0 + 1Z^{-5} \qquad ...(3)$$

Using Equations (1) and (2) we can write,

$$Z\{x_2(n)\} = Z^2\left[1 + 2Z^{-1} + 3Z^{-2} + 4Z^{-3} + Z^{-5}\right]$$

$$\therefore \quad X_2(Z) = Z^2 + 2Z + 3 + 4Z^{-1} + Z^{-3}$$

ROC : This is two sided finite duration sequence. Thus ROC is entire Z-plane except Z = 0 and Z = ∞.

Prob. 4 : Determine the Z-transform of the following finite duration sequence

$$x(n) = \begin{cases} 1 & \text{for } 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

Soln. : Since the magnitude of x (n) is 1, it is an unit step but having a finite duration. This signal is generated from unit step as shown in Fig. U-21.

Thus the given signal can be expressed as,

$$x(n) = u(n) - u(n-N) \qquad ...(1)$$

Here u (n) is unit step and u (n − N) is delayed unit step.

According to linearity property the Z-transform of Equation (1) can be written as,

$$X(Z) = Z\{u(n)\} - Z\{u(n-N)\} \qquad ...(2)$$

We know that Z-transform of unit step is,

$$Z\{u(n)\} = \frac{Z}{Z-1} \qquad \textbf{ROC : } |Z| > 1 \qquad ...(3)$$

Now consider second term u (n − N). It is unit step delayed by 'N' samples. According to time shifting property we can write,

$$Z\{u(n-N)\} = Z^{-N} \cdot \frac{Z}{Z-1} \text{ : ROC } |Z| > 1 \qquad ...(4)$$

Putting Equations (3) and (4) in Equation (2) we get,

$$X(Z) = \frac{Z}{Z-1} - \frac{Z^{-N} \cdot Z}{Z-1} = \frac{Z(1-Z^{-N})}{Z-1}$$

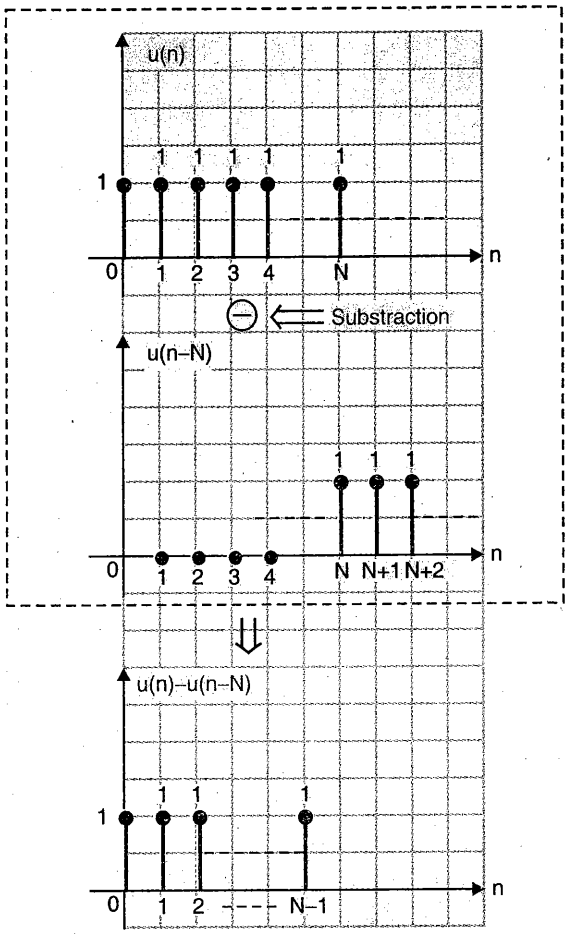


Fig. U-21 : Generation of x (n)

This is standard Z-transform pair

$$\therefore u(n) - u(n-N) \overset{Z}{\longleftrightarrow} \frac{Z(1-Z^{-N})}{Z-1}$$

Note : If combination of several signals has finite duration then ROC is exclusively dictated by the finite duration of the signal and not by the ROC of individual transform. This is finite duration unit step and it is causal. So its ROC is entire Z plane except Z = 0.

Prob. 5 : Express the Z transform of

$$y(n) = \sum_{k=-\infty}^{n} x(k) \text{ in terms of } X(Z).$$

Soln. : The given expression is,

$$y(n) = \sum_{k=-\infty}^{n} x(k) \qquad ...(1)$$

Let us expand the summation

$$\therefore \quad y(n) = x(-\infty)+\ldots+x(0)+x(1)+\ldots+x(n-1)+x(n) \qquad ...(2)$$

Replace 'n' by 'n – 1' in Equation (1)

$$\therefore \quad y(n-1) = \sum_{k=-\infty}^{n-1} x(k) \qquad ...(3)$$

Expanding Equation (3) we get,

$$y(n-1) = x(-\infty)+\ldots+x(0)+x(1)+\ldots+x(n-1) \qquad ...(4)$$

Subtracting Equation (4) from Equation (2) we get,

$$y(n)-y(n-1) = x(n) \qquad ...(5)$$

Taking Z transform of both sides,

$$Y(Z)-Z^{-1}Y(Z) = X(Z)$$

$$\therefore \quad Y(Z)[1-Z^{-1}] = X(Z)$$

$$\therefore \quad X(Z) = Y(Z)[1-Z^{-1}]$$

1.3.3 Scaling in the Z-domain (Multiplication by Exponential Sequence) :

Statement : If $x(n) \overset{Z}{\longleftrightarrow} X(Z)$ ROC : $r_1 < |Z| < r_2$

then $a^n x(n) \overset{Z}{\longleftrightarrow} X\left(\frac{Z}{a}\right)$ ROC : $|a|r_1 < |Z| < |a|r_2$

For any constant 'a' real or complex.

Proof : According to definition of Z-transform.

$$Z\{x(n)\} = X(Z) = \sum_{n=-\infty}^{\infty} x(n)Z^{-n} \qquad ...(1)$$

$$\therefore \quad Z\{a^n x(n)\} = \sum_{n=-\infty}^{\infty} a^n x(n)Z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) a^n Z^{-n} = \sum_{n=-\infty}^{\infty} x(n) (a^{-1} Z)^{-n}$$

$$\therefore \quad Z\{a^n x(n)\} = \sum_{n=-\infty}^{\infty} x(n) \left(\frac{Z}{a}\right)^{-n} \qquad \ldots(2)$$

Comparing Equations (1) and (2),

$$Z\{a^n x(n)\} = X\left(\frac{Z}{a}\right)$$

$$\therefore \quad a^n x(n) \overset{Z}{\longleftrightarrow} X\left(\frac{Z}{a}\right)$$

ROC : ROC of $X(Z)$ is $r_1 < |Z| < r_2$

To obtain ROC of $X\left(\frac{Z}{a}\right)$, replace 'Z' by $\frac{Z}{a}$.

$\therefore$ ROC of $X\left(\frac{Z}{a}\right)$: $r_1 < \left|\frac{Z}{a}\right| < r_2$

$$\therefore \quad \text{ROC of } X\left(\frac{Z}{a}\right): \quad |a| r_1 < |Z| < |a| r_2$$

Note : (1) Scaling in the time domain corresponds to shrinking or expanding the Z-plane.
(2) By taking 'a' as complex variable one can change scale of radius and ω in the Z-plane.

Solved Problems on Scaling Property :

Prob. 1 : Obtain Z-transform of $x(n) = a^n u(n)$ using scaling property.

Soln. : Here $u(n)$ is a unit step. Its Z- transform is,

$$Z\{u(n)\} = \frac{Z}{Z-1} \qquad \textbf{ROC : } |Z| > 1 \qquad \ldots(1)$$

According to linearity property we have,

$$a^n x(n) \overset{Z}{\longleftrightarrow} X\left(\frac{Z}{a}\right) \qquad \ldots(2)$$

Let $x(n) = u(n)$. Thus $Z\{a^n x(n)\} = Z\{a^n u(n)\}$ is obtained by replacing Z by $\frac{Z}{a}$ in Equation (1).

$$\therefore \quad Z\{a^n u(n)\} = \frac{Z/a}{Z/a - 1} \qquad ROC \left|\frac{Z}{a}\right| > 1$$

$$\therefore Z\{a^n u(n)\} = \frac{Z}{Z-a} \qquad ROC : |Z| > |a|$$

We have already obtained Z-transform of this sequence. Thus using scaling property, same result is obtained.

Prob. 2 : Obtain Z-transform of $x(n) = a^n \cos \omega_0 n\, u(n)$.

Soln. : We have already obtained the Z-transform of $\cos \omega_0 n\, u(n)$.

Recall that result,

$$Z\{\cos \omega_0 n \; u(n)\} = \frac{Z^2 - Z\cos\omega_0}{Z^2 - 2Z\cos\omega_0 + 1} \qquad ROC : |Z| > 1 \qquad ...(1)$$

According to scaling property we have,

$$a^n x(n) \overset{Z}{\longleftrightarrow} X\left(\frac{Z}{a}\right) \qquad ...(2)$$

Remember that whenever we use scaling property; we will have to replace Z by $\frac{Z}{a}$ in the equation of $X(Z)$. Thus replacing Z by $\frac{Z}{a}$ in Equation (1) we get,

$$Z\{a^n \cos \omega_0 n\} = \frac{\left(\frac{Z}{a}\right)^2 - \left(\frac{Z}{a}\right)\cos\omega_0}{\left(\frac{Z}{a}\right)^2 - 2\left(\frac{Z}{a}\right)\cos\omega_0 + 1} \qquad ROC : \left|\frac{Z}{a}\right| > 1$$

Multiplying and dividing numerator and denominator by a^2 we get,

$$Z\{a^n \cos \omega_0 n\} = \frac{Z^2 - aZ\cos\omega_0}{Z^2 - 2aZ\cos\omega_0 + a^2} \qquad ROC : |Z| > |a|$$

Here ROC is $|Z| > |a|$. That means it is exterior part of circle having radius r = a. This is shown in Fig. U-22.

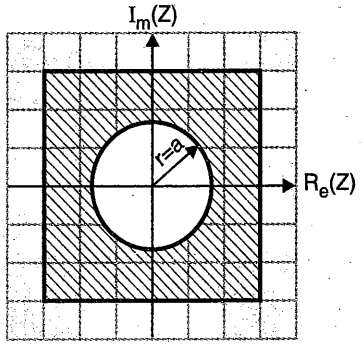


Fig. U-22 : ROC of $(a^n \cos \omega_0 n)\, u(n)$

Prob. 3 : Find Z transform of x (n) and draw its ROC,

$$x(n) = \left[(0.5)^n \sin \frac{\pi n}{4} \right] u(n)$$

Soln. : The given function is,

$$x(n) = \left[(0.5)^n \sin \frac{\pi n}{4} \right] u(n)$$

$$\therefore \quad x(n) = (0.5)^n \sin \frac{\pi n}{4}\, u(n) \qquad \ldots(1)$$

We have standard Z transform pair,

$$\sin \omega_0 n\, u(n) \overset{Z}{\longleftrightarrow} \frac{Z \sin \omega_0}{Z^2 - 2Z \cos \omega_0 + 1} \qquad \text{ROC} : |Z| > 1 \qquad \ldots(2)$$

Consider the term $\sin \frac{\pi n}{4}\, u(n)$. Here, let $\omega_0 = \frac{\pi}{4}$. Thus Equation (2) becomes,

$$\sin \frac{\pi n}{4}\, u(n) \overset{Z}{\longleftrightarrow} \frac{Z \sin \frac{\pi}{4}}{Z^2 - 2Z \cos \frac{\pi}{4} + 1}, \qquad \text{ROC} : |Z| > 1$$

$$\therefore \quad Z\left\{ \sin \frac{\pi}{4} n\, u(n) \right\} = \frac{0.707\, Z}{Z^2 - 1.414\, Z + 1}, \qquad \text{ROC} : |Z| > 1 \qquad \ldots(3)$$

According to the scaling property we have,

$$a^n u(n) \overset{Z}{\longleftrightarrow} X\left(\frac{Z}{a}\right)$$

Here a = 0.5. Thus Z transform of given function, x (n) is obtained by replacing Z by $\frac{Z}{0.5}$ that means by 2Z in Equation (3) we get,

$$\therefore \quad Z\left\{\left[(0.5)^n \sin\frac{\pi n}{4}\right] u(n)\right\} = \frac{0.707\,(2Z)}{(2Z)^2 - 1.414\,(2Z) + 1}, \qquad \text{ROC} : |2Z| > 1$$

$$\therefore \quad X(Z) = \frac{1.414\,Z}{4Z^2 - 2.828 + 1}, \quad \text{ROC} : |Z| > \frac{1}{2}$$

Thus ROC is exterior part of circle having radius $\frac{1}{2}$ as shown in Fig. U-23.

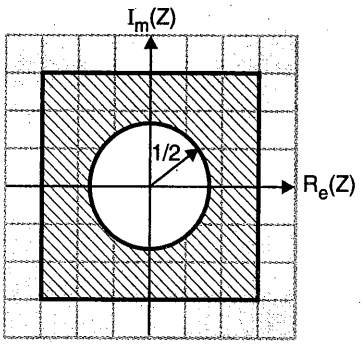


Fig. U-23

Prob. 4 : Obtain Z-transform of $x(n) = a^n \sin \omega_0 n\, u(n)$.

Soln. : We have already obtained Z-transform of $\sin \omega_0 n$. Recall that result,

$$Z\left\{\sin \omega_0 n\, u(n)\right\} = \frac{Z \sin \omega_0}{Z^2 - 2Z\cos \omega_0 + 1} \qquad \text{ROC} : |Z| > 1 \qquad \ldots(1)$$

Here we have,

$x(n) = a^n \sin \omega_0 n\, u(n)$. Thus according to scaling property, X (Z) can be obtained by replacing Z by $\frac{Z}{a}$ in Equation (1).

$$\therefore \quad Z\left\{a^n \sin \omega_0 n\, u(n)\right\} = \frac{\left(\frac{Z}{a}\right)\sin \omega_0}{\left(\frac{Z}{a}\right)^2 - 2\left(\frac{Z}{a}\right)\cos \omega_0 + 1} \qquad \text{ROC} : \left|\frac{Z}{a}\right| > 1.$$

Dividing numerator and denominator by a^2 we get,

$$Z\left\{a^n \sin \omega_0 n\, u(n)\right\} = \frac{aZ \sin \omega_0}{Z^2 - 2aZ \cos \omega_0 + a^2} \qquad ROC : |Z| > |a|$$

Prob. 5 : Find Z-transform and sketch the ROC

$x(n) = (-1)^n 2^{-n} u(n)$

Soln. : Given $x(n) = (-1)^n 2^{-n} u(n)$

$$\therefore \quad x(n) = (-1)^n \frac{1}{(2)^n} u(n)$$

$$\therefore \quad x(n) = \left(-\frac{1}{2}\right)^n u(n) \qquad ...(1)$$

Here $u(n)$ is unit step and its Z-transform is given by,

$$Z\{u(n)\} = \frac{Z}{Z-1} \qquad ROC : |Z| > 1 \qquad ...(2)$$

According to scaling property, we can write,

$$Z\left\{a^n u(n)\right\} = \frac{Z/a}{Z/a - 1} \quad : \quad ROC \left|\frac{Z}{a}\right| > 1 \qquad ...(3)$$

Observe Equation (1). Here $a = -1/2$. Putting this value in Equation (3) we get,

$$Z\left\{\left(-\frac{1}{2}\right)^n u(n)\right\} = \frac{Z/(-1/2)}{\left(\frac{Z}{-1/2}\right) - 1} \quad : \quad ROC\ |Z| > \left|\left(-\frac{1}{2}\right)\right|$$

$$= \frac{-2Z}{-2Z-1} \quad : \quad ROC\ |Z| > \left|-\frac{1}{2}\right|$$

$$\therefore \quad Z\left\{\left(-\frac{1}{2}\right)^n u(n)\right\} = \frac{2Z}{2Z+1} \quad : ROC\ |Z| > 1/2$$

Thus ROC is exterior part of circle having radius $r = 1/2$.

Prob. 6 : Determine Z-transform including ROC of the following,

$$x(n) = \left(\frac{1}{2}\right)^n \{u(n) - u(n-10)\}$$

Soln. : The given expression can be written as,

$$x(n) = \left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{2}\right)^n u(n-10) \qquad ...(1)$$

According to linearity property we have,

$$X(Z) = a_1 X_1(Z) + a_2 X_2(Z) \qquad ...(2)$$

Recall the standard Z-transform pair,

$$\alpha^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha} \qquad \textbf{ROC} : |Z| > |\alpha| \qquad ...(3)$$

Consider first term of Equation (1),

$$\text{Let,} \quad x_1(n) = \left(\frac{1}{2}\right)^n u(n)$$

Using Equation (3) we can write,

$$X_1(Z) = Z\left\{\left(\frac{1}{2}\right)^n u(n)\right\} = \frac{Z}{Z-1/2} = \frac{2Z}{2Z-1} \qquad \text{ROC} : |Z| > \frac{1}{2} \qquad ...(4)$$

Now consider second term of Equation (1),

$$\textbf{Let} \quad x_2(n) = \left(\frac{1}{2}\right)^n u(n-10) \qquad ...(5)$$

First we will obtain Z-transform of $u(n-10)$

Recall the time shifting property,

$$x(n-k) \overset{Z}{\longleftrightarrow} Z^{-K} \cdot X(Z)$$

Now we have, $$u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-1}; \qquad \text{ROC} : |Z| > 1$$

$$\therefore \; Z\{u(n-10)\} = Z^{-10}\left(\frac{Z}{Z-1}\right); \qquad \text{ROC } |Z| > 1$$

$$\therefore \; Z\{u(n-10)\} = \frac{Z^{-9}}{Z-1} \qquad ...(6)$$

According to scaling property we have,

$$Z\{a^n x(n)\} = X\left(\frac{Z}{a}\right)$$

That means we have to replace Z by $\frac{Z}{a}$.

In this case, $a = \frac{1}{2}$. Thus applying scaling property to Equation (6) we can write,

$$Z\left\{\left(\frac{1}{2}\right)^n u(n-10)\right\} = \frac{\left[\frac{Z}{(1/2)}\right]^{-9}}{\frac{Z}{1/2}-1} = \frac{(Z^{-9})\left(\frac{1}{1/2}\right)^{-9}}{\frac{Z}{1/2}-1}; \qquad \text{ROC} : \left|\frac{Z}{1/2}\right| > 1$$

$$= \frac{Z^{-9} \cdot \left(\frac{1}{2}\right)^{+9}}{2Z-1}; \qquad ROC : |Z| > \frac{1}{2}$$

$$X_2(Z) = Z\left\{\left(\frac{1}{2}\right)^n u(n-10)\right\} = \frac{\left(\frac{1}{2}\right)^9}{Z^9(2Z-1)} \qquad ROC : |Z| > \frac{1}{2} \qquad ...(7)$$

Putting Equations (4) and (7) in Equation (2) we get,

$$X(Z) = \frac{2Z}{2Z-1} - \frac{\left(\frac{1}{2}\right)^9}{Z^9(2Z-1)}; \qquad ROC : |Z| > \frac{1}{2}$$

$$\therefore \quad X(Z) = \frac{2Z \times Z^9 - \left(\frac{1}{2}\right)^9}{Z^9(2Z-1)}; \qquad ROC : |Z| > \frac{1}{2}$$

$$\therefore \quad X(Z) = \frac{2Z^{10} - \left(\frac{1}{2}\right)^9}{Z^9(2Z-1)}; \qquad ROC : |Z| > \frac{1}{2}$$

Thus ROC is exterior part of circle having radius r = 1/2.

Prob. 7 : Determine Z transform and draw ROC of the following signal $x(n) = (2)^{n+2} u(n-1)$. Is the signal causal ?

Soln. : The given signal is,

$$x(n) = (2)^{n+2} u(n-1)$$

$$\therefore \quad x(n) = 2^n \cdot 2^2 u(n-1)$$

$$\therefore \quad x(n) = 4 \cdot 2^n u(n-1) \qquad ...(1)$$

First we will obtain Z transform of $u(n-1)$. We have,

$$u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-1}, \qquad ROC : |Z| > 1$$

According to shifting property we have,

$$u(n-1) \overset{Z}{\longleftrightarrow} Z^{-1}\left[\frac{Z}{Z-1}\right], \qquad ROC : |Z| > 1$$

$$\therefore \quad Z\{u(n-1)\} = \frac{1}{Z-1}, \qquad ROC : |Z| > 1 \qquad ...(2)$$

Now according to the scaling property, we have,

$$Z\left\{a^n u(n)\right\} = X\left(\frac{Z}{a}\right)$$

Here a = 2. Thus applying scaling property to Equation (2).

$$Z\left\{2^n u(n-1)\right\} = \frac{1}{\frac{Z}{2}-1}, \qquad \text{ROC} : \left|\frac{Z}{2}\right| > 1$$

$$\therefore \quad Z\left\{2^n u(n-1)\right\} = \frac{2}{Z-2}, \qquad \text{ROC} : |Z| > 2$$

Thus for Equation (1) we can write,

$$Z\left\{4 \cdot 2^n u(n-1)\right\} = 4 \cdot \frac{2}{Z-2}, \qquad \text{ROC} : |Z| > 2$$

$$\therefore \quad X(Z) = \frac{8}{Z-2}, \qquad \text{ROC} : |Z| > 2$$

Here ROC is exterior part of circle having radius 2, as shown in Fig. U-24. Since ROC is exterior part of circle; the given sequence is causal.

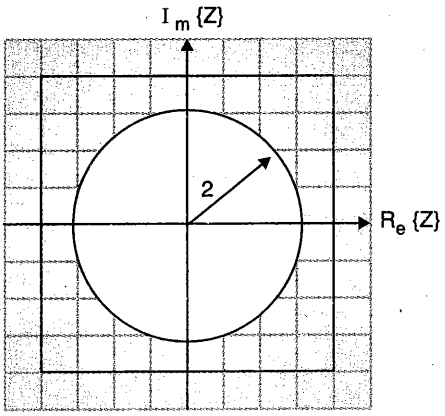


Fig. U-24

1.3.4 Time Reversal Property :

Statement : If $x(n) \overset{Z}{\longleftrightarrow} X(Z)$ ROC : $r_1 < |Z| < r_2$

Then $x(-n) \overset{Z}{\longleftrightarrow} X(Z^{-1})$ ROC : $\frac{1}{r_2} < |Z| < \frac{1}{r_1}$

Proof : According to definition of Z-transform we have,

$$Z\left\{x(n)\right\} = X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

$$\therefore \quad Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n)Z^{-n} \qquad ...(1)$$

Put $l = -n$; the limits will change as follows :
when $n = -\infty$, $l = \infty$ and when $n = \infty$, $l = -\infty$.

$$\therefore \quad Z\{x(-n)\} = \sum_{l=\infty}^{-\infty} x(l)Z^{l}$$

$$\therefore \quad Z\{x(-n)\} = \sum_{l=-\infty}^{\infty} x(l)(Z^{-1})^{-l} \qquad ...(2)$$

Comparing this equation with the definition of Z-transform we get,

$$\boxed{Z\{x(-n)\} = X(Z^{-1})} \qquad ...(3)$$

Note : Here $x(-n)$ is the folded version of $x(n)$. So $x(-n)$ is the time reversed signal. Thus the folding of signal in time domain is equivalent to replacing Z by Z^{-1} in the Z-domain. Replacing Z by Z^{-1} is called as inversion. Hence folding in time domain is equivalent to inversion in the Z-domain.

Comment on ROC : ROC of $x(n)$ is inverse of that of $x(-n)$. This means that, if Z_0 belongs to the ROC of $x(n)$ then $1/Z_0$ is in the ROC for $x(-n)$.

Given ROC is $r_1 < |Z| < r_2$. Thus ROC of $X(Z^{-1})$ is obtained by replacing Z by Z^{-1}.

Thus ROC of Z^{-1} : $r_1 < |Z^{-1}| < r_2$

$$\therefore \quad r_1 < \left|\frac{1}{Z}\right| < r_2$$

$$\therefore \quad |Z| < \frac{1}{r_1} \text{ and } |Z| > \frac{1}{r_2}$$

$$\boxed{\therefore \frac{1}{r_2} < |Z| < \frac{1}{r_1}}$$

Solved Problems on Time Reversal Property :

Prob. 1 : Obtain the Z-transform of signal $x(n) = u(-n)$.

Soln. : We have the Z-transform of unit step,

$$Z\{u(n)\} = \frac{Z}{Z-1} \qquad \text{ROC} : |Z| > 1 \qquad ..(1)$$

According to the time reversal property,

$$Z\{x(-n)\} = X(Z^{-1})$$

Thus Z-transform of $x(-n)$ is obtained by replacing Z by Z^{-1} in Equation (1).

$$\therefore Z\{u(-n)\} = \frac{Z^{-1}}{Z^{-1}-1} \qquad ROC : |Z^{-1}| > 1$$

Multiplying numerator and denominator by Z we get,

$$Z\{u(-n)\} = \frac{1}{1-Z} \qquad ROC : |Z^{-1}| > 1$$

Here ROC is $|Z^{-1}| > 1$

$\therefore \left|\frac{1}{Z}\right| > 1$ that means $|Z| < 1$.

This is the standard Z-transform pair,

$$\therefore u(-n) \overset{Z}{\longleftrightarrow} \frac{1}{1-Z} \qquad ROC : |Z| < 1$$

Thus ROC is interior part of circle having radius r = 1. ***The circle or radius = 1 is also called as unit circle***. Thus ROC is interior part of unit circle as shown in Fig. U-25.

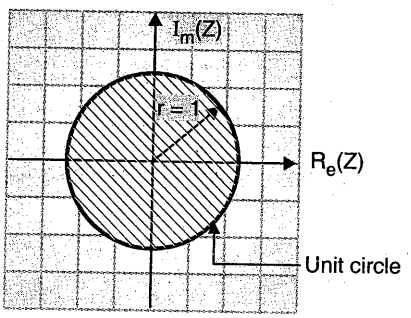


Fig. U-25 : ROC of $u(-n)$

Prob. 2 : Determine Z-transform and ROC of :

$$x(n) = \left(\frac{1}{2}\right)^n u(-n)$$

Soln. :

We have $Z\{u(-n)\} = \frac{1}{1-Z}$

ROC : $|Z| < 1$

Here the given signal is $\left(\frac{1}{2}\right)^n u(-n)$. According to scaling property we have,

$$Z\{a^n x(n)\} = X\left(\frac{Z}{a}\right)$$

That means we will have to replace Z by $\frac{Z}{a}$. In this example $a = \frac{1}{2}$.

$$\text{Thus, } Z\left\{\left(\frac{1}{2}\right)^n u(-n)\right\} = \frac{1}{1-\frac{Z}{1/2}}$$

$$\text{ROC} : \left|\frac{Z}{1/2}\right| < 1$$

$$\therefore Z\left\{\left(\frac{1}{2}\right)^n u(-n)\right\} = \frac{1}{1-2Z} \qquad \text{ROC} : |Z| < \frac{1}{2}$$

Thus ROC is interior part of circle having radius r = 1/2. This ROC is similar to Fig. U-22. Only difference is here, r = 1/2.

Prob. 3 : The Z-transform of DT signal $x(n)$ is given by $x(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z^2+4}$ ROC $|Z| > 2$. Determine Z transform and ROC of the following signals using properties of Z-transform.

(i) $2^n x(n)$ (ii) $nx(n)$ (iii) $x(-n)$ (iv) $x(n-4)$.

Soln. :

We have,

$$Z\{u(n)\} = \frac{Z}{Z-1}$$

$$\text{ROC } |Z| > 1$$

$$\text{and } Z\{u(-n)\} = \frac{Z^{-1}}{Z^{-1}-1}$$

$$\text{ROC } |Z^{-1}| > 1$$

$$= \frac{1}{1-Z}$$

$$\text{ROC} \left| \frac{1}{Z} \right| < 1$$

$$\text{i.e. } 1 < |Z|$$

$$\therefore \quad X(Z) = \frac{Z}{Z-1} + \frac{1}{1-Z}$$

$$\text{ROC} : |Z| > 1$$

$$\therefore \quad X(Z) = \frac{Z}{Z-1} - \frac{(-1)}{1-Z} = \frac{Z}{Z-1} + \frac{(-1)}{Z-1} = \frac{Z-1}{Z-1}$$

$$\therefore \quad X(Z) = 1$$

Thus ROC is entire Z plane.

Prob. 4 : Find Z-transform of following signals and comment on the results.

(i) $x(n) = a^n u(n-1)$ (ii) $x(n) = a^n u(-n-1)$

(iii) $x(n) = -(a^n) u(n-1)$ (iv) $x(n) = -(a^n) u(-n-1)$

Soln. :

(i) $\mathbf{x(n) = a^n u(n-1)}$

Here $u(n-1)$ is delayed unit step. First we will obtain $Z\{u(n-1)\}$.

Now we have,

$$Z\{u(n)\} = \frac{Z}{Z-1} \qquad \text{ROC} : |Z| > 1$$

Applying time shifting property we can write,

$$Z\{u(n-1)\} = Z^{-1} \times \frac{Z}{Z-1} = \frac{1}{Z-1} \qquad \text{ROC} : |Z| > 1$$

Now we have to obtain $Z\{a^n u(n-1)\}$.

Applying scaling property we get,

$$Z\{a^n u(n-1)\} = \frac{1}{\frac{Z}{a} - 1} \qquad \text{ROC} : \left| \frac{Z}{a} \right| > 1$$

$$\therefore Z\{a^n u(n-1)\} = \frac{a}{Z-a} \qquad \text{ROC} : |Z| > a \qquad \ldots(1)$$

Thus ROC is exterior part of circle having radius, r = a.

(ii) $\mathbf{x(n) = a^n u(-n-1)}$

First we will obtain $Z\{u(-n-1)\}$. Here $u(-n)$ is folded unit step. The Z- transform of folded unit step is given by,

$$Z\{u(-n)\} = \frac{1}{1-Z} \qquad \text{ROC} : |Z| < 1$$

Now recall time shifting property, it is

$$\text{If } x(n) \overset{Z}{\longleftrightarrow} X(Z) \text{ then } x(n-k) \overset{Z}{\longleftrightarrow} Z^{-k}X(Z).$$

Here we have folded sequence. For folded sequence this property can be written as,

$$x(-n-k) \overset{Z}{\longleftrightarrow} Z^{+k}X(Z)$$

This is because folded sequence is mirror image of original sequence.

Thus we can write,

$$Z\{u(-n-1)\} = Z^{+1} \cdot Z\{u(-n)\}$$

$$= Z^{1} \cdot \frac{1}{1-Z} \qquad \text{ROC} : |Z| < 1$$

$$\therefore \quad Z\{u(-n-1)\} = \frac{Z}{1-Z} \qquad \text{ROC} : |Z| < 1$$

Now we have to obtain $Z\{a^n u(-n-1)\}$. Using scaling property we get,

$$Z\{a^n u(-n-1)\} = \frac{Z/a}{1-Z/a} \qquad \text{ROC} : \left|\frac{Z}{a}\right| < 1$$

$$\therefore \quad Z\{a^n u(-n-1)\} = \frac{Z}{a-Z} \qquad \text{ROC} : |Z| < |a| \qquad \ldots(2)$$

Thus ROC is interior part of circle having radius r = a.

(iii) $\mathbf{x(n) = -a^n u(n-1)}$

From Equation (1) we have,

$$Z\{a^n u(n-1)\} = \frac{a}{Z-a} \qquad \text{ROC} : |Z| > |a|$$

Thus $Z\{-a^n u(n-1)\} = -\frac{a}{Z-a} \qquad \text{ROC} : |Z| > |a|$

$$\therefore \quad Z\{-a^n u(n-1)\} = \frac{a}{a-Z} \qquad \text{ROC} : |Z| > |a| \qquad \ldots(3)$$

(iv) $\mathbf{x(n) = -(a)^n u(-n-1)}$

From Equation (2) we have,

$$Z\{a^n u(-n-1)\} = \frac{Z}{a-Z} \qquad \text{ROC} : |Z| < |a|$$

Thus $Z\{-a^n u(-n-1)\} = -\frac{Z}{a-Z}$ ROC : $|Z| < |a|$

$$Z\{a^n u(-n-1)\} = \frac{Z}{Z-a} \quad \text{ROC} : |Z| < |a| \qquad \ldots(4)$$

Note : Compare given Equations (1) and (3) as well as (2) and (4). Here magnitude of sequence is changed in time domain. From the result of Z-transform, Equations (1) and (3), and Equations (2) and (4), we can say that the magnitude in Z-domain is also reversed. Thus changing the magnitude in time domain, changes the magnitude in Z-domain also; while the ROC remains same.

1.3.5 Differentiation in Z domain :

Statement : If $x(n) \overset{Z}{\longleftrightarrow} X(Z)$

then $n\,x(n) \overset{Z}{\longleftrightarrow} -Z\frac{dX(Z)}{dZ}$

Proof : According to the definition of Z-transform,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \qquad \ldots(1)$$

Differentiate both sides with respect to Z,

$$\frac{dX(Z)}{dZ} = \frac{d}{dZ}\left[\sum_{n=-\infty}^{\infty} x(n) Z^{-n}\right]$$

We can transfer d/dZ inside the summation sign,

$$\therefore \frac{dX(Z)}{dZ} = \sum_{n=-\infty}^{\infty} \frac{d}{dZ}[x(n) Z^{-n}] = \sum_{n=-\infty}^{\infty} (-n) \cdot x(n) Z^{-n-1}$$

But Z^{-n-1} can be written as $Z^{-n} \cdot Z^{-1}$

$$\therefore \frac{dX(Z)}{dZ} = \sum_{n=-\infty}^{\infty} (-n)\, x(n) Z^{-n} \cdot Z^{-1}$$

Since the limits of summation are in terms of n ; we can take Z^{-1} outside the summation.

$$\therefore \frac{dX(Z)}{dZ} = Z^{-1} \sum_{n=-\infty}^{\infty} (-n)\, x(n) Z^{-n}$$

Taking negative sign outside the summation,

$$\frac{dX(Z)}{dZ} = -Z^{-1} \sum_{n=-\infty}^{\infty} [n x(n)] Z^{-n}$$

$$\therefore \frac{dX(Z)}{dZ} = -\frac{1}{Z} \sum_{n=-\infty}^{\infty} [n x(n)] Z^{-n}$$

$$\therefore -Z \frac{dX(Z)}{dZ} = \sum_{n=-\infty}^{\infty} [n x(n)] Z^{-n} \quad ...(2)$$

Comparing R.H.S. with Equation (1), we can say that $\sum_{n=-\infty}^{\infty} [n x(n)] Z^{-n}$ is the Z- transform of $n x(n)$. Thus Equation (2) becomes,

$$-Z \frac{dX(Z)}{dZ} = Z\{n x(n)\}$$

$$\therefore n x(n) \overset{Z}{\longleftrightarrow} -Z \frac{dX(Z)}{dZ}$$

Hence proved. Note that both transforms have the same ROC.

Note : Multiplying the sequence in time domain by 'n' is equivalent to multiplying the derivative of its Z-transform, by – Z in the 'Z' domain.

Solved Problems on Differentiation Property :

Prob. 1 : Find Z transform and ROC of following sequence

$$x(n) = \frac{1}{2}\delta(n+1) + 5\left(\frac{1}{2}\right)^{-n} u(-n) + u(-n-1)$$

Soln. : The given expression is,

$$x(n) = \frac{1}{2}\delta(n+1) + 5\left(\frac{1}{2}\right)^{-n} u(n) + u(-n-1) \quad ...(1)$$

It can be written as,

$$x(n) = x_1(n) + x_2(n) + x_3(n) \quad ...(2)$$

Here $x_1(n) = \frac{1}{2}\delta(n+1)$ ...

We have, $\delta(n) \overset{Z}{\longleftrightarrow} 1$ ROC : Entire Z plane except Z = 0

According to time shifting property,

$$\delta(n+1) \overset{Z}{\longleftrightarrow} Z^1 \quad \text{ROC : Entire Z plane except Z = 0}$$

$$\therefore \quad Z\left\{\frac{1}{2}\delta(n+1)\right\} = X_1(Z) = \frac{1}{2}Z \quad \text{ROC : Entire Z plane except Z = 0.} \qquad ...(4)$$

$$\text{Now} \quad x_2(n) = 5\left(\frac{1}{2}\right)^{-n} u(n)$$

$$\therefore \quad x_2(n) = 5 \cdot \frac{1}{\left(\frac{1}{2}\right)^n} u(n) = 5 \cdot \frac{2^n}{(1)^n} u(n)$$

$$\therefore \quad x_2(n) = 5 \cdot 2^n u(n) \qquad ...(5)$$

We have standard Z transform pair,

$$u(-n) \overset{Z}{\longleftrightarrow} \frac{1}{1-Z}, \qquad \text{ROC} : |Z| < 1$$

$$\therefore \quad a^n u(-n) \overset{Z}{\longleftrightarrow} \frac{1}{1-\frac{Z}{a}} \qquad \text{ROC} : |Z| < a$$

Here a = 2

$$\therefore \quad 2^n u(-n) \overset{Z}{\longleftrightarrow} \frac{1}{1-\frac{Z}{2}} \qquad \text{ROC} : |Z| < 2$$

$$\therefore \quad Z\left\{5 \cdot 2^n u(n)\right\} = X_2(Z) = \frac{5}{1-\frac{Z}{2}} \qquad \text{ROC} : |Z| < 2 \qquad ...(6)$$

$$\text{Now} \quad x_3(n) = u(-n-1)$$

First we will obtain $Z\{u(-n)\}$. Here $u(-n)$ is folded unit step. The Z-transform of folded unit step is,

$$Z\{u(-n)\} = \frac{1}{1-Z}, \qquad \text{ROC} : |Z| < 1$$

According to the time shifting property we have,

$$x(n-k) \overset{Z}{\longleftrightarrow} Z^{-k} X(Z)$$

For folded sequence this property can be written as,

$$x(-n-k) \overset{Z}{\longleftrightarrow} Z^{+k} X(Z)$$

$$\therefore \quad Z\{u(-n-1)\} = Z^{+1} \cdot Z\{u(-n)\}$$

$$\therefore \quad Z\{u(-n-1)\} = Z^{+1} \cdot \frac{1}{1-Z} \qquad \text{ROC} : |Z| < 1$$

$$\therefore \quad Z\{u(-n-1)\} = X_3(Z) = \frac{Z}{1-Z}, \qquad \text{ROC} : |Z| < 1 \qquad \ldots(7)$$

The total Z-transform is obtained by applying linearity property to Equation (2).

$$\therefore \quad X(Z) = X_1(Z) + X_2(Z) + X_3(Z)$$

Thus from Equations (4), (6) and (7) we can write,

$$X(Z) = \frac{1}{2} Z + \frac{5}{1 - \frac{Z}{2}} + \frac{Z}{1-Z}$$

We have ROCs, $|Z| < 1$ and $|Z| < 2$. Thus combined ROC is $|Z| < 1$ and it is shown in Fig. U-26.

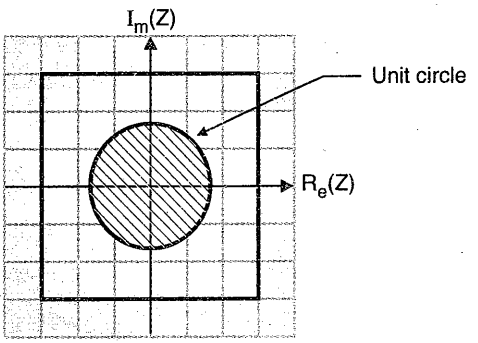


Fig. U-26

Prob. 2 : Using differentiation property obtain the Z-transform of unit ramp sequence.

Soln. : We know that unit ramp sequence is given by,

$$x(n) = n\,u(n) \qquad \ldots(1)$$

According to differentiation property we have,

$$Z\{n\,x(n)\} = -Z \frac{dX(Z)}{dZ} \qquad \ldots(2)$$

In this case we can write,

$$Z\{n\,u(n)\} = -Z \frac{d}{dZ} [Z(u(n))]$$

$$\text{We have} \quad Z\{u(n)\} = \frac{Z}{Z-1} \qquad \text{ROC} : |Z| > 1$$

$$\therefore \quad Z\{nu(n)\} = -Z\frac{d}{dZ}\left[\frac{Z}{Z-1}\right]$$

$$\therefore \quad Z\{nu(n)\} = -Z\frac{d}{dZ}\left[(Z)\cdot(Z-1)^{-1}\right]$$

$$= -Z\left[Z\cdot\frac{d}{dZ}\left[(Z-1)^{-1}\right]+(Z-1)^{-1}\cdot\frac{d}{dZ}Z\right]$$

$$= -Z\left[Z(-1)(Z-1)^{-2}+(Z-1)^{-1}\cdot 1\right]$$

$$= -Z\left[\frac{-Z}{(Z-1)^2}+\frac{1}{(Z-1)}\right] = -Z\left[\frac{-Z+Z-1}{(Z-1)^2}\right]$$

$$= -Z\left[\frac{-1}{(Z-1)^2}\right]$$

$$\therefore \quad Z\{nu(n)\} = \frac{Z}{(Z-1)^2} \qquad \text{ROC} : |Z| > 1$$

This is the standard Z-transform pair. Earlier we had obtained the Z-transform of $nu(n)$ without using differentiation property.

Prob. 3 : Obtain the Z-transform of signal,

$x(n) = n a^n u(n)$

Soln. :

Let $x_1(n) = a^n u(n)$ $\quad\therefore\quad x(n) = n x_1(n)$

According to differentiation property,

$$n x_1(n) \overset{Z}{\longleftrightarrow} -Z\frac{d}{dZ}X_1(Z) \qquad \ldots(1)$$

In this case we can write,

$$Z\{n a^n u(n)\} = -Z\frac{d}{dZ}\left[Z\{a^n u(n)\}\right] \qquad \ldots(2)$$

Here the Z-transform of $a^n u(n)$ is given by,

$$Z\{a^n u(n)\} = \frac{Z}{Z-a} \qquad \text{ROC} : |Z| > |a|$$

Putting this value in Equation (2) we get,

$$Z\{n a^n u(n)\} = -Z\frac{d}{dZ}\left[\frac{Z}{Z-a}\right]$$

$$\therefore \quad Z\{n a^n u(n)\} = -Z\frac{d}{dZ}\left[Z\cdot(Z-a)^{-1}\right]$$

$$= -Z\left\{ Z \cdot \frac{d}{dZ}\left[(Z-a)^{-1} \right] + (Z-a)^{-1} \cdot \frac{d}{dZ} Z \right\}$$

$$= -Z\left[Z \cdot (Z-a)^{-2} \cdot (-1) + (Z-a)^{-1} \cdot 1 \right]$$

$$= -Z\left[\frac{-Z}{(Z-a)^2} + \frac{1}{(Z-a)} \right] = -Z\left[\frac{-Z+Z-a}{(Z-a)^2} \right]$$

$$= -Z\left[\frac{-a}{(Z-a)^2} \right]$$

$$\therefore \quad Z\{ n a^n u(n) \} = \frac{aZ}{(Z-a)^2} \qquad \text{ROC} : |Z| > |a| \qquad \ldots(3)$$

This is also standard Z-transform pair.

$$\therefore \; n a^n u(n) \overset{Z}{\longleftrightarrow} \frac{aZ}{(Z-a)^2} \qquad \text{ROC} : |Z| > |a|$$

Note : From this equation we can obtain Z- transform of unit ramp sequence, $n\,u(n)$. By putting $a = 1$ in Equation (3) we get,

$$Z\{ n\,u(n) \} = \frac{Z}{(Z-1)^2} \qquad \text{ROC} : |Z| > |a|$$

Prob. 4 : The z transform of DT signal $x(n)$ is given by $x(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z^2+4}$ ROC $|Z| > 2$. Determine Z transform and ROC of the following signals using properties of Z transform.

(i) $2^n x(n)$ (ii) $nx(n)$ (iii) $x(-n)$ (iv) $x(n-4)$.

Soln. :

(i) $2^n x(n)$

According to the scaling property.

$$a^n x(n) \overset{Z}{\longleftrightarrow} X\left(\frac{Z}{a}\right)$$

Here $X(Z) = \frac{Z}{Z^2+4}$, ROC $|Z| > 2$

$$\therefore \quad 2^n x(n) \overset{Z}{\longleftrightarrow} \frac{Z_{12}}{\left(\frac{Z}{2}\right)^2 + 4} \text{ ROC } \left|\frac{Z}{2}\right| > 2$$

$$\therefore \quad Z\{ 2^n x(n) \} = \frac{Z_{12}}{\frac{Z^2}{4} + 4}, \quad \text{ROC } |Z| > 4$$

(ii) $n\,x(n)$

According to the differentiation property,

$$n\,x(n) \overset{Z}{\longleftrightarrow} -Z\frac{d}{dz}X(Z)$$

$$\therefore \quad Z\{n\,x(n)\} = -Z\frac{d}{dz}\left\{\frac{Z}{Z^2+4}\right\} = -Z\left\{Z(-1)(Z^2+4)^{-2}+(Z^2+4)^{-1}\right\}$$

$$\therefore \quad Z\{n\,x(n)\} = \frac{Z^2}{Z^2+4}-\frac{Z}{Z^2+4} = \frac{Z^2-Z}{Z^2+4}$$

The ROC remains same as $X(Z)$.

(iii) $x(-n)$

According to time reversal property,

$$x(-n) \overset{Z}{\longleftrightarrow} X(Z^{-1})$$

$$\therefore \quad Z\{x(-n)\} = \frac{Z^{-1}}{(Z^{-1})^2+4} \quad \text{ROC } |Z^{-1}| > 2$$

$$= \frac{Z^{-1}}{Z^{-2}+4} \quad \text{ROC } \left|\frac{1}{Z}\right| > 2 \text{ i.e. } 1 > 2|Z| \text{ i.e. } |Z| > \frac{1}{2}$$

$$\therefore \quad Z\{x(-n)\} = \frac{Z}{1+4Z^2} \quad \text{ROC } |Z| > \frac{1}{2}$$

(iv) $x(n-4)$

According to time shifting property,

$$x(n-k) \overset{Z}{\longleftrightarrow} Z^{-k}X(Z)$$

$$\therefore \quad Z\{x(n-4)\} = Z^{-4}\left[\frac{Z}{Z^2+4}\right] = \frac{Z^{-3}}{Z^2+4}$$

Since k = 4 which is greater than zero, ROC is same as $X(Z)$ except $Z = 0$.

1.3.6 Convolution of Two Sequences :

Statement : If $x_1(n) \overset{Z}{\longleftrightarrow} X_1(Z)$ and $x_2(n) \overset{Z}{\longleftrightarrow} X_2(Z)$

Then $x_1(n) * x_2(n) \overset{Z}{\longleftrightarrow} X_1(Z) \cdot X_2(Z)$

and ROC is atleast the intersection of ROC of $X_1(Z)$ and $X_2(Z)$.

Proof : According to the definition of convolution, the convolution of $x_1(n)$ and $x_2(n)$ can be written as,

$$x(n) = x_1(n) * x_2(n) \qquad \ldots(1)$$

$$\therefore \quad x(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k) \qquad \ldots(2)$$

Taking Z-transform of x (n),

$$X(Z) = Z\left\{ \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k) \right\} \qquad \ldots(3)$$

According to the definition of Z-transform we have,

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) \cdot Z^{-n} \qquad \ldots(4)$$

In Equation (3) take the bracket term as x (n). Thus Equation (4) becomes,

$$\therefore \quad Z\{x(n)\} = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k) \right] \cdot Z^{-n}$$

Rearranging the summation terms,

$$Z\{x(n)\} = \sum_{k=-\infty}^{\infty} x_1(k) \left[\sum_{n=-\infty}^{\infty} x_2(n-k)\, Z^{-n} \right] \qquad \ldots(5)$$

Now Z^{-n} can be written as,

$$Z^{-n} = Z^{-n} \times Z^{k} \times Z^{-k} = Z^{-(n-k)} \times Z^{-k}$$

Putting this value in Equation (5) we get,

$$Z\{x(n)\} = \sum_{k=-\infty}^{\infty} x_1(k) \left[\sum_{n=-\infty}^{\infty} x_2(n-k)\, Z^{-(n-k)} \cdot Z^{-k} \right]$$

In the second summation; the limits of summation are in terms of 'n'. So we can take Z^{-k} out of the summation sign.

$$\therefore \quad Z\{x(n)\} = \left[\sum_{k=-\infty}^{\infty} x_1(k) \cdot Z^{-k} \right] \left[\sum_{n=-\infty}^{\infty} x_2(n-k)\, Z^{-(n-k)} \right]$$

In the second summation put $n - k = m$. The limits will change as follows :

when $n = -\infty \Rightarrow -\infty - k = m \qquad \therefore \quad m = -\infty$

and when $n = +\infty \Rightarrow \infty - k = m \quad \therefore \quad m = \infty$

$$\therefore \quad Z\{x(n)\} = \left[\sum_{k=-\infty}^{\infty} x_1(k) \times Z^{-k}\right]\left[\sum_{m=-\infty}^{\infty} x_2(m) Z^{-m}\right] \qquad ...(6)$$

Compare each bracket of R.H.S. with the definition of Z- transform.

$$\therefore \quad Z\{x(n)\} = X_1(Z) \cdot X_2(Z) \qquad ...(7)$$

But here $x(n) = x_1(n) * x_2(n)$

$$\therefore \quad Z\{x_1(n) * x_2(n)\} = X_1(Z) \cdot X_2(Z)$$

$$\therefore \quad x_1(n) * x_2(n) \overset{Z}{\longleftrightarrow} X_1(Z) \cdot X_2(Z)$$

ROC is atleast the intersection of $X_1(Z)$ and $X_2(Z)$.

Note : Convolution of two sequences in time domain is equivalent to multiplication of its Z- transforms in Z domain.

Solved Problems :

Prob. 1 : Find the linear convolution of $x_1(n)$ and $x_2(n)$ using Z-transform

$x_1(n) = \{1, 2, 3, 4\}$ and $x_2(n) = \{1, 2, 0, 2, 1\}$.
$\uparrow$ $\uparrow$

Soln. : According to the property of linear convolution in Z- domain we have,

$$Z\{x_1(n) * x_2(n)\} = X_1(Z) \cdot X_2(Z) \qquad ...(1)$$

Step I : We have $x_1(n) = \{1, 2, 3, 4\}$
$\uparrow$

According to the definition of Z-transform,

$$X_1(Z) = \sum_{n=-\infty}^{\infty} x_1(n) Z^{-n}$$

Here limits of summation are from $n = -2$ to $n = 1$.

$$\therefore \quad X_1(Z) = \sum_{n=-2}^{1} x_1(n)\, Z^{-n}.$$

$$= x_1(-2)Z^2 + x_1(-1)Z + x_1(0)Z^0 + x_1(1)Z^{-1}$$

$$\therefore \quad X_1(Z) = 1Z^2 + 2Z + 3 + 4Z^{-1} \qquad ...(2)$$

Let us determine the ROC of $X_1(Z)$.

We have $X_1(Z) = Z^2 + 2Z + 3 + \frac{4}{Z}$

ROC : (i) Putting $Z = 0$ we get,

$$X_1(Z) = 0 + 0 + 3 + \frac{4}{0} = 0 + 0 + 3 + \infty = \infty$$

Thus $Z = 0$ is not allowed.

(ii) Putting $Z = \infty$ we get,

$$X_1(Z) = \infty + \infty + 3 + \frac{4}{\infty} = \infty + \infty + 3 + 0 = \infty$$

Thus $Z = \infty$ is not allowed.

Therefore ROC is entire Z plane except $Z = 0$ and $Z = \infty$

Step II : Now we will obtain Z-transform of $x_2(n)$.

We have $x_2(n) = \{1, 2, 0, 2, 1\}$

$\uparrow$

Using definition of Z-transform,

$$X_2(Z) = \sum_{n=-\infty}^{\infty} x_2(n) Z^{-n}$$

Here the range of n is $n = -2$ to $n = +2$

$$\therefore X_2(Z) = \sum_{n=-2}^{2} x_2(n) Z^{-n}$$

$$\therefore X_2(Z) = x_2(-2)Z^2 + x_2(-1)Z^1 + x_2(0)Z^0 + x_2(1)Z^{-1} + x_2(2)Z^{-2}$$

$$\therefore X_2(Z) = 1Z^2 + 2Z + 0Z^0 + 2Z^{-1} + 1Z^{-2}$$

$$\therefore X_2(Z) = Z^2 + 2Z + 2Z^{-1} + Z^{-2} \quad \ldots(3)$$

Let us determine the ROC of $X_2(Z)$

We have $X_2(Z) = Z^2 + 2Z + \frac{2}{Z} + \frac{1}{Z^2}$

ROC :

(i) Putting $Z = 0$ we get,

$$X_2(Z) = 0 + 0 + \frac{2}{0} + \frac{1}{0} = 0 + 0 + \infty + \infty = \infty.$$

Thus Z = 0 is not allowed.

(ii) Putting Z = ∞ we get,

$$X_2(Z) = \infty + \infty + \frac{2}{\infty} + \frac{1}{\infty} = \infty + \infty + 0 + 0 = \infty.$$

Thus Z = ∞ is not allowed.

Therefore the ROC is entire Z plane except Z = 0 and Z = ∞.

Step III : Now we have,

$$X(Z) = X_1(Z) \cdot X_2(Z)$$

$$\therefore X(Z) = (Z^2 + 2Z + 3 + 4Z^{-1}) \cdot (Z^2 + 2Z + 2Z^{-1} + Z^{-2})$$

$$\therefore X(Z) = Z^4 + 2Z^3 + 2Z + 1 + 2Z^3 + 4Z^2 + 4 + 2Z^{-1} + 3Z^2 + 6Z + 6Z^{-1} + 3Z^{-2} + 4Z + 8 + 8Z^{-2} + 4Z^{-3}$$

$$\therefore X(Z) = Z^4 + 4Z^3 + 7Z^2 + 12Z + 13 + 8Z^{-1} + 11Z^{-2} + 4Z^{-3} \quad ...(4)$$

Step IV : According to definition of Z-transform,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

From Equation (4), we get the range of n. It is from n = −4 to n = 3.

$$\therefore X(Z) = \sum_{n=-4}^{3} x(n) Z^{-n}$$

Expanding the summation we get,

$$X(Z) = x(-4)Z^4 + x(-3)Z^3 + x(-2)Z^2 + x(-1)Z + x(0) + x(1)Z^{-1} + x(2)Z^{-2} + x(3)Z^{-3} \quad ...(5)$$

Comparing Equations (4) and (5),

x(−4) = 1, x(−3) = 4, x(−2) = 7, x(−1) = 12,
x(0) = 13, x(1) = 8, x(2) = 11, x(3) = 4.

Thus we can write,

$$x(n) = \{x(-4), x(-3), x(-2), x(-1), x(0), x(1), x(2), x(3)\}$$
$$\uparrow$$

$$\therefore x(n) = \{1, 4, 7, 12, 13, 8, 11, 4\}$$
$$\uparrow$$

1.3.7 Initial Value Theorem :

Statement : If $x(n)$ is a causal sequence then its initial value is given by,

$$x(0) = \lim_{Z \to \infty} X(Z)$$

Proof : According to the definition of Z-transform.

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \qquad ...(1)$$

But if $x(n)$ is a causal sequence then limits of summation will be from $n = 0$ to $n = \infty$.

$$\therefore \quad X(Z) = \sum_{n=0}^{\infty} x(n) Z^{-n} \qquad ...(2)$$

Expanding the summation we get,

$$X(Z) = x(0)Z^0 + x(1)Z^{-1} + x(2)Z^{-2} + x(3)Z^{-3} + \ldots$$

Applying limits as $Z \to \infty$ we get,

$$\lim_{Z \to \infty} X(Z) = \lim_{Z \to \infty} x(0) + \lim_{Z \to \infty} x(1)Z^{-1} + \lim_{Z \to \infty} x(2)Z^{-2} + \ldots. \text{ (as } Z^0 = 1)$$

$$\therefore \quad \lim_{Z \to \infty} X(Z) = \lim_{Z \to \infty} x(0) + \lim_{Z \to \infty} x(1) \cdot \frac{1}{Z} + \lim_{Z \to \infty} x(2) \frac{1}{Z^2} + \ldots \qquad ...(3)$$

Now as $Z \to \infty$ the terms $\frac{1}{Z} = \frac{1}{Z^2}$ etc. will be $\frac{1}{\infty}$ which is zero. Thus Equation (3) becomes,

$$\lim_{Z \to \infty} X(Z) = x(0) \qquad ...(4)$$

But $x(0)$ is called as initial value of $x(n)$.

$$\therefore \text{ Initial value } = x(0) = \lim_{Z \to \infty} X(Z)$$

Hence proved.

1.3.8 Final Value Theorem :

Statement : If $x(n) \overset{Z}{\longleftrightarrow} X(Z)$ then,

$$x(\infty) = \lim_{Z \to 1} [(Z-1)X(Z)]$$

Proof : According to the definition of Z-transform for causal sequence we have,

$$Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) Z^{-n} \quad \ldots(1)$$

Consider the term $x(n)-x(n-1)$, using Equation (1), Z-transform of this term can be expressed as,

$$Z\{x(n)-x(n-1)\} = \sum_{n=0}^{\infty} [x(n)-x(n-1)]Z^{-n}$$

$$\therefore \quad Z\{x(n)\}-Z\{x(n-1)\} = \sum_{n=0}^{\infty} x(n)Z^{-n} - \sum_{n=0}^{\infty} x(n-1)Z^{-n} \quad \ldots(2)$$

Now we have $Z\{x(n)\} = X(Z)$ and

According to shifting property, $Z\{x(n-1)\} = Z^{-1}X(Z)$

$$\therefore \quad X(Z)-Z^{-1}X(Z) = \sum_{n=0}^{\infty} x(n)Z^{-n} - \sum_{n=0}^{\infty} x(n-1)Z^{-n}$$

$$\therefore \quad X(Z)(1-Z^{-1}) = \sum_{n=0}^{\infty} x(n)Z^{-n} - \sum_{n=0}^{\infty} x(n-1)Z^{-n} \quad \ldots(3)$$

Now consider the second summation. Here Z^{-n} can be written as,

$Z^{-n} = Z^{-n} \cdot Z^{1} \cdot Z^{-1} = Z^{-(n-1)} \cdot Z^{-1}$

$$\therefore \quad X(Z)(1-Z^{-1}) = \sum_{n=0}^{\infty} x(n)Z^{-n} - \sum_{n=0}^{\infty} x(n-1)Z^{-(n-1)}Z^{-1}$$

$$\therefore \quad X(Z)(1-Z^{-1}) = \sum_{n=0}^{\infty} x(n)Z^{-n} - Z^{-1}\sum_{n=0}^{\infty} x(n-1)Z^{-(n-1)}$$

Taking limit as $Z \to 1$ on both sides,

$$\lim_{Z\to 1} X(Z)(1-Z^{-1}) = \lim_{Z\to 1} \sum_{n=0}^{\infty} x(n)Z^{-n} - \lim_{Z\to 1} Z^{-1} \cdot \sum_{n=0}^{\infty} x(n-1)Z^{-(n-1)}$$

Remember that the first summation is $Z\{x(n)\}$ and second summation is $Z\{x(n-1)\}$. Now expanding the summations we can write,

$$\therefore \lim_{Z\to 1} X(Z)(1-Z^{-1}) = \lim_{Z\to 1} \Big[(X(0)-Z^{-1}X(-1))+(X(1)-Z^{-1}X(0)) + (X(2)-Z^{-1}X(1))+\ldots+[X(\infty)-Z^{-1}X(\infty-1))\Big] \ldots(4)$$

Here as $\lim_{Z\to 1}$ we get $\lim_{Z\to 1} Z^{-1} = \lim_{Z\to 1} \frac{1}{Z} = 1$ and $X(-1) = 0$

because $x(n) = 0$ for $n = -1$ since it is a causal sequence.

$$\therefore \lim_{Z\to 1} X(Z)(1-Z^{-1}) = [X(0)-0]+[X(1)-X(0)]+\ldots.+[X(\infty)-X(\infty-1)]$$

$= X(\infty)$ ... (Here we are adding the difference between two terms and this addition is upto ∞).

$$\therefore X(\infty) = \lim_{Z\to 1} X(Z)(1-Z^{-1})$$

Hence proved.

Solved Problems :

Prob. 1 : Find initial and final values of $x(n)$ if Z transform is,

$$X(Z) = \frac{2}{Z^2+\frac{1}{6}Z-\frac{1}{6}}$$

Soln. : According to initial value theorem,

$$x(0) = \lim_{Z\to\infty} X(Z) = \lim_{Z\to\infty} \frac{2}{Z^2+\frac{1}{6}Z-\frac{1}{6}}$$

$$\therefore x(0) = 0$$

According final value theorem,

$$x(\infty) = \lim_{Z\to 1} (Z-1)\cdot X(Z) = \lim_{Z\to 1} (Z-1)\cdot \frac{2}{Z^2+\frac{1}{6}Z-\frac{1}{6}}$$

We will obtain roots of denominator term.

$$Z^2+\frac{1}{6}Z-\frac{1}{6} \Rightarrow \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-\frac{1}{6}\pm\sqrt{\frac{1}{36}+\frac{4}{6}}}{2}$$

$$\therefore \quad \text{roots} = \frac{-\frac{1}{6} \pm \sqrt{\frac{1+24}{36}}}{2} = \frac{-\frac{1}{6} \pm \frac{5}{6}}{2}$$

$$\therefore \quad \text{roots} = \frac{1}{3}, -\frac{1}{2}$$

$$\therefore \quad x(\infty) = \lim_{Z \to 1} (Z-1) \cdot \frac{2}{\left(Z-\frac{1}{3}\right)\left(Z+\frac{1}{2}\right)}$$

$$\therefore \quad x(\infty) = 0$$

Prob. 2 : Determine the value of signal $x(n)$ at $n = 0$ and $n = \infty$ if $X(Z) = \dfrac{2Z^2 + 0.25}{(Z+0.25)(Z-1)}$

Soln. : According to initial value theorem,

$$x(0) = \lim_{Z \to \infty} X(Z)$$

$$x(0) = \lim_{Z \to \infty} \frac{2Z^2 + 0.25}{(Z+0.25)(Z-1)}$$

Multiplying numerator and denominator by Z^{-2};

$$x(0) = \lim_{Z \to \infty} \frac{2 + 0.25Z^{-2}}{Z^{-1}(Z+0.25) \cdot Z^{-1}(Z-1)} = \lim_{Z \to \infty} \frac{2 + \frac{0.25}{Z^2}}{\left(1+\frac{0.25}{Z}\right)\left(1-\frac{1}{Z}\right)}$$

$$\therefore \quad x(0) = \frac{2}{1} = 2$$

According to final value theorem,

$$x(\infty) = \lim_{Z \to 1} (Z-1) X(Z)$$

$$\therefore \quad x(\infty) = \lim_{Z \to 1} (Z-1) \cdot \frac{2Z^2 + 0.25}{(Z+0.25)(Z-1)} = \lim_{Z \to 1} \frac{2Z^2 + 0.25}{Z+0.25} = \frac{2+0.25}{1+0.25}$$

$$\therefore \quad x(\infty) = 1.8$$

Summary of the properties of Z-transform :

Table U-3 shows the summary of Z-transform properties.

Table U-3

	Property	Equation Time domain Z-domain	ROC
1.	Linearity	$a_1 x_1(n) + a_2 x_2(n) \overset{Z}{\longleftrightarrow} a_1 X_1(Z) + a_2 X_2(Z)$	Intersection of the ROC of $X_1(Z)$ and $X_2(Z)$
2.	Time shifting	$x(n-k) \overset{Z}{\longleftrightarrow} Z^{-k} X(Z)$	Same as ROC of $X(Z)$ Except $Z = 0$ if $K > 0$ and $Z = \infty$ if $K < 0$
3.	Scaling in Z domain	$a^n x(n) \overset{Z}{\longleftrightarrow} X\left(\frac{Z}{a}\right)$	$\lvert a \rvert r_1 < \lvert Z \rvert < \lvert a \rvert r_2$
4.	Time reversal	$x(-n) \overset{Z}{\longleftrightarrow} X(Z^{-1})$	$\frac{1}{r_2} < \lvert Z \rvert < \frac{1}{r_1}$
5.	Differentiation in Z domain	$nx(n) \overset{Z}{\longleftrightarrow} -Z\frac{d}{dZ}X(Z)$	Same as ROC of $X(Z)$
6.	Convolution	$x_1(n) * x_2(n) \overset{Z}{\longleftrightarrow} X_1(Z) \cdot X_2(Z)$	ROC is atleast intersection of ROC of $X_1(Z)$ and $X_2(Z)$
7.	Correlation	$r_{x_1 x_2}(l) \overset{Z}{\longleftrightarrow} X_1(Z) \cdot X_2(Z^{-1})$	ROC is intersection of ROC of $X_1(Z)$ and $X_2(Z)$
8.	Multiplication	$x_1(n) \cdot x_2(n) \overset{Z}{\longleftrightarrow} \frac{1}{2\pi j} \oint_c X_1(v) X_2\left(\frac{Z}{v}\right) v^{-1} dv$	ROC is atleast $r_{1l} r_{2l} < \lvert Z \rvert < r_{1u} r_{2u}$

	Property	Equation Time domain Z-domain	ROC
9.	Parseval's relation	$\sum_{n=-\infty}^{\infty} x_1(n)\,x_2^*(n) \overset{Z}{\longleftrightarrow} \frac{1}{2\pi j}\oint_c X_1(v)\,X_2^*\left(\frac{1}{v^*}\right)v^{-1}\,dv$	–
10.	Initial value theorem	Causal $x(n)$ $x(0) = \lim_{Z\to\infty} X(Z)$	–
11.	Final value theorem	Causal $x(n)$: $x(\infty) = \lim_{Z\to 1} X(Z)$	–

Summary of Z-transform pairs :

Table U-4

	Signal $x(n)$	Z-transform $X(Z)$	ROC
(1)	$\delta(n)$	1	Entire Z-plane
(2)	$\delta(n-k)$	Z^{-k}	Entire Z-plane except Z = 0
(3)	$\delta(n+k)$	Z^k	Entire Z-plane except Z = ∞
(4)	$u(n)$	$\frac{Z}{Z-1}$	$\|Z\| > 1$
(5)	$nu(n)$	$\frac{Z}{(Z-1)^2}$	$\|Z\| > 1$
(6)	$\alpha^n u(n)$	$\frac{Z}{Z-\alpha}$	$\|Z\| > \|\alpha\|$
(7)	$-\alpha^n u(-n-1)$	$\frac{Z}{Z-\alpha}$	$\|Z\| < \|\alpha\|$
(8)	$\alpha^n u(n) + \beta^n u(-n-1)$	$\frac{Z}{Z-\alpha} - \frac{Z}{Z-\beta}$	$\|\alpha\| < \|Z\| < \|\beta\|$
(9)	$\cos\omega_0 n\ u(n)$	$\frac{Z^2 - Z\cos\omega_0}{Z^2 - 2Z\cos\omega_0 + 1}$	; ROC : $\|Z\| > 1$
(10)	$\sin\omega_0 n\ u(n)$	$\frac{Z\sin\omega_0}{Z^2 - 2Z\cos\omega_0 + 1}$	; ROC : $\|Z\| > 1$

	Signal x (n)	Z-transform X(Z)	ROC
(11)	$a^n \cos \omega_0 n$	$\frac{Z^2 - aZ\cos\omega_0}{Z^2 - 2aZ\cos\omega_0 + a^2}$	; ROC : $\|Z\| > \|a\|$
(12)	$a^n \sin \omega_0 n \cdot u(n)$	$\frac{aZ\sin\omega_0}{Z^2 - 2aZ\cos\omega_0 + a^2}$	; ROC : $\|Z\| > \|a\|$
(13)	$u(-n)$	$\frac{1}{1-Z}$	; ROC : $\|Z\| > 1$
(14)	$n a^n u(n)$	$\frac{aZ}{(Z-a)^2}$	; ROC : $\|Z\| > \|a\|$

Solved Problems :

Prob. 1 : Find Z-transform of

$x(n) = a^{|n|}$, $0 < |a| < 1$

Soln. : The given signal is,

$$x(n) = a^{|n|}, \quad 0 < |a| < 1 \qquad ...(1)$$

First we will decide the nature of x (n). Let $a = \frac{1}{2}$.

$$\text{For} \quad n = 0 \Rightarrow x(0) = a^{|0|} = \left(\frac{1}{2}\right)^0 = 1$$

$$\text{For} \quad n = 1 \Rightarrow x(1) = a^{|1|} = \left(\frac{1}{2}\right)^1 = 1/2$$

$$\text{For} \quad n = 2 \Rightarrow x(2) = a^{|2|} = \left(\frac{1}{2}\right)^2 = 1/4$$

$$\text{For} \quad n = 3 \Rightarrow x(3) = a^{|3|} = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

Now we will calculate x (n) for negative values of n.

$$\text{For} \quad n = -1 \Rightarrow x(-1) = a^{|-1|} = \left(\frac{1}{2}\right)^1 = \frac{1}{2}$$

$$\text{For} \quad n = -2 \Rightarrow x(-2) = a^{|-2|} = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$\text{For} \quad n = -3 \Rightarrow x(-3) = a^{|-3|} = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

The signal $x(n) = a^{|n|}$ is plotted as shown in Fig. U-27.

Thus it is two sided sequence. It can be expressed as,

$$x(n) = a^{|n|} = a^n u(n) + a^{-n} u(-n-1) \qquad ...(2)$$

This is because at left hand side we are getting negative values of n.

According to the definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \qquad ...(3)$$

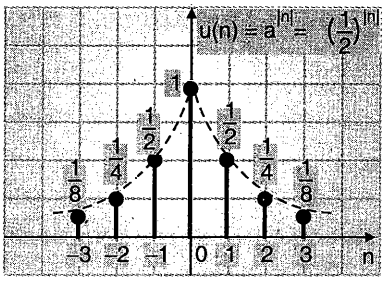


Fig. U-27 : Sequence $x(n) = a^{|n|}$

$$\therefore \quad X(Z) = \sum_{n=-\infty}^{\infty} [a^n u(n) + a^{-n} u(-n-1)] Z^{-n}$$

Now the summation can be divided into two terms as follows :

$$X(Z) = \sum_{n=0}^{\infty} a^n Z^{-n} + \sum_{n=-\infty}^{-1} a^{-n} Z^{-n} \qquad ...(4)$$

Let $$X(Z) = X_1(Z) + X_2(Z) \qquad ...(5)$$

$$\therefore \quad X_1(Z) = \sum_{n=0}^{\infty} a^n Z^{-n}$$

We have standard Z-transform pair,

$$a^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-a} \qquad \text{ROC} : |Z| > |a|$$

$$\therefore \quad X_1(Z) = \frac{Z}{Z-a} \qquad \text{ROC} : |Z| > |a| \qquad ...(6)$$

Now consider second term of Equation (4).

$$X_2(Z) = \sum_{n=-\infty}^{-1} a^{-n} Z^{-n}$$

Put $n = -l$ then the limits will change as follows :

when $n = -\infty \Rightarrow -\infty = -l \quad \therefore \; l = \infty$

when $n = -1 \Rightarrow -1 = -l \quad \therefore \; l = 1$

$$\therefore \quad X_2(Z) = \sum_{l=1}^{\infty} a^l Z^l = \sum_{l=1}^{\infty} (aZ)^l$$

$$\therefore \quad X_2(Z) = (aZ) + (aZ)^2 + (aZ)^3 + (aZ)^4 + \ldots$$

Taking (aZ) common we get,

$$X_2(Z) = aZ\,[1 + (aZ) + (aZ)^2 + (aZ)^3 + \ldots.]$$

Here the bracket term is standard geometric series which converges to, $\frac{1}{1-aZ}$ if $|aZ| < 1$.

Thus we have, $X_2(Z) = aZ\left(\frac{1}{1-aZ}\right)$ if $|aZ| < 1$ that means $|Z| < \frac{1}{|a|}$

$$\therefore \quad X_2(Z) = \frac{aZ}{1-aZ} \qquad \text{ROC} : |Z| < \frac{1}{|a|} \qquad \ldots(7)$$

Putting Equations (6) and (7) in Equation (5) we get,

$$X(Z) = \frac{Z}{Z-a} + \frac{aZ}{1-aZ} \qquad \text{ROC} : |Z| > |a| \text{ and } |Z| < \frac{1}{|a|}$$

This is two sided sequence. The ROC is as shown in Fig. U-28.

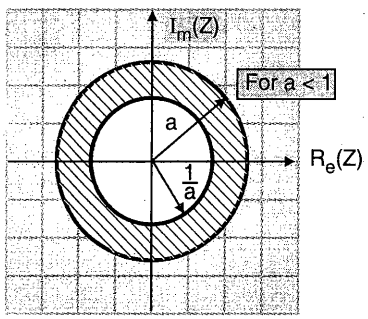


Fig. U-28 : ROC : $|a| < |Z| < \frac{1}{|a|}$

Prob. 2 : Find the Z-transform of

$$x(n) = \frac{a^n}{n!},\ n \geq 0$$

Soln. : According to definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \qquad ...(1)$$

But the given range of n is $n \geq 0$. That means it is a causal sequence. The limits of summation will be from n = 0 to $n = \infty$.

$$\therefore \quad X(Z) = \sum_{n=0}^{\infty} \left(\frac{a^n}{n!}\right) Z^{-n}$$

$$\therefore \quad X(Z) = \sum_{n=0}^{\infty} \frac{(a Z^{-1})^n}{n!} \qquad ...(2)$$

Now we have standard series of e^x,

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x^2 + \frac{x^3}{2!} + \ldots = e^x \qquad ...(3)$$

Thus Equation (2) can be written as,

$$X(Z) = e^{a Z^{-1}}$$

$$\therefore \quad X(Z) = e^{a/Z}$$

Prob. 3 : Find the Z-transform using frequency shifting property $x(n) = e^{-an} \cos(\omega_0 n)$

Soln. : We have obtained Z-transform of $a^n \cos \omega_0 n$ using scaling property.

It is,

$$Z\left\{a^n \cos \omega_0 n\right\} = \frac{Z^2 - a Z \cos \omega_0}{Z^2 - 2 a Z \cos \omega_0 + a^2} \qquad \textbf{ROC :} |Z| > |a| \qquad ...(1)$$

The given equation is,

$$x(n) = e^{-an} \cos \omega_0 n \qquad ...(2)$$

$$\therefore \quad x(n) = (e^{-a})^n \cos \omega_0 n$$

Let $a = e^{-a}$. Thus from Equation (1) we get,

$$Z\left\{(e^{-a})^n \cos \omega_0 n\right\} = \frac{Z^2 - e^{-a} Z \cos \omega_0}{Z^2 - 2e^{-a} Z \cos \omega_0 + (e^{-a})^2} \quad \text{ROC} : |Z| > |e^{-a}|$$

$$Z\left\{e^{-an} \cos \omega_0 n\right\} = \frac{Z^2 - Z e^{-a} \cos \omega_0}{Z^2 - 2Z e^{-a} \cos \omega_0 + e^{-2a}} \quad \text{ROC} : |Z| > \frac{1}{|e^{-a}|}$$

Prob. 4 : Determine the Z-transform of the sequence :

$$x(n) = \begin{cases} n & 0 \le n \le N-1 \\ N & N \le n \end{cases}$$

Soln. : According to the definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \qquad ...(1)$$

The summation can be split up into two parts as follows :

$$X(Z) = \sum_{n=0}^{N-1} x(n) Z^{-n} + \sum_{n=N}^{\infty} x(n) Z^{-n} \qquad ...(2)$$

Putting the corresponding values of $x(n)$ we get,

$$X(Z) = \sum_{n=0}^{N-1} n Z^{-n} + \sum_{n=N}^{\infty} N Z^{-n} \qquad ...(3)$$

$$\textbf{Let} \quad X(Z) = X_1(Z) + X_2(Z) \qquad ...(4)$$

$$\text{Here} \quad X_1(Z) = \sum_{n=0}^{N-1} n Z^{-n}$$

Here 'n' can be expressed as, $n = (n^{1/n})^n$.

$$\therefore \quad X_1(Z) = \sum_{n=0}^{N-1} (n^{1/n})^n \cdot (z^{-1})^n$$

$$\therefore \quad X_1(Z) = \sum_{n=0}^{N-1} \left[(n)^{1/n} \cdot Z^{-1}\right]^n \qquad ...(5)$$

Recall the standard formula of infinite geometric sum series,

$$\sum_{n=N_1}^{N_2} a^n = \begin{cases} \dfrac{a^{N_1} - a^{N_2+1}}{1-a} & a \neq 1 \\ N_2 - N_1 + 1 & a = 1 \end{cases} \quad ...(6)$$

Compare Equations (5) and (6)

Here, $a = (n)^{1/n} \cdot Z^{-1}$, $N_1 = 0$, $N_2 = N-1$

$$\therefore X_1(Z) = \frac{\left[(n)^{1/n} \cdot Z^{-1}\right]^0 - \left[(n)^{1/n} \cdot Z^{-1}\right]^{N-1+1}}{1-(n)^{1/n} \cdot Z^{-1}} \text{ ... for } (n)^{1/n} \cdot Z^{-1} \neq 1.$$

$$\therefore X_1(Z) = \frac{1-\left[(n)^{1/n} \cdot Z^{-1}\right]^N}{1-(n)^{1/n} \cdot Z^{-1}}$$

$$\therefore X_1(Z) = \frac{1-(n)^{N/n} \cdot Z^{-N}}{1-(n)^{1/n} \cdot Z^{-1}} \quad ...(7)$$

Now $$X_2(Z) = \sum_{n=N}^{\infty} NZ^{-n} \quad ...(8)$$

Here N is constant so it can be taken out of the summation sign.

$$\therefore X_2(Z) = N \sum_{n=N}^{\infty} Z^{-n}$$

$$\therefore X_2(Z) = N \sum_{n=N}^{\infty} (Z^{-1})^n \quad ...(9)$$

Compare Equations (6) and (9)

Here, $a = Z^{-1}$, $N_1 = N$ and $N_2 = \infty$

$$\therefore X_2(Z) = N \cdot \left[\frac{(Z^{-1})^N - (Z^{-1})^{\infty+1}}{1-Z^{-1}}\right]$$

Here, $\infty + 1 = \infty$ and $Z^{-1} = \dfrac{1}{Z}$

$$\therefore\ X_2(Z) = N\left[\frac{Z^{-N} - \left(\frac{1}{Z}\right)^{\infty}}{1 - Z^{-1}}\right] = N\left[\frac{Z^{-N} - \left(\frac{1}{Z^{\infty}}\right)}{1 - Z^{-1}}\right]$$

$$\text{But}\quad \frac{1}{Z^{\infty}} = \frac{1}{\infty} = 0$$

$$\therefore\ X_2(Z) = N\left[\frac{Z^{-N}}{1 - Z^{-1}}\right]$$

$$\therefore\ X_2(Z) = \frac{NZ^{-N}}{1 - Z^{-1}} \qquad \ldots(10)$$

Putting Equations (7) and (10) in Equation (4) we get,

$$X(Z) = \frac{1 - (n)^{N/n} Z^{-N}}{1 - (n)^{1/n} Z^{-1}} + \frac{NZ^{-N}}{1 - Z^{-1}}$$

Prob. 5 : Find the Z-transform and plot the ROC of $x(n) = n^2 u(n)$.

Soln. :

Let $x(n) = n\, x_1(n)$ ...(1)

where $x_1(n) = n u(n)$

We have, $Z[n u(n)] = \frac{Z}{(Z-1)^2} = X_1(Z)$ ...(2)

According to the differentiation property we have,

$$n x_1(n) \overset{Z}{\longleftrightarrow} -Z \frac{d}{dZ} X_1(Z)$$

$$\therefore\quad Z\{n x_1(n)\} = Z\{n \cdot n u(n)\} = -Z\frac{d}{dZ}\left[\frac{Z}{(Z-1)^2}\right]$$

$$= -Z\frac{d}{dZ}\left[\frac{Z}{(Z^2 - 2Z + 1)}\right] = -Z\frac{d}{dZ}\left[Z \cdot (Z^2 - 2Z + 1)^{-1}\right]$$

$$= -Z\left\{Z \cdot \frac{d}{dZ}\left[(Z^2 - 2Z + 1)^{-1}\right] + (Z^2 - 2Z + 1)^{-1} \cdot \frac{d}{dZ}(Z)\right\}$$

$$= -Z\left[Z(-1)(Z^2-2Z+1)^{-2}\cdot(2Z-2)+(z^2-2z+1)^{-1}\cdot 1 \right]$$

$$= -Z\left[-Z(Z^2-2Z+1)^{-2}\cdot(2Z-2)+\frac{1}{(Z-1)^2}\cdot 1 \right]$$

$$= \left\{ \frac{z^2(2z-2)}{(z-1)^2}-\frac{z}{(z-1)^2} \right\}$$

$$\therefore \quad Z\{n\,x_1(n)\} = \left\{ \frac{Z^2(2Z-2)}{(Z-1)^2}+\frac{z}{(Z-1)^2} \right\}$$

ROC is same as the ROC of $n\,u(n)$. Thus ROC is $|Z|>1$.

That means ROC is exterior part of unit circle.

Theory Questions

Q. 1 Explain the various properties of ROC of Z-transform.

Q. 2 What are advantages of Z-transform ?

Q. 3 State and prove convolution property of Z-transform.

Unsolved Numericals

Q. 1 Find Z transform of following function along with ROC, $x(n) = a^n u(n)+\delta(n-5)$

Ans. :

$$X(Z) = \frac{Z}{Z-a}+z^{-5}$$

Q. 2 Determine the Z-transform of :

$x(n) = (\cos \omega_0 n)\,u(n)$

Ans. :

$$(\cos \omega_0 n)\,u(n) \overset{Z}{\longleftrightarrow} \frac{Z^2-Z\cos\omega_0}{Z^2-2Z\cos\omega_0+1}, \qquad \text{ROC is} \quad |Z|>1$$

Q. 3 Obtain Z-transform of $x(n) = a^n \cos \omega_0 n\,u(n)$.

Ans. :

$$Z\{a^n \cos\omega_0 n\,u(n)\} = \frac{Z^2-aZ\cos\omega_0}{Z^2-2\,aZ\cos\omega_0+a^2} \qquad \text{ROC} : |Z|>|a|$$

Q. 4 Determine Z transform and draw ROC of the following signal $x(n) = (2)^{n+2} u(n-1)$. Is the signal causal ?

Ans. :

$$X(Z) = \frac{8}{Z-2}, \qquad ROC : |Z| > 2$$

The given signal is causal.

Q. 5 Obtain the Z-transform of signal,

$x(n) = n a^n u(n)$

Ans. :

$$n a^n u(n) \overset{Z}{\longleftrightarrow} \frac{aZ}{(Z-a)^2} \qquad ROC : |Z| > |a|$$

Q. 6 Find Z-transform of

$x(n) = a^{|n|}$, $0 < |a| < 1$.

Ans. :

$$X(Z) = \frac{Z}{Z-a} + \frac{aZ}{1-aZ} \quad ROC : |Z| > |a| \text{ and } |Z| < \frac{1}{|a|}$$

Q. 7 Find the Z-transform of

$x(n) = \frac{a^n}{n!}$, $n \geq 0$

Ans. :

$$X(Z) = e^{a/Z}$$

Q. 8 Determine the Z-transform of the sequence :

$$x(n) = \begin{cases} n & 0 \leq n \leq N-1 \\ N & N \leq n \end{cases}$$

Ans. :

$$X(Z) = \frac{1-(n)^{N/n} Z^{-N}}{1-(n)^{1/n} Z^{-1}} + \frac{N Z^{-N}}{1-Z^{-1}}$$

❑❑❑

Analysis of LTI Systems Using Z Transform

Section No.	Name of the Topic
1.1	Introduction
1.2	Inverse Z Transform
1.3	LTI System Analysis
1.4	Transient and Steady State Response
1.5	Introduction to Analog Filter Design

• Code : DSP17

1.1 Introduction :

In this chapter we will discuss, the use of Z transform for analysis of LTI systems. If the signals are in Z domain then the stability of LTI system can be easily determined with the help of pole-zero plot. Similarly in the Z domain, causality of LTI system can be easily determined.

The solution of differential equations becomes simple in the Z domain. Using unilateral Z transform the transient response of zero state response of LTI system can be obtained. After completing analysis of systems in Z domain; it is required to convert the signals into discrete domain. For this inverse Z transform is used. Before studying analysis of system in Z domain; we will discuss different methods of inverse Z transform.

1.2 Inverse Z Transform :

In this section we will discuss how to obtain sequence x(n) from given X(Z). This procedure of obtaining x(n) from its Z-transform X(Z) is called as inverse Z-transform (IZT).

Methods for obtaining IZT :

The different methods for obtaining IZT from X(Z) are as follows :

(1) Inverse Z transform by inspection.

(2) Power series expansion.

(3) Partial fraction expansion.

(4) Residue method.

1.2.1 Inverse Z Transform by Inspection :

This is the simplest method of obtaining inverse Z transform. We have derived the equations for standard Z transform pairs. By using these standard pairs; inverse Z transform can be directly obtained.

For example if we want to obtain inverse Z transform of, $X(Z) = \frac{Z}{Z-3}, \; |Z| > 3$

Then we can use standard Z transform pair,

$$\alpha^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha}, \quad |Z| > |\alpha|$$

Here $\alpha = 3$. Thus inverse Z of $X(Z)$ is,

$$x(n) = 3^n u(n)$$

1.2.2 Power Series Expansion Method :

This method is also called as *long division method or direct division method.* Generally X(Z) is expressed as power series in 'Z'.

$$\therefore \; X(Z) = a_0 + a_1 Z^{-1} + a_2 Z^{-2} + ... + a_n Z^{-n} \qquad ...(1)$$

Equation (1) can be easily obtained by dividing numerator by the denominator of X(Z). Now according to the definition of 'Z' transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \qquad ...(2)$$

If the sequence is causal then the limits of 'n' will be from n = 0 to n = ∞.

$$\therefore \quad X(Z) = \sum_{n=0}^{\infty} x(n) Z^{-n} \qquad \ldots(3)$$

Expanding Equation (3) we get,

$$X(Z) = x(0)Z^0 + x(1)Z^{-1} + x(2)Z^{-2} + \ldots\ldots + x(n)Z^{-n}$$

But $Z^0 = 1.$

$$\therefore \quad X(Z) = x(0) + x(1)Z^{-1} + x(2)Z^{-2} + \ldots + x(n)Z^{-n} \qquad \ldots(4)$$

Now by comparing Equations (1) and (4) we can write,

$$x(0) = a_0$$

$$x(1) = a_1$$

$$x(2) = a_2$$

$$\vdots$$

$$x(n) = a_n$$

Thus the general expression of discrete time causal sequence x (n) is,

$$\boxed{x(n) = a_n \qquad n \geq 0} \qquad \ldots(5)$$

Solved Problems on IZT :

Prob. 1 : Determine the inverse Z-transform of

$$X(Z) = \frac{Z^2}{(0.5 - 1.5Z + Z^2)} \text{ for ROC } |Z| < 0.5 \text{ using long division method.}$$

Soln. : The given ROC is $|Z| < 0.5$. That means ROC is interior part of circle having radius r = 0.5. Now while obtaining the long divisions remember the following instructions.

In case of anti-causal sequence :

(i) Carryout the long division by writing the polynomials in the reverse order that means starting with the most negative term on the left.

(ii) In case of causal sequence, carryout the long division without changing order of the polynomial. That means starting with the most positive term on the left.

The given expression is,

$$X(Z) = \frac{Z^2}{0.5 - 1.5\,Z + Z^2} \qquad \ldots(1)$$

While carrying out the long division; always convert the given expression in the simplest form. That means as far as possible it should be in the form of 1 divided by some polynomial. Thus dividing numerator and denominator by Z^{-2} we get,

$$X(Z) = \frac{1}{0.5Z^{-2} - 1.5Z^{-1} + 1} \quad \ldots(2)$$

This is non-causal sequence. So numerator polynomial should start from maximum negative value of Z. Observe Equation (2). Here the polynomial is in the proper order. So we do not have to change the order of denominator polynomial. Now we will perform the long division as follows :

$$\begin{array}{l|l}
 & 2Z^2 + 6Z^3 + 14Z^4 + 30Z^5 \\
0.5Z^{-2} - 1.5Z^{-1} + 1 & \overline{)\ 1} \\
 & 1 - 3Z + 2Z^2 \\
 & \underline{-\quad +\quad -} \\
 & 0 + 3Z - 2Z^2 \\
 & \quad 3Z - 9Z^2 + 6Z^3 \\
 & \quad \underline{-\quad +\quad -} \\
 & \quad 7Z^2 - 6Z^3 \\
 & \quad 7Z^2 - 21Z^3 + 14Z^4 \\
 & \quad \underline{-\quad +\quad -} \\
 & \quad 15Z^3 - 14Z^4 \\
 & \quad 15Z^3 - 45Z^4 + 30Z^5 \\
 & \quad \underline{-\quad +\quad -} \\
 & \quad 31Z^4 - 30Z^5
\end{array}$$

Thus $X(Z)$ can be approximately written as,

$$X(Z) = 2Z^2 + 6Z^3 + 14Z^4 + 30Z^5 + \ldots\ldots \quad \ldots(3)$$

Now according to the definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

For anti-causal sequence; the limits of summation will be from $n = -1$ to $n = -\infty$.

$$\therefore \quad X(Z) = \sum_{n=-1}^{\infty} x(n) Z^{-n}$$

Expanding the summation we get,

$$X(Z) = x(-1)Z^1 + x(-2)Z^2 + x(-3)Z^3 + x(-4)Z^4 + x(-5)Z^5 + \ldots\ldots \quad \ldots(4)$$

Comparing Equations (3) and (4) we have,

$x(0) = 0$ $\qquad$ $x(-3) = 6$

$x(-1) = 0$ $\qquad$ $x(-4) = 14$

$x(-2) = 2$ $\qquad$ $x(-5) = 30$ ....

Thus sequence x (n) is,

$$x(n) = \{\ldots x(-5), x(-4), x(-3), x(-2), x(-1), \underset{\uparrow}{x(0)}\}$$

$$\therefore x(n) = \{\ldots 30, 14, 6, 2, \underset{\uparrow}{0}, 0\}$$

Prob. 2 :

Find inverse Z of $X(Z) = \dfrac{1-\frac{1}{2}Z^{-1}}{1-\frac{1}{4}Z^{-2}}$ $\quad |Z| > 1/2$

Soln. :

The given ROC is $|Z| > 1/2$. That means it is causal sequence. Now for the causal sequence the denominator polynomial should have the maximum power of Z on its left. The given expression has the denominator in the proper form. So we don't have to rearrange the expression. Now we will perform the long division as follows :

$$\begin{array}{r|l}
 & 1-\frac{1}{2}Z^{-1}+\frac{1}{4}Z^{-2}-\frac{1}{8}Z^{-3}\ldots\ldots \\
\hline
1-\frac{1}{4}Z^{-2} & 1-\frac{1}{2}Z^{-1} \\
 & 1-\frac{1}{4}Z^{-2} \\
 & \text{- \quad +} \\
\hline
 & -\frac{1}{2}Z^{-1}+\frac{1}{4}Z^{-2} \\
 & -\frac{1}{2}Z^{-1}+\frac{1}{8}Z^{-3} \\
 & \text{+ \quad -} \\
\hline
 & \frac{1}{4}Z^{-2}-\frac{1}{8}Z^{-3} \\
 & \frac{1}{4}Z^{-2}-\frac{1}{16}Z^{-4} \\
 & \text{- \quad +} \\
\hline
 & -\frac{1}{8}Z^{-3}+\frac{1}{16}Z^{-4} \\
 & -\frac{1}{8}Z^{-3}+\frac{1}{32}Z^{-5} \\
 & \text{+ \quad -} \\
\hline
 & \frac{1}{16}Z^{-4}-\frac{1}{32}Z^{-5}\ldots\ldots
\end{array}$$

Thus the expression X (Z) can be approximately written as,

$$X(Z) = 1-\frac{1}{2}Z^{-1}+\frac{1}{4}Z^{-2}-\frac{1}{8}Z^{-3}+\ldots\ldots \qquad \ldots(1)$$

According to the definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}.$$

The given sequence is a causal sequence. For causal sequence the limits of summation will be from n = 0 to n = ∞.

$$\therefore \quad X(Z) = \sum_{n=0}^{\infty} x(n) Z^{-n}$$

Expanding the summation we get,

$$X(Z) = x(0)Z^0 + x(1)Z^{-1} + x(2)Z^{-2} + x(3)Z^{-3} + \ldots\ldots$$

$$\therefore \quad X(Z) = x(0) + x(1)Z^{-1} + x(2)Z^{-2} + x(3)Z^{-3} + \ldots\ldots \qquad (2)$$

Comparing Equations (1) and (2) we get,

$$x(0) = 1 \qquad x(2) = \frac{1}{4}$$

$$x(1) = -\frac{1}{2} \qquad x(3) = -\frac{1}{8}$$

Thus the sequence x (n) is written as,

$$x(n) = \{x(0), x(1), x(2), x(3) \ldots\ldots\}$$

$$\therefore \quad x(n) = \left\{1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8} \ldots\ldots\right\} \qquad \ldots(3)$$

$$\uparrow$$

This is the required sequence x(n). Now Equation (3) can also be written as,

$$x(n) = \left(-\frac{1}{2}\right)^n u(n)$$

Prob. 3 : Find the inverse Z of :

$$X(Z) = \frac{Z}{Z-1} \qquad |Z| > 1$$

Soln. :

Since the given ROC is |Z| > 1; it is causal sequence. We know that for the causal sequence the denominator polynomial should have maximum power of Z on its left. Thus the given expression is in the proper form.

Note : We can also convert $X(Z) = \frac{Z}{Z-1}$ into negative powers of Z by multiplying and dividing by Z^{-1}. Thus $X(Z) = \frac{1}{1-Z^{-1}}$ and we can solve using the same method.

Now performing the long division we get,

$$
\begin{array}{r|l}
 & 1+Z^{-1}+Z^{-2}+Z^{-3}+\ldots\ldots \\
\hline
Z-1 & Z \\
 & Z-1 \\
 & - \quad + \\
\hline
 & 1 \\
 & 1-Z^{-1} \\
 & - \quad + \\
\hline
 & Z^{-1} \\
 & Z^{-1}-Z^{-2} \\
 & - \quad + \\
\hline
 & Z^{-2} \\
 & Z^{-2}-Z^{-3} \\
 & - \quad + \\
\hline
 & Z^{-3}
\end{array}
$$

Thus $X(Z)$ can be approximately written as,

$$X(Z) = 1+Z^{-1}+Z^{-2}+Z^{-3}+\ldots\ldots \quad \ldots(1)$$

Now the Z-transform of causal sequence is given by,

$$X(Z) = \sum_{n=0}^{\infty} x(n)Z^{-n}$$

Expanding the summation,

$$X(Z) = x(0)+x(1)Z^{-1}+x(2)Z^{-2}+x(3)Z^{-3}+\ldots\ldots \quad \ldots(2)$$

Comparing Equations (1) and (2) we get,

$$x(n) = \{x(0), x(1), x(2), x(3)\ldots\ldots\}$$

$$\therefore \quad x(n) = \{1, 1, 1, 1, \ldots\ldots\}$$

$$\uparrow$$

$$\therefore \quad x(n) = u(n)$$

$$\therefore \quad Z^{-1}\left\{\frac{Z}{Z-1}\right\} = u(n)$$

This is standard Z-transform pair.

Prob. 4 : Obtain inverse Z of :

$$X(Z) = \frac{Z}{Z-a} \quad \text{if} \quad |Z| < |a|$$

Soln. : Given ROC is $|Z| < |a|$. So the sequence is anti-causal. That means the numerator polynomial should have minimum power of Z on its left. Thus we can write $X(Z)$ as :

$$X(Z) = \frac{Z}{-a+Z} \quad \text{if} \quad |Z| < |a| \qquad ...(1)$$

Now we will perform the long division as follows :

$$\begin{array}{r} -a^{-1}Z - a^{-2}Z^2 - a^{-3}Z^3 \ldots. \\ \hline -a+Z \;\big)\; Z \\ Z - a^{-1}Z^2 \\ \hline a^{-1}Z^2 \\ a^{-1}Z^2 - a^{-2}Z^3 \\ \hline a^{-2}Z^3 \\ a^{-2}Z^3 - a^{-3}Z^4 \\ \hline a^{-3}Z^4 \\ \vdots \\ \vdots \end{array}$$

Thus $X(Z)$ can be approximately written as,

$$X(Z) = -a^{-1}Z^1 - a^{-2}Z^2 - a^{-3}Z^3 - \ldots\ldots \qquad ...(2)$$

According to definition of Z-transform,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n)Z^{-n}$$

For anti-causal sequence, the range of n is from $n = -1$ to $n = -\infty$

$$\therefore \; X(Z) = \sum_{n=-1}^{-\infty} x(n)Z^{-n}$$

Expanding the summation,

$$X(Z) = x(-1)Z^1 + x(-2)Z^2 + x(-3)Z^3 + \ldots\ldots \qquad ...(3)$$

Comparing Equations (2) and (3) we get,

$$x(-1) = -a^{-1}$$

$$x(-2) = -a^{-2}$$

$$x(-3) = -a^{-3} \ldots\ldots$$

Thus sequence x(n) can be written as,

$$x(n) = \left\{ \ldots\ldots -a^{-3}, -a^{-2}, -a^{-1}, \underset{\uparrow}{0} \right\} \quad \ldots(4)$$

Equation (4) can be written as,

$$x(n) = -a^n u(-n-1)$$

This is the standard Z-transform pair.

Prob. 5 : Find inverse Z-transform :

$$X(Z) = \log(1 + aZ^{-1}), \quad |Z| > |a|$$

Soln. : The given expression is,

$$X(Z) = \log(1 + aZ^{-1}), \quad |Z| > |a| \quad \ldots(1)$$

Now the power series expansion for $\log(1+x)$ with $|x| < 1$ is given by,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots\ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n \quad \ldots(2)$$

Let $x = aZ^{-1}$ Thus Equation (2) becomes,

$$X(Z) = \log(1 + aZ^{-1}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} [aZ^{-1}]^n$$

$$\therefore \quad X(Z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} a^n \cdot Z^{-n} \quad \ldots(3)$$

Now according to the definition of Z-transform,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \quad \ldots(4)$$

But the given sequence is causal, since $|Z| > |a|$. And we will assume that the sequence is present from n = 1 to n = ∞. Thus changing the limits of summation in Equation (4) we get,

$$X(Z) = \sum_{n=1}^{\infty} x(n) Z^{-n} \quad \ldots(5)$$

Now comparing Equations (3) and (5), we get,

$$x(n) = \frac{(-1)^{n+1}}{n} a^n \quad \text{if } n \geq 1$$

This is the required IZT.

Limitations of power series expansion method :

If 'n' is large then the long division will become tedious. The method is used only when the resulting pattern is simple enough or if we want the values of only first few samples.

1.2.3 Partial Fraction Expansion (P.F.E.) Method :

This is possible for the Z transforms which are rational in nature, that means they are expressed as the ratio of two polynomials.

That means

$$X(Z) = \frac{N(Z)}{D(Z)} = \frac{b_0 + b_1 Z^{-1} + b_2 Z^{-2} + ... + b_M Z^{-M}}{a_0 + a_1 Z^{-1} + a_2 Z^{-2} + ... a_N Z^{-N}} \quad ...(1)$$

Here $N(Z)$ = Numerator polynomial

$D(Z)$ = Denominator polynomial

$b_0, b_1 .... b_M$ = Coefficients of numerator

$a_0, a_1 ... a_N$ = Coefficients of denominator

M = Degree of numerator

N = Degree of denominator

Steps to follow in partial fraction expansion :

(1) Check whether the given function is in proper form or not. The function is said to be in proper form when following conditions are satisfied.

(i) The coefficient a_0 in Equation (1) should be equal to 1 ($a_0 = 1$). If $a_0 \neq 1$ then the polynomials are adjusted accordingly.

(ii) In Equation (1) $a_n \neq 1$ and the degree of numerator should be less than the degree of denominator ($M < N$). If this condition is not satisfied then the long division is carried out to make $M < N$.

Thus the first step is to obtain the function in the proper form.

(2) Multiply the numerator and denominator by Z^N. That means convert the function in terms of positive powers of Z.

(3) Obtain the equation $\frac{X(Z)}{Z}$.

(4) Factorise the denominator and obtain the roots. Then the denominator will be in the form $(Z - P_1)(Z - P_2) .... (Z - P_N)$. Here $P_1, P_2 .... P_N$ are called as poles.

(5) Write down the equation in partial fraction expansion form as follows.

$$\frac{X(Z)}{Z} = \frac{A_1}{Z - P_1} + \frac{A_2}{Z - P_2} + ... + \frac{A_N}{Z - P_N} \quad ...(2)$$

Here $A_1, A_2, ... A_N$ are coefficients. The coefficient A_K is calculated as,

$$A_K = (Z - P_K) \cdot \frac{X(Z)}{Z} \Bigg|_{Z = P_K} \quad ...(3)$$

Equation (3) indicates that the calculation $(Z - P_K) \cdot \frac{X(Z)}{Z}$ is performed by putting $Z = P_K$.

Here $P_K = P_1, P_2, \ldots$

(6) By calculating values of $A_1, A_2, \ldots A_N$, transfer 'Z' to the R.H.S. of Equation (2). Now we standard Z transform pairs to obtain inverse Z transform.

$$a^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-a}, \qquad \textbf{ROC } |Z| > |a| \qquad ...(4)$$

$$\text{and} \quad -a^n u(-n-1) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-a}, \qquad \text{ROC } |Z| < |a| \qquad ...(5)$$

Thus depending on given ROC and by using Equations (4) and (5), the sequence $x(n)$ is obtained as follows :

$$x(n) = IZT\left\{\frac{Z}{Z-P_K}\right\} = (P_K)^n u(n) \quad \text{if ROC : } |Z| > |P_K|$$

That means causal sequence. ...(6)

$$\text{and } x(n) = IZT\left\{\frac{Z}{Z-P_K}\right\} = -(P_K)^n u(-n-1) \quad \text{if ROC : } |Z| < |P_K|$$

That means anticausal sequence. ...(7)

Solved Problems :

While solving the numericals we will consider different cases depending on the nature of poles.

Case I : When there are simple poles.

Prob. 1 : Find inverse Z of the following :

$$X(Z) = \frac{1 - 1/2\,Z^{-1}}{1 + \frac{3}{4}Z^{-1} + \frac{1}{8}Z^{-2}}, \quad |Z| > \frac{1}{2}$$

Soln. :

Step I : We will check whether the given function is in proper form or not.

Here $a_0 = 1$, M = Degree of numerator = 1, N = Degree of denominator = 2

$\therefore$ M < N and $a_0 = 1$

Hence the given function is in proper form.

Note : While checking the degree of numerator and denominator, do not consider the sign.

Step II : We will convert the given function in terms of positive powers of Z. The given function is,

$$X(Z) = \frac{1 - \frac{1}{2}Z^{-1}}{1 + \frac{3}{4}Z^{-1} + \frac{1}{8}Z^{-2}} \qquad ...(1)$$

To obtain positive powers of Z; multiply numerators and denominator by Z^2.

$$\therefore \quad X(Z) = \frac{Z^2\left[1 - 1/2\,Z^{-1}\right]}{Z^2\left[1 + \frac{3}{4}Z^{-1} + \frac{1}{8}Z^{-2}\right]}$$

$$\therefore \quad X(Z) = \frac{Z^2 - \frac{1}{2}Z}{Z^2 + \frac{3}{4}Z + \frac{1}{8}} \qquad ...(2)$$

Step III : We will convert Equation (2) in the form of $\frac{X(Z)}{Z}$. To do this, take Z common from the numerator.

$$\therefore \quad X(Z) = \frac{Z\left(Z - \frac{1}{2}\right)}{Z^2 + \frac{3}{4}Z + \frac{1}{8}}$$

$$\therefore \quad \frac{X(Z)}{Z} = \frac{Z - 1/2}{Z^2 + \frac{3}{4}Z + \frac{1}{8}} \qquad ...(3)$$

Step IV : We will obtain the roots (poles) of the denominator. To obtain poles P_1 and P_2 use following equation,

$$P_1, P_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad ...(4)$$

The equation of denominator is, $Z^2 + \frac{3}{4}Z + \frac{1}{8}$

Thus here a = 1, b = $\frac{3}{4}$ and c = $\frac{1}{8}$

Putting these values in Equation (4) we get,

$$P_1, P_2 = \frac{-\frac{3}{4} \pm \sqrt{\frac{9}{16} - 4\left(1 \times \frac{1}{8}\right)}}{(2 \times 1)} = \frac{-\frac{3}{4} \pm \sqrt{\frac{9}{16} - \frac{1}{2}}}{2}$$

$$\therefore \quad P_1, P_2 = \frac{-\frac{3}{4} \pm \sqrt{\frac{9-8}{16}}}{2} = \frac{-\frac{3}{4} \pm \frac{1}{4}}{2} = -\frac{3}{8} \pm \frac{1}{8}$$

$$\therefore \quad P_1 = -\frac{3}{8} - \frac{1}{8} = -\frac{4}{8} = -\frac{1}{2}$$

and $\quad P_2 = -\frac{3}{8} + \frac{1}{8} = \frac{-3+1}{8} = -\frac{1}{4}$

Step V : We will write down Equation (3) in the partial fraction expansion form.

$$\therefore \quad \frac{X(Z)}{Z} = \frac{Z-\frac{1}{2}}{(Z-P_1)(Z-P_2)} = \frac{Z-\frac{1}{2}}{\left(Z+\frac{1}{2}\right)\left(Z+\frac{1}{4}\right)} \qquad ...(5)$$

Thus in terms of coefficients A_1 and A_2 we can write,

$$\frac{X(Z)}{Z} = \frac{A_1}{\left(Z+\frac{1}{2}\right)} + \frac{A_2}{\left(Z+\frac{1}{4}\right)} \qquad ...(6)$$

Step VI : We will obtain values of A_1 and A_2. The general expression to calculate A_K is,

$$A_K = (Z-P_K)\frac{X(Z)}{Z}\Bigg|_{Z=P_K}$$

$$\therefore \quad A_1 = (Z-P_1)\frac{X(Z)}{Z}\Bigg|_{Z=P_1}$$

Here $P_1 = -1/2$

$$\therefore \quad A_1 = \left(Z+\frac{1}{2}\right)\cdot\frac{X(Z)}{Z}\Bigg|_{Z=-\frac{1}{2}}$$

Putting the value of $\frac{X(Z)}{Z}$ from Equation (5) we get,

$$A_1 = \left(Z+\frac{1}{2}\right)\cdot\frac{Z-\frac{1}{2}}{\left(Z+\frac{1}{2}\right)\left(Z+\frac{1}{4}\right)}\Bigg|_{Z=-\frac{1}{2}}$$

$$\therefore \quad A_1 = \frac{Z-\frac{1}{2}}{Z+\frac{1}{4}}\Bigg|_{Z=-\frac{1}{2}} = \frac{-\frac{1}{2}-\frac{1}{2}}{-\frac{1}{2}+\frac{1}{4}} = \frac{-1}{\frac{-2+1}{4}} = \frac{-1}{-\frac{1}{4}} = 4$$

Similarly, $A_2 = (Z-P_2)\frac{X(Z)}{Z}\Big|_{Z=P_2}$

Here $P_2 = -\frac{1}{4}$

$$\therefore \quad A_2 = \left(Z+\frac{1}{4}\right)\cdot\frac{Z-\frac{1}{2}}{\left(Z+\frac{1}{2}\right)\left(Z+\frac{1}{4}\right)}\Bigg|_{Z=-\frac{1}{4}}$$

$$\therefore \quad A_2 = \left.\frac{Z-\frac{1}{2}}{Z+\frac{1}{2}}\right|_{Z=-\frac{1}{4}} = \frac{-\frac{1}{4}-\frac{1}{2}}{-\frac{1}{4}+\frac{1}{2}} = \frac{-\frac{3}{4}}{1/4} = -3$$

Thus $A_1 = 4$ and $A_2 = -3$.

Step VII : Putting values of A_1 and A_2 in Equation (6) we get,

$$\frac{X(Z)}{Z} = \frac{4}{Z+\frac{1}{2}} - \frac{3}{Z+\frac{1}{4}}$$

Now we will obtain equation of $X(Z)$ by transferring Z to the R.H.S.

$$\therefore \quad X(Z) = \frac{4Z}{Z+\frac{1}{2}} - \frac{3Z}{Z+\frac{1}{4}} \qquad ...(7)$$

Step VIII : The first term of Equation (7) is $4 \cdot \frac{Z}{Z+\frac{1}{2}}$. This term is causal when $|Z| > \frac{1}{2}$.

Note : The standard form is $\frac{Z}{Z+a}$. It is causal when $|Z| > |a|$.

Now the given ROC is $|Z| > \frac{1}{2}$. So first term is causal.

Now recall the standard equation,

$$IZT\left\{\frac{Z}{Z-P_K}\right\} = (P_K)^n u(n) \text{ when } |Z| > |P_K| \qquad ..(8)$$

Here $P_K = P_1 = -\frac{1}{2}$

$$\therefore \quad IZT\left\{\frac{Z}{Z+\frac{1}{2}}\right\} = \left(-\frac{1}{2}\right)^n u(n) \qquad ...(9)$$

Now consider second term. It is $\frac{Z}{Z+\frac{1}{4}}$. This term is causal when $|Z| > \frac{1}{4}$. Given ROC is $|Z| > \frac{1}{2}$. That means by default it is $> \frac{1}{4}$. Thus the second term is also causal. Thus using Equation (8).

$$IZT\left\{\frac{Z}{Z+\frac{1}{4}}\right\} = \left(-\frac{1}{4}\right)^n u(n) \qquad ...(10)$$

... since $P_2 = -\frac{1}{4}$

Thus from Equations (9) and (10) we can write IZT of given function as,

$$x(n) = 4\left(-\frac{1}{2}\right)^n u(n) - 3\left(-\frac{1}{4}\right)^n u(n)$$

Prob. 2 : Obtain inverse Z of the following :

$$X(Z) = \frac{1-\frac{1}{2}Z^{-1}}{1-\frac{1}{4}Z^{-2}}, \qquad |Z| > \frac{1}{2}$$

Soln. :

Step I : First we will check whether given function is in the proper form or not.

Here $a_0 = 1$, $M = 1$ and $N = 2$.

Since $M < N$; the function is in the proper form.

Step II : The given function is,

$$X(Z) = \frac{1-\frac{1}{2}Z^{-1}}{1-\frac{1}{4}Z^{-2}} \qquad ...(1)$$

Multiply numerator and denominator by Z^2.

$$\therefore \quad X(Z) = \frac{Z^2-\frac{1}{2}Z}{Z^2-\frac{1}{4}} \qquad ...(2)$$

Step III : We will obtain the equation of $\frac{X(Z)}{Z}$ as follows :

$$X(Z) = \frac{Z\left(Z-\frac{1}{2}\right)}{Z^2-\frac{1}{4}}$$

$$\therefore \quad \frac{X(Z)}{Z} = \frac{\left(Z-\frac{1}{2}\right)}{Z^2-\frac{1}{4}} \qquad ...(3)$$

Step IV : We will obtain the roots of denominator $Z^2-\frac{1}{4}$.

Here $Z^2-\frac{1}{4} = Z^2-\left(\frac{1}{2}\right)^2$

$\therefore$ Roots are $\left(Z-\frac{1}{2}\right)$ and $\left(Z+\frac{1}{2}\right)$

Thus the poles are,

$$\left(Z-\frac{1}{2}\right) = (Z-P_1) \Rightarrow P_1 = \frac{1}{2}$$

and $$\left(Z+\frac{1}{2}\right) = (Z-P_2) \Rightarrow P_2 = -\frac{1}{2}$$

Step V : Thus Equation (3) can be written as,

$$\frac{X(Z)}{Z} = \frac{Z-\frac{1}{2}}{\left(Z-\frac{1}{2}\right)\left(Z+\frac{1}{2}\right)} = \frac{1}{Z+\frac{1}{2}} \qquad ...(4)$$

Step VI : The partial fraction expansion form is,

$\frac{X(Z)}{Z} = \frac{A_1}{Z-P_1}$. Thus $A_1 = 1$. We do not have to calculate this value.

Step VII : Now from Equation (4);

$$X(Z) = \frac{Z}{Z+\frac{1}{2}} \qquad ...(5)$$

This function is causal if $|Z| > \frac{1}{2}$. The given ROC is $|Z| > \frac{1}{2}$.

Thus given function is causal. Now we will use the standard formula of causal sequence.

$$IZT\left\{\frac{Z}{Z-P_K}\right\} = (P_K)^n u(n)$$

$$\therefore x(n) = \left(-\frac{1}{2}\right)^n u(n)$$

Prob. 3 : Determine IZT of the following :

$$X(Z) = \frac{3}{Z-\frac{1}{4}-\frac{1}{8}Z^{-1}} \text{ ; } x(n) \text{ is causal.}$$

Soln. :

Step I : Here $a_0 = 1$, M = 0 and N = 1.

Since M < N the function is in the proper form.

Step II : The given equation is,

$$X(Z) = \frac{3}{Z-\frac{1}{4}-\frac{1}{8}Z^{-1}} \qquad ...(1)$$

To convert it into positive terms of Z, multiply and divide by Z.

$$\therefore \quad X(Z) = \frac{3Z}{Z\left(Z-\frac{1}{4}-\frac{1}{8}Z^{-1}\right)}$$

$$\therefore \quad X(Z) = \frac{3Z}{Z^2-\frac{1}{4}Z-\frac{1}{8}} \qquad ...(2)$$

Step III : The equation of $\frac{X(Z)}{Z}$ is,

$$\frac{X(Z)}{Z} = \frac{3}{Z^2 - \frac{1}{4}Z - \frac{1}{8}} \quad \ldots(3)$$

Step IV : The denominator equation is, $Z^2 - \frac{1}{4}Z - \frac{1}{8}$.

Here $a = 1$, $b = -\frac{1}{4}$ and $c = -\frac{1}{8}$

Thus poles $P_1, P_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\therefore \quad P_1, P_2 = \frac{\frac{1}{4} \pm \sqrt{\frac{1}{16} + \frac{1}{2}}}{2} = \frac{\frac{1}{4} \pm \sqrt{\frac{9}{16}}}{2} = \frac{\frac{1}{4} \pm \frac{3}{4}}{2}$$

$$\therefore \quad P_1 = \frac{1}{8} + \frac{3}{8} = \frac{4}{8} = \frac{1}{2} \text{ and } P_2 = \frac{1}{8} - \frac{3}{8} = \frac{-2}{8} = -\frac{1}{4}.$$

Thus $P_1 = \frac{1}{2}$ and $P_2 = -\frac{1}{4}$

Step V : Thus Equation (3) can be written as,

$$\frac{X(Z)}{Z} = \frac{3}{\left(Z - \frac{1}{2}\right)\left(Z + \frac{1}{4}\right)} \quad \ldots(4)$$

$$\therefore \quad \frac{X(Z)}{Z} = \frac{A_1}{\left(Z - \frac{1}{2}\right)} + \frac{A_2}{\left(Z + \frac{1}{4}\right)} \quad \ldots(5)$$

Step VI : Now we will calculate values of A_1 and A_2.

$$A_1 = (Z - P_1)\frac{X(Z)}{Z}\Big|_{Z = P_1}$$

$$\therefore \quad A_1 = \left(Z - \frac{1}{2}\right)\frac{3}{\left(Z - \frac{1}{2}\right)\left(Z + \frac{1}{4}\right)}\Bigg|_{Z = \frac{1}{2}}$$

$$\therefore \quad A_1 = \frac{3}{\frac{1}{2} + \frac{1}{4}} = \frac{3}{3/4} = 4$$

$$\text{and } A_2 = (Z - P_2)\frac{X(Z)}{Z}\Big|_{Z = P_2}$$

$$\therefore \quad A_2 = \left(Z + \frac{1}{4}\right)\frac{3}{\left(Z - \frac{1}{2}\right)\left(Z + \frac{1}{4}\right)}\Bigg|_{Z = -\frac{1}{4}}$$

$$\therefore \quad A_2 = \frac{3}{-\frac{1}{4}-\frac{1}{2}} = \frac{3}{-3/4} = -4$$

Step VII : Putting values of A_1 and A_2 in Equation (5),

$$\frac{X(Z)}{Z} = \frac{4}{Z-\frac{1}{2}} - \frac{4}{Z+\frac{1}{4}}$$

$$\therefore \quad X(Z) = 4\frac{Z}{Z-\frac{1}{2}} - 4\frac{Z}{Z+\frac{1}{4}} \qquad ...(6)$$

Step VIII : The given function is stable ; thus we will use the formula

$$IZT\left\{\frac{Z}{Z-P_K}\right\} = (P_K)^n u(n)$$

$$\therefore \quad x(n) = 4(P_1)^n u(n) - 4(P_2)^n u(n)$$

$$\therefore \quad x(n) = 4\left(\frac{1}{2}\right)^n u(n) - 4\left(-\frac{1}{4}\right)^n u(n)$$

$$\therefore \quad x(n) = 4\left[\left(\frac{1}{2}\right)^n - \left(-\frac{1}{4}\right)^n\right] u(n)$$

Prob. 4 : Determine all possible sequences $x(n)$ associated with Z transform

$$X(Z) = \frac{5Z^{-1}}{(1-2Z^{-1})(3-Z^{-1})}$$

Soln. : The given function is,

$$X(Z) = \frac{5Z^{-1}}{(1-2Z^{-1})(3-Z^{-1})} = \frac{5Z^{-1}}{3-Z^{-1}-6Z^{-1}-2Z^{-2}}$$

$$\therefore \quad X(Z) = \frac{5Z^{-1}}{3-7Z^{-1}-2Z^{-2}} \qquad ...(1)$$

Step I : Here M = 1 and N = 2 ; M < N but $a_0 = 3$. We want $a_0 = 1$.

Thus to convert the given function into proper form; multiply numerator and denominator by $\frac{1}{3}$.

$$\therefore \quad X(Z) = \frac{\frac{5}{3}Z^{-1}}{1-\frac{7}{3}Z^{-1}-\frac{2}{3}Z^{-2}}$$

Step II : Multiply numerator and denominator by Z^2

$$\therefore \quad X(Z) = \frac{\frac{5}{3}Z}{Z^2 - \frac{7}{3}Z - \frac{2}{3}}$$

Step III : The equation of $\frac{X(Z)}{Z}$ is,

$$\frac{X(Z)}{Z} = \frac{\frac{5}{3}}{Z^2 - \frac{7}{3}Z - \frac{2}{3}} \qquad ...(2)$$

Step IV : We will obtain roots of denominator

$$P_1, P_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{\frac{7}{3} \pm \sqrt{\frac{49}{9} - \frac{8}{3}}}{2} = \frac{7}{6} \pm \frac{5}{6}$$

$$\therefore \quad P_1 = \frac{7}{6} + \frac{5}{6} = 2 \text{ and } P_2 = \frac{7}{6} - \frac{5}{6} = \frac{1}{3}.$$

Step V : Thus Equation (2) can be written as,

$$\therefore \quad \frac{X(Z)}{Z} = \frac{\frac{5}{3}}{(Z-2)\left(Z - \frac{1}{3}\right)} \qquad ...(3)$$

In the form of partial fraction expansion we can write,

$$\therefore \quad \frac{X(Z)}{Z} = \frac{A_1}{Z-2} + \frac{A_2}{Z - \frac{1}{3}} \qquad ...(4)$$

Step VI : Now, $$A_1 = (Z - P_1) \cdot \frac{X(Z)}{Z} \Big|_{Z = P_1}$$

$$\therefore \quad A_1 = (Z-2) \frac{5/3}{(Z-2)(Z - \frac{1}{3})} \Big|_{Z=2}$$

$$= \frac{5/3}{2 - 1/3} = \frac{5/3}{5/3} = \mathbf{1}$$

and $$A_2 = (Z - \frac{1}{3}) \cdot \frac{5/3}{(Z-2)(Z - \frac{1}{3})} \Big|_{Z = \frac{1}{3}}$$

$$= \frac{5/3}{1/3 - 2} = \frac{5/3}{-5/3} = -1$$

Step VII : Putting values of A_1 and A_2 in Equation (4) we get,

$$\frac{X(Z)}{Z} = \frac{1}{Z-2} - \frac{1}{Z - \frac{1}{3}}$$

$$\therefore \quad X(Z) = \frac{Z}{Z-2} - \frac{Z}{Z-1/3} \qquad \ldots(5)$$

Step VIII : To obtain all possible sequences we will consider the following ROC.

(i) When ROC is $|Z| > 2$:

The first term of Equation (5) will be causal if $|Z| > 2$ and the second term will be causal if $|Z| > 1/3$. Now if ROC is $|Z| > 2$ then by default it is greater than 1/3. Thus for this ROC both the terms are causal.

$$\therefore \quad x(n) = IZT\left\{\frac{Z}{Z-P_K}\right\} = (P_K)^n u(n) \qquad \ldots(6)$$

$$\therefore \quad x(n) = (2)^n u(n) - (1/3)^n u(n)$$

(ii) When ROC is $2 > |Z| > 1/3$:

For the first term to be causal, the required ROC is $|Z| > 2$. But the given ROC is $2 > |Z|$ that means $|Z| < 2$. Thus first term is anticausal. We will use the standard formula of IZT for anticausal sequence

$$x(n) = IZT\left\{\frac{Z}{Z-P_K}\right\} = -(P_K)^n u(-n-1) \qquad \ldots(7)$$

Thus IZT of first term is,

$$x_1(n) = -(2)^n u(-n-1)$$

Now for the second term to be causal the required ROC is $|Z| > \frac{1}{3}$ and the given ROC is $|Z| > \frac{1}{3}$. So second term is causal. Thus using Equation (6) IZT of second term is,

$$x_2(n) = \left(\frac{1}{3}\right)^n u(n)$$

Thus IZT of total function is,

$$x(n) = x_1(n) + x_2(n)$$

$$\therefore \quad x(n) = -(2)^n u(-n-1) + \left(\frac{1}{3}\right)^n u(n)$$

(iii) When ROC is $|Z| < \frac{1}{3}$:

We know that the required ROC for first and second term to be causal are $|Z| > 2$ and $|Z| > \frac{1}{3}$ respectively. But the given ROC is $|Z| < \frac{1}{3}$. Thus both the terms will be anticausal. Now using Equation (7),

$$x(n) = -(2)^n u(-n-1) - \left(\frac{1}{3}\right)^n u(-n-1)$$

Prob. 5 : Using partial fraction expansion method find inverse Z transform of following system given by,

$$X(Z) = \frac{Z(Z^2 - 4Z + 5)}{(Z-1)(Z-2)(Z-3)}$$

for ROC being $|Z| > 3$, $2 < |Z| < 3$, $|Z| < 2$.

Soln. : The given function is,

$$X(Z) = \frac{Z(Z^2 - 4Z + 5)}{(Z-1)(Z-2)(Z-3)} \quad ...(1)$$

Step I :

$$\frac{X(Z)}{Z} = \frac{Z^2 - 4Z + 5}{(Z-1)(Z-2)(Z-3)} \quad ...(2)$$

Equation (2) is in the proper form. The roots of denominator are already given; so the poles are $P_1 = 1$, $P_2 = 2$ and $P_3 = 3$.

Step II : In the P.F.E. form Equation (2) can be written as,

$$\frac{X(Z)}{Z} = \frac{A_1}{Z-1} + \frac{A_2}{Z-2} + \frac{A_3}{Z-3} \quad ...(3)$$

We will obtain values of A_1, A_2 and A_3 as follows :

$$A_1 = (Z - P_1)\frac{X(Z)}{Z}\Big|_{Z = P_1}$$

$$\therefore A_1 = (Z-1)\cdot\frac{Z^2 - 4Z + 5}{(Z-1)(Z-2)(Z-3)}\Big|_{Z = 1}$$

$$\therefore A_1 = \frac{1 - 4 + 5}{(1-2)(1-3)} = \frac{2}{(-1)(-2)}$$

$$\therefore A_1 = +1$$

$$A_2 = (Z - P_2)\cdot\frac{X(Z)}{Z}\Big|_{Z = P_2}$$

$$\therefore A_2 = (Z-2)\cdot\frac{Z^2 - 4Z + 5}{(Z-1)(Z-2)(Z-3)}\Big|_{Z = 2}$$

$$\therefore A_2 = \frac{4 - 8 + 5}{(2-1)(2-3)} = \frac{1}{(1)(-1)}$$

$$\therefore A_2 = -1$$

$$\text{And} \quad A_3 = (Z-P_3)\frac{X(Z)}{Z}\Big|_{Z=P_3}$$

$$\therefore \quad A_3 = (Z-3)\cdot\frac{Z^2-4Z+5}{(Z-1)(Z-2)(Z-3)}\Big|_{Z=3}$$

$$\therefore \quad A_3 = \frac{9-12+5}{(3-1)(3-2)} = \frac{2}{(2)(1)}$$

$$\therefore \quad A_3 = 1$$

Step III : Putting values of A_1, A_2 and A_3 in Equation (3) we get,

$$\therefore \quad \frac{X(Z)}{Z} = \frac{1}{Z-1} - \frac{1}{Z-2} + \frac{1}{Z-3}$$

$$\therefore \quad X(Z) = \frac{Z}{Z-1} - \frac{Z}{Z-2} + \frac{Z}{Z-3} \qquad \ldots(4)$$

Step IV : We will obtain x (n) for different ROC conditions as follows :

i) For ROC : $|Z| > 3$:

Since ROC is $|Z| > 3$ then all three terms in Equation (4) are causal.

$$\therefore \quad x(n) = (1)^n u(n) - (2)^n u(n) + (3)^n u(n)$$

ii) For ROC : $2 < |Z| < 3$:

That means $|Z| > 2$ and $|Z| < 3$

For this ROC; first term is causal, second term is causal and the third term is anticausal.

$$\therefore \quad x(n) = (1)^n u(n) - (2)^n u(n) - (3)^n u(-n-1)$$

iii) For ROC : $|Z| < 2$:

In this case all terms of Equation (4), will be anti-causal.

$$\therefore \quad x(n) = -(1)^n u(-n-1) + (2)^n u(-n-1) - (3)^n u(-n-1)$$

Prob. 6 : Determine inverse Z-transform of

$$X(Z) = \frac{1}{1-1.5Z^{-1}+0.5Z^{-2}}$$

if (i) ROC : $|Z| > 1$

(ii) ROC : $|Z| < 0.5$

(iii) ROC : $0.5 < |Z| < 1.0$

Soln. :

Step I : Here $a_0 = 1$, M = 0, N = 2

Thus the given function is in the proper form.

Step II : To convert X(Z) into positive powers of Z; multiply numerator and denominator by Z^2.

$$\therefore \quad X(Z) = \frac{Z^2}{Z^2 - 1.5Z + 0.5} \qquad ...(1)$$

Step III : We will obtain the expression $\frac{X(Z)}{Z}$

$$\therefore \quad \frac{X(Z)}{Z} = \frac{Z}{Z^2 - 1.5Z + 0.5} \qquad ...(2)$$

Step IV : We will obtain the roots of denominator $Z^2 - 1.5Z + 0.5$

Here a = 1, b = – 1.5 and c = 0.5

$$\therefore \quad P_1, P_2 = \frac{1.5 \pm \sqrt{(1.5)^2 - 4 \times 0.5}}{2}$$

$$\therefore \quad P_1 = 1 \text{ and } P_2 = 0.5$$

Step V : In PFE form Equation (2) can be written as,

$$\frac{X(Z)}{Z} = \frac{A_1}{Z-1} + \frac{A_2}{Z-0.5} \qquad ...(3)$$

Step VI : We will calculate the values of A_1 and A_2 as follows :

$$A_1 = (Z - P_1)\frac{X(Z)}{Z} \cdot \Big|_{Z = P_1}$$

$$\therefore \quad A_1 = (Z-1) \cdot \frac{Z}{(Z-1)(Z-0.5)} \cdot \Big|_{Z = 1}$$

$$\therefore \quad A_1 = \frac{1}{1-0.5} = \frac{1}{0.5}$$

$$\therefore \quad A_1 = 2$$

and $$A_2 = (Z - P_2)\frac{X(Z)}{Z} \cdot \Big|_{Z = P_2}$$

$$\therefore \quad A_2 = (Z-0.5)\frac{Z}{(Z-1)(Z-0.5)} \cdot \Big|_{Z = 0.5}$$

$$\therefore \quad A_2 = \frac{0.5}{0.5-1} = \frac{0.5}{-0.5}$$

$$\therefore \quad A_2 = -1$$

Step VII : Putting values of A_1 and A_2 in Equation (3) we get,

$$\frac{X(Z)}{Z} = \frac{2}{Z-1} - \frac{1}{Z-0.5}$$

$$\therefore \quad X(Z) = \frac{2Z}{Z-1} - \frac{Z}{Z-0.5} \qquad ...(4)$$

Step VIII : We will obtain x (n) according to given ROC conditions.

(i) ROC : | Z | > 1 :

It means that | Z | > 0.5 also. So both the terms of Equation (4) are causal. For the causal sequence we have,

$$x(n) = IZT\left\{\frac{Z}{Z-P_k}\right\} = (P_k)^n u(n)$$

$$\therefore \; x(n) = 2(1)^n u(n) - (0.5)^n u(n)$$

(ii) ROC : | Z | < 0.5 :

For this ROC both the terms of Equation (4) become anticausal. And for anticausal sequence we have,

$$x(n) = IZT\left\{\frac{Z}{Z-P_k}\right\} = -(P_k)^n u(-n-1)$$

$$\therefore \; x(n) = 2[-(1)^n u(-n-1)] - [-(0.5)^n u(-n-1)]$$

(iii) ROC : 0.5 < | Z | < 1 :

For this ROC ; the first term is anticausal and second term is causal.

$$\therefore \; x(n) = 2[-(1)^n u(-n-1)] - (0.5)^n u(n)$$

Prob. 7 : Determine IZT of :

$$X(Z) = \frac{4Z^{-2} + 3Z^{-1} + 2}{\frac{1}{2}Z^{-2} - \frac{3}{2}Z^{-1} + 1} \; ; \text{ For causal sequence}$$

Soln. :

Step I : Here $a_0 = 1$, M = 2, N = 2

Since M = 2 (M not less than N); the given function is not in the proper form. To obtain it in the proper form we will perform the long division.

$$\frac{1}{2}Z^{-2}-\frac{3}{2}Z^{-1}+1 \;\Big)\; \begin{array}{r} 8 \\ \hline 4Z^{-2}+3Z^{-1}+2 \\ 4Z^{-2}-12Z^{-1}+8 \\ - \quad + \quad - \\ \hline 15Z^{-1}-6 \end{array}$$

Thus we can write, $X(Z) = Q+\frac{R}{D}$

$$\therefore \quad X(Z) = 8+\frac{15Z^{-1}-6}{\frac{1}{2}Z^{-2}-\frac{3}{2}Z^{-1}+1} \qquad ...(1)$$

Now consider,

$$X_1(Z) = \frac{15Z^{-1}-6}{\frac{1}{2}Z^{-2}-\frac{3}{2}Z^{-1}+1} \qquad ...(2)$$

First we will obtain inverse Z of $X_1(Z)$. Here M = 1 and N = 2. Thus $X_1(Z)$ is in the proper form.

Step II : Multiply numerator and denominator by Z^2.

$$\therefore \quad X_1(Z) = \frac{15Z-6Z^2}{\frac{1}{2}-\frac{3}{2}Z+Z^2} = \frac{-6Z^2+15Z}{Z^2-\frac{3}{2}Z+\frac{1}{2}}$$

Step III :
$$\therefore \quad \frac{X_1(Z)}{Z} = \frac{-6Z+15}{Z^2-\frac{3}{2}Z+\frac{1}{2}} \qquad ...(3)$$

Step IV : Now we will obtain roots of $Z^2-\frac{3}{2}Z+\frac{1}{2}$

Here a = 1, b = $-\frac{3}{2}$ and c = $\frac{1}{2}$

$$P_1, P_2 = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{\frac{3}{2}\pm\sqrt{\frac{9}{4}-2}}{2} = \frac{\frac{3}{2}\pm\sqrt{\frac{1}{4}}}{2} = \frac{\frac{3}{2}\pm\frac{1}{2}}{2}$$

$$\therefore \quad P_1, P_2 = \frac{3}{4}\pm\frac{1}{4}$$

$$\therefore \quad P_1 = \frac{3}{4}+\frac{1}{4} = 1 \quad \text{and} \quad P_2 = \frac{3}{4}-\frac{1}{4} = \frac{1}{2}$$

Step V : Thus Equation (3) can be written as,

$$\text{Thus} \quad \frac{X_1(Z)}{Z} = \frac{-6Z+15}{(Z-1)(Z-\frac{1}{2})} \qquad ...(4)$$

In the form of partial fraction expansion we can write,

$$\frac{X_1(Z)}{Z} = \frac{A_1}{(Z-P_1)} + \frac{A_2}{(Z-P_2)} \qquad ...(5)$$

Step VI :

$$\text{Now,} \quad A_1 = (Z-P_1)\frac{X_1(Z)}{Z}\Bigg|_{Z=P_1}$$

$$\therefore \quad A_1 = \frac{(Z-1)(-6Z+15)}{(Z-1)(Z-\frac{1}{2})}\Bigg|_{Z=1} = \frac{-6+15}{1-1/2} = \frac{9}{1/2} = 18$$

$$\text{and} \quad A_2 = (Z-P_2)\frac{X_1(Z)}{Z}\Bigg|_{Z=P_2}$$

$$\therefore \quad A_2 = \frac{(Z-1/2)(-6Z+15)}{(Z-1)(Z-1/2)}\Bigg|_{Z=1/2} = \frac{-6\left(\frac{1}{2}\right)+15}{\frac{1}{2}-1} = \frac{12}{-1/2} = -24$$

Step VII :

$$\text{Thus} \quad \frac{X_1(Z)}{Z} = \frac{18}{(Z-1)} - \frac{24}{(Z-1/2)}$$

$$\therefore \quad X_1(Z) = \frac{18Z}{Z-1} - \frac{24Z}{Z-1/2} \qquad ...(6)$$

Step VIII : Now given sequence is causal. Thus

$$x(n) = IZT\left\{\frac{Z}{Z-P_K}\right\} = (P_K)^n u(n)$$

∴ Thus IZT of Equation (6) is,

$$x_1(n) = 18(1)^n u(n) - 24\left(\frac{1}{2}\right)^n u(n) \qquad ...(7)$$

Now the first term in Equation (1) is 8. We have standard Z transform pair $\delta(n) \overset{Z}{\longleftrightarrow} 1$

$$\text{That means} \quad IZT\{1\} = \delta(n)$$

$$\therefore \quad IZT\{8\} = 8\delta(n)$$

Thus the total IZT of given function is,

$$\boxed{x(n) = 8\delta(n) + 18(1)^n u(n) - 24\left(\frac{1}{2}\right)^n u(n)}$$

Prob. 8 : Obtain inverse Z-transform (IZT) using PFE where

$$X(Z) = \frac{1}{(Z-1)(Z-3)}$$

Comment on ROC's of $X(Z)$.

Soln. : The given function is,

$$X(Z) = \frac{1}{(Z-1)(Z-3)}$$

$$\therefore \quad X(Z) = \frac{1}{Z^2-4Z+3} \qquad ...(1)$$

Step I : Here $a_0 = 1$, $M = 0$ and $N = 2$

Since $M < N$ the function is in the proper form.

Step II : The given function $X(Z)$ is already in terms of positive powers of Z.

Step III : To obtain $\frac{X(Z)}{Z}$ we will multiply both sides of Equation (1) by $\frac{1}{Z}$.

$$\therefore \quad \frac{X(Z)}{Z} = \frac{1}{Z(Z^2-4Z+3)}$$

$$\therefore \quad \frac{X(Z)}{Z} = \frac{1}{Z(Z-1)(Z-3)} \qquad ...(2)$$

Step IV : Equation (2) can be written as,

$$\frac{X(Z)}{Z} = \frac{1}{(Z-0)(Z-1)(Z-3)} \qquad ...(3)$$

Thus poles are, $P_1 = 0$, $P_2 = 1$ and $P_2 = 3$

Step V : In the partial fraction expansion form we can write,

$$\frac{X(Z)}{Z} = \frac{A_1}{Z} + \frac{A_2}{(Z-1)} + \frac{A_3}{(Z-3)} \qquad ...(4)$$

Step VI : We will calculate values of A_1, A_2 and A_3.

$$A_1 = (Z-P_1)\frac{X(Z)}{Z}\Bigg|_{Z=P_1} = (Z-0)\times\frac{1}{(Z-0)(Z-1)(Z-3)}\Bigg|_{Z=0} = \frac{1}{(-1)(-3)} = \frac{1}{3}$$

$$A_2 = (Z-P_2)\frac{X(Z)}{Z}\Bigg|_{Z=P_2}$$

$$\therefore A_2 = (Z-1)\frac{1}{Z(Z-1)(Z-3)}\Bigg|_{Z=1} = \frac{1}{(Z)(Z-3)}\Bigg|_{Z=1} = \frac{1}{1(1-3)} = -\frac{1}{2}$$

$$\text{and } A_3 = (Z-P_3)\frac{X(Z)}{Z}\Bigg|_{Z=P_3}$$

$$\therefore A_3 = (Z-3)\frac{1}{Z(Z-1)(Z-3)}\Bigg|_{Z=3} = \frac{1}{Z(Z-1)}\Bigg|_{Z=3} = \frac{1}{3(3-1)} = \frac{1}{6}$$

Step VII : Putting A_1, A_2 and A_3 in Equation (4) we get,

$$\frac{X(Z)}{Z} = \frac{1/3}{Z} - \frac{1/2}{(Z-1)} + \frac{1/6}{(Z-3)}$$

$$\therefore \quad X(Z) = \frac{1}{3} - \frac{1}{2}\cdot\frac{Z}{Z-1} + \frac{1}{6}\frac{Z}{Z-3} \qquad ...(5)$$

Step VIII : Comments on ROC :

The first term in equation is constant. So for this term the ROC is entire Z-plane. For the second term to be causal ROC should be $|Z| > 1$ and for the third term to be causal ROC should be $|Z| > 3$. Thus for the entire function to be causal ROC is $|Z| > 1$.

Assuming sequence is causal; IZT of Equation (5) is,

$$x(n) = \frac{1}{3}\delta(n) - \frac{1}{2}(1)^n u(n) + \frac{1}{6}(3)^n u(n)$$

Case II : When there are repeated poles

When there are repeated poles then, for obtaining A_K we have to use another method. Let us solve one example.

Prob. 9 : Determine IZT of the following :

$$X(Z) = \frac{3Z^{-3}}{\left(1-\frac{1}{4}Z^{-1}\right)^2}\ ;\ x(n) \text{ left handed system.}$$

Soln. :

Observe the denominator; there are repeated poles. Now first we will convert given function into positive powers of Z.

Step I : Multiply numerator and denominator by Z^3.

$$\therefore \quad X(Z) = \frac{Z^3 \times 3Z^{-3}}{Z^3\left(1-\frac{1}{4}Z^{-1}\right)^2}$$

$$\therefore \quad X(Z) = \frac{3}{Z \times Z^2\left(1-\frac{1}{4}Z^{-1}\right)^2}$$

Remember that when we transfer Z^2 inside the square bracket, it becomes Z

$$\therefore \quad X(Z) = \frac{3}{Z(Z\times 1 - Z\times 4Z^{-1})^2}$$

$$\therefore \quad X(Z) = \frac{3}{Z\left(Z-\frac{1}{4}\right)^2} \qquad ...(1)$$

Step II : Now we will obtain $\frac{X(Z)}{Z}$

$$\therefore \quad \frac{X(Z)}{Z} = \frac{3}{Z^2\left(Z-\frac{1}{4}\right)^2} \qquad ...(2)$$

Step III : Since there are repeated roots, partial fraction expansion form of Equation (2) can be written as :

$$\frac{X(Z)}{Z} = \frac{A_1}{Z} + \frac{A_2}{Z^2} + \frac{A_3}{\left(Z-\frac{1}{4}\right)} + \frac{A_4}{\left(Z-\frac{1}{4}\right)^2} \qquad ...(3)$$

Step IV : In Equation (3), Z^2 and $\left(Z \cdot -\frac{1}{4}\right)^2$ are original terms. So value of A_2 and A_4 is obtained by using normal method. While Z and $\left(Z-\frac{1}{4}\right)$ are extra terms. Thus value of A_1 and A_3 is obtained as follows :

$$A_1 = \frac{d}{dZ}\left[Z^2 \frac{X(Z)}{Z}\right]\Bigg|_{Z=0}$$

$$\therefore \quad A_1 = \frac{d}{dZ}\left[Z^2 \times \frac{3}{Z^2(Z-1/4)^2}\right]\Bigg|_{Z=0}$$

$$\therefore \quad A_1 = \frac{d}{dZ}\left[\frac{3}{\left(Z-\frac{1}{4}\right)^2}\right]\Bigg|_{Z=0}$$

$$\therefore \quad A_1 = \left[3(-2)\left(Z-\frac{1}{4}\right)^{-3}\right]\Bigg|_{Z=0} = 384$$

and $$A_3 = \frac{d}{dZ}\left[\left(Z-\frac{1}{4}\right)^2 \frac{X(Z)}{Z}\right]\Bigg|_{Z=\frac{1}{4}} = \frac{d}{dZ}\left[\left(Z-\frac{1}{4}\right)^2 \times \frac{3}{Z^2\left(Z-\frac{1}{4}\right)^2}\right]\Bigg|_{Z=\frac{1}{4}}$$

$$\therefore \quad A_3 = \frac{d}{dZ}\left[\frac{3}{Z^2}\right]\Bigg|_{Z=\frac{1}{4}} = [3\times(-2)Z^{-3}]\Big|_{Z=\frac{1}{4}}$$

$$\therefore \quad A_3 = \left[3(-2)\left(\frac{1}{4}\right)^{-3}\right] = -384$$

$$\text{Now} \quad A_2 = Z^2 \frac{X(Z)}{Z}\Big|_{Z=0} = Z^2\left[\frac{3}{Z^2\left(Z-\frac{1}{4}\right)^2}\right]\Bigg|_{Z=0}$$

$$\therefore \quad A_2 = \frac{3}{\left(Z-\frac{1}{4}\right)^2}\Bigg|_{Z=0} = 48$$

$$\text{And} \quad A_4 = \left(Z-\frac{1}{4}\right)^2 \frac{X(Z)}{Z}\Big|_{Z=\frac{1}{4}} = \left(Z-\frac{1}{4}\right)^2 \times \frac{3}{Z^2\left(Z-\frac{1}{4}\right)^2}\Bigg|_{Z=\frac{1}{4}}$$

$$\therefore \quad A_4 = \frac{3}{(1/4)^2} = 48$$

Step V : Now putting values of A_1, A_2, A_3 and A_4 in Equation (3).

$$\frac{X(Z)}{Z} = \frac{384}{Z} + \frac{48}{Z^2} - \frac{384}{Z-\frac{1}{4}} + \frac{48}{\left(Z-\frac{1}{4}\right)^2}$$

$$\therefore \quad X(Z) = 384 + \frac{48}{Z} - \frac{384\,Z}{Z-\frac{1}{4}} + \frac{48\,Z}{\left(Z-\frac{1}{4}\right)^2} \qquad \ldots(4)$$

Step VI : The given sequence is left-handed that means anticausal. Now we will calculate IZT of each term separately.

(i) $\text{IZT}\{384\} = 384\ \text{IZT}\{1\}$

$$\text{We know that } \delta(n) \xleftrightarrow{Z} 1 \therefore \text{IZT}\{1\} = \delta(n)$$

$$\therefore \text{IZT}\{384\} = 384\,\delta(n) \qquad \ldots(5)$$

(ii) $$\text{IZT}\left\{\frac{48}{Z}\right\} = 48\ \text{IZT}\left\{\frac{1}{Z}\right\} = 48\ \text{IZT}\left\{Z^{-1}\right\}$$

$$\text{Now we have } \delta(n-1) \xleftrightarrow{Z} Z^{-1} \therefore \text{IZT}\left\{Z^{-1}\right\} = \delta(n-1)$$

$$\therefore \quad \text{IZT}\left\{\frac{48}{Z}\right\} = 48\,\delta(n-1) \qquad \ldots(6)$$

(iii) $$IZT\left\{\frac{-384Z}{Z-\frac{1}{4}}\right\} = -384\,IZT\left\{\frac{Z}{Z-\frac{1}{4}}\right\}$$

Since the sequence is anticausal we have,

$$x(n) = IZT\left\{\frac{Z}{Z-P_K}\right\} = -(P_K)^n u(-n-1)$$

Here $P_K = \frac{1}{4}$

$$\therefore \quad IZT\left\{\frac{-384\,Z}{Z-\frac{1}{4}}\right\} = +384\left(\frac{1}{4}\right)^n u(-n-1) \qquad ...(7)$$

(iv) $$IZT\left\{\frac{48\,Z}{\left(Z-\frac{1}{4}\right)^2}\right\} = 48\,IZT\left\{\frac{Z}{\left(Z-\frac{1}{4}\right)^2}\right\}$$

Now we have standard Z-transform pair,

$$-n\,a^n u(-n-1) = \frac{a\,Z}{(Z-a)^2} \qquad \text{if } |a| < 1$$

Here $a = 1/4$

$$\therefore \quad IZT\left\{\frac{48\,Z}{\left(Z-\frac{1}{4}\right)^2}\right\} = 48\times 4\,IZT\left\{\frac{\frac{1}{4}Z}{\left(Z-\frac{1}{4}\right)^2}\right\} = -192\,n\left(\frac{1}{4}\right)^n u(-n-1) \qquad ...(8)$$

Thus IZT of total function is,

$$x(n) = 384\,\delta(n) + 48\,\delta(n-1) + 384\left(\frac{1}{4}\right)^n u(-n-1) - 192\,n\left(\frac{1}{4}\right)^n u(-n-1)$$

Prob. 10 : Determine convolution of following pairs of signals by means of Z transform :

$$x_1(n) = \left(\frac{1}{4}\right)^n u(n-1),\ x_2(n) = \left[1+\left(\frac{1}{2}\right)^n\right] u(n)$$

Soln. : According to the property of linear convolution.

$$\therefore \quad Z\left\{x_1(n) * x_2(n)\right\} = X_1(Z)\cdot X_2(Z) \qquad ...(1)$$

Step I : We will obtain Z transform of $x_1(n)$.

We have,

$$x_1(n) = \left(\frac{1}{4}\right)^n u(n-1)$$

Now according to the standard Z transform pair,

$$u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-1}, \qquad \text{ROC} : |Z| > 1$$

According to time shifting property,

$$u(n-1) \overset{Z}{\longleftrightarrow} Z^{-1}\left[\frac{Z}{Z-1}\right], \qquad \text{ROC} : |Z| > 1$$

$$\therefore \quad Z\{u(n-1)\} = \frac{1}{Z-1}, \qquad \text{ROC} : |Z| > 1$$

Using scaling property,

$$Z\left\{\left(\frac{1}{4}\right)^n u(n-1)\right\} = \frac{1}{\frac{Z}{1/4}-1}, \qquad \text{ROC} : \left|\frac{Z}{1/4}\right| > 1$$

$$\therefore \quad Z\left\{\left(\frac{1}{4}\right)^n u(n-1)\right\} = X_1(Z) = \frac{1}{4Z-1}, \quad \text{ROC} : |Z| > \frac{1}{4} \qquad \ldots(2)$$

Step II : We will obtain Z-transform of $x_2(n)$.

We have,

$$x_2(n) = \left[1+\left(\frac{1}{2}\right)^n\right]u(n)$$

$$\therefore \quad x_2(n) = u(n)+\left(\frac{1}{2}\right)^n u(n) \qquad \ldots(3)$$

We have,

$$u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-1}, \qquad \text{ROC} : |Z| > 1$$

And using scaling property we can write,

$$\left(\frac{1}{2}\right)^n u(n) \overset{Z}{\longleftrightarrow} \frac{\frac{Z}{1/2}}{\frac{Z}{1/2}-1}, \qquad \text{ROC} : \left|\frac{Z}{1/2}\right| > 1$$

$$\therefore \quad Z\left\{\left(\frac{1}{2}\right)^n u(n)\right\} = \frac{2Z}{2Z-1}, \qquad \text{ROC} : |Z| > \frac{1}{2}$$

Applying linearity property to Equation (3) we get,

$$X_2(Z) = Z\{u(n)\}+Z\left\{\left(\frac{1}{2}\right)^n u(n)\right\}$$

$$\therefore \quad X_2(Z) = \frac{Z}{Z-1}+\frac{2Z}{2Z-1} \qquad \ldots(4)$$

We have individual ROCs, $|Z| > 1$ and $|Z| > \frac{1}{2}$. Thus combined ROC is, $|Z| > 1$.

Step III : Using Equation (1),

$$X(Z) = Z\left\{x_1(n) * x_2(n)\right\} = X_1(Z) \cdot X_2(Z)$$

$$\therefore \quad X(Z) = \frac{1}{4Z-1}\left[\frac{Z}{Z-1} + \frac{2Z}{2Z-1}\right]$$

$$\therefore \quad X(Z) = \frac{1}{4Z-1}\left[\frac{Z(2Z-1)+2Z(Z-1)}{(Z-1)(2Z-1)}\right]$$

$$\therefore \quad X(Z) = \frac{1}{4Z-1}\left[\frac{2Z^2-Z+2Z^2-2Z}{(Z-1)(2Z-1)}\right]$$

$$\therefore \quad X(Z) = \frac{1}{4Z-1}\left[\frac{4Z^2-3Z}{(Z-1)(2Z-1)}\right]$$

$$\therefore \quad X(Z) = \frac{4Z^2-3Z}{(4Z-1)(Z-1)(2Z-1)} \qquad ...(5)$$

Step IV : We will perform Partial Fraction Expansion method (PFE) to obtain $x(n)$, which is the result of linear convolution.

To adjust denominator terms, Equation (5) can be written as,

$$X(Z) = \frac{\left(\frac{1}{4}\times\frac{1}{2}\right)(4Z^2-3Z)}{\frac{1}{4}(4Z-1)(Z-1)\cdot\frac{1}{2}(2Z-1)}$$

$$\therefore \quad X(Z) = \frac{\frac{1}{8}(4Z^2-3Z)}{\left(Z-\frac{1}{4}\right)(Z-1)\left(Z-\frac{1}{2}\right)}$$

$$\therefore \quad X(Z) = \frac{\frac{1}{2}Z^2-\frac{3}{8}Z}{\left(Z-\frac{1}{4}\right)\left(Z-\frac{1}{2}\right)(Z-1)}$$

$$\therefore \quad \frac{X(Z)}{Z} = \frac{\frac{1}{2}Z-\frac{3}{8}}{\left(Z-\frac{1}{4}\right)\left(Z-\frac{1}{2}\right)(Z-1)} \qquad ...(6)$$

In PFE form Equation (6) can be written as,

$$\frac{X(Z)}{Z} = \frac{A_1}{\left(Z-\frac{1}{4}\right)} + \frac{A_2}{\left(Z-\frac{1}{2}\right)} + \frac{A_3}{(Z-1)} \qquad ..(7)$$

The values of coefficients A_1, A_2 and A_3 are calculated as follows :

$$A_1 = \left(Z-\frac{1}{4}\right)\cdot\frac{X(Z)}{Z}\cdot\Bigg|_{Z=\frac{1}{4}}$$

$$\therefore\quad A_1 = \frac{\left(Z-\frac{1}{4}\right)\left(\frac{1}{2}Z-\frac{3}{8}\right)}{\left(Z-\frac{1}{4}\right)\left(Z-\frac{1}{2}\right)(Z-1)}\cdot\Bigg|_{Z=\frac{1}{4}}$$

$$\therefore\quad A_1 = \frac{\left(\frac{1}{2}\times\frac{1}{4}\right)-\frac{3}{8}}{\left(\frac{1}{4}-\frac{1}{2}\right)\left(\frac{1}{4}-1\right)} = \frac{\frac{1}{8}-\frac{3}{8}}{\left(-\frac{1}{4}\right)\left(-\frac{3}{4}\right)} = \frac{-\frac{2}{8}}{\frac{3}{16}}$$

$$\therefore\quad A_1 = -\frac{2}{8}\times\frac{16}{3} = -\frac{4}{3}$$

$$\therefore\quad A_1 = -\frac{4}{3}$$

Similarly,

$$A_2 = \left(Z-\frac{1}{2}\right)\frac{X(Z)}{Z}\cdot\Bigg|_{Z=\frac{1}{2}}$$

$$\therefore\quad A_2 = \frac{\left(Z-\frac{1}{2}\right)\left(\frac{1}{2}Z-\frac{3}{8}\right)}{\left(Z-\frac{1}{4}\right)\left(Z-\frac{1}{2}\right)(Z-1)}\cdot\Bigg|_{Z=\frac{1}{2}}$$

$$\therefore\quad A_2 = \frac{\frac{1}{4}-\frac{3}{8}}{\left(\frac{1}{2}-\frac{1}{4}\right)\left(\frac{1}{2}-1\right)} = \frac{\frac{-1}{8}}{\left(\frac{1}{4}\right)\left(-\frac{1}{2}\right)} = \frac{-\frac{1}{8}}{-\frac{1}{8}}$$

$$\therefore\quad A_2 = 1$$

And $$A_3 = (Z-1)\frac{X(Z)}{Z}\cdot\Bigg|_{Z=1}$$

$$\therefore \quad A_3 = \left.\frac{(Z-1)\left(\frac{1}{2}Z-\frac{3}{8}\right)}{\left(Z-\frac{1}{4}\right)\left(Z-\frac{1}{2}\right)(Z-1)}\right|_{Z=1}$$

$$\therefore \quad A_3 = \frac{\frac{1}{2}-\frac{3}{8}}{\left(1-\frac{1}{4}\right)\left(1-\frac{1}{2}\right)} = \frac{\frac{1}{8}}{\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)} = \frac{\frac{1}{8}}{\frac{3}{8}}$$

$$\therefore A_3 = \frac{1}{3}$$

Putting these values in Equation (7) we get,

$$\therefore \quad \frac{X(Z)}{Z} = \frac{-4/3}{Z-\frac{1}{4}} + \frac{1}{Z-\frac{1}{2}} + \frac{1/3}{Z-1}$$

$$\therefore \quad X(Z) = -\frac{4}{3}\cdot\left(\frac{Z}{Z-\frac{1}{4}}\right) + 1\left(\frac{Z}{Z-\frac{1}{2}}\right) + \frac{1}{3}\left(\frac{1}{Z-1}\right) \qquad ...(8)$$

The combined ROC is $|Z| > 1$. So every term in Equation (8) is causal. Now we have,

$$IZT\left\{\frac{Z}{Z-P_k}\right\} = \left(p_k\right)^n u(n)$$

Thus Equation (8) becomes,

$$x(n) = -\frac{4}{3}\left(\frac{1}{4}\right)^n u(n) + 1\cdot\left(\frac{1}{2}\right)^n u(n) + \frac{1}{3}(1)^n u(n)$$

Prob. 11 : Find inverse Z of :

$$X(Z) = \frac{Z^3}{(Z-1)\left(Z-\frac{1}{2}\right)^2}, \qquad |Z| > 1$$

Soln. : There are repeated poles in the denominator.

Step I : We will obtain $\frac{X(Z)}{Z}$ as follows :

$$\frac{X(Z)}{Z} = \frac{Z^2}{(Z-1)\left(Z-\frac{1}{2}\right)^2} \qquad ...(1)$$

Step II : In the partial fraction expansion from Equation (1) can be written as,

$$\frac{X(Z)}{Z} = \frac{A_1}{Z-1} + \frac{A_2}{Z-\frac{1}{2}} + \frac{A_3}{\left(Z-\frac{1}{2}\right)^2} \qquad ...(2)$$

Step III : A_1 and A_3 are calculated using normal method. While A_2 can be calculated as follows :

$$A_2 = \frac{d}{dZ}\left[\left(Z-\frac{1}{2}\right)^2 \frac{X(Z)}{Z}\right]\Bigg|_{Z=\frac{1}{2}}$$

$$\therefore \quad A_2 = \frac{d}{dZ}\left[\frac{\left(Z-\frac{1}{2}\right)^2 \times Z^2}{(Z-1)\left(Z-\frac{1}{2}\right)^2}\right]\Bigg|_{Z=1/2} = \frac{d}{dZ}\left[\frac{Z^2}{Z-1}\right]\Bigg|_{Z=1/2}$$

$$\therefore \quad A_2 = \left[Z^2 \times \frac{d}{dZ}(Z-1)^{-1} + (Z-1)^{-1}\frac{d}{dZ}Z^2\right]\Bigg|_{Z=1/2}$$

$$\therefore \quad A_2 = \left[Z^2(-1)\times(Z-1)^{-2} + (Z-1)^{-1}\times 2Z\right]\Bigg|_{Z=1/2}$$

$$\therefore \quad A_2 = \left[\frac{1}{4}(-1)\times\frac{1}{\left(\frac{1}{2}-1\right)^2} + \frac{1}{\left(\frac{1}{2}-1\right)}\times 2\times\frac{1}{2}\right] = \left[\frac{-1}{4}\times(4) + \left(\frac{1}{-\frac{1}{2}}\times 1\right)\right]$$

$$\therefore \quad A_2 = -3$$

$$\text{Now, } A_1 = (Z-1)\frac{X(Z)}{Z}\Bigg|_{Z=1} = \frac{(Z-1)Z^2}{(Z-1)\left(Z-\frac{1}{2}\right)^2}\Bigg|_{Z=1}$$

$$\therefore \quad A_1 = \frac{Z^2}{\left(Z-\frac{1}{2}\right)^2}\Bigg|_{Z=1} = \frac{1}{\left(1-\frac{1}{2}\right)^2} = 4.$$

$$\text{And} \quad A_3 = (Z-1/2)^2\frac{X(Z)}{Z}\Bigg|_{Z=\frac{1}{2}} = \frac{(Z-1/2)^2\cdot Z^2}{(Z-1)\left(Z-\frac{1}{2}\right)^2}\Bigg|_{Z=1/2}$$

$$\therefore \quad A_3 = \left.\frac{Z^2}{Z-1}\right|_{Z=1/2}$$

$$\therefore \quad A_3 = \frac{1/4}{1/2-1} = \frac{1/4}{-1/2} = -1/2$$

Step IV : Putting the value of A_1, A_2 and A_3 in Equation (2) we get,

$$\frac{X(Z)}{Z} = \frac{4}{Z-1} + \frac{-3}{(Z-1/2)} + \frac{-1/2}{(Z-1/2)^2}$$

$$\therefore \quad X(Z) = \frac{4Z}{Z-1} - \frac{3Z}{Z-1/2} - \frac{1/2\,Z}{(Z-1/2)^2} \qquad ...(3)$$

Step V : Given ROC is $|Z| > 1$ thus each term in Equation (3) is causal. For the last term use the standard Z-transform pair.

$$n\,a^n u(n) \overset{Z}{\longleftrightarrow} \frac{aZ}{(Z-a)^2} \text{ if } |Z| > |a|$$

$$\therefore x(n) = 4(1)^n u(n) - 3\left(\frac{1}{2}\right)^n u(n) - n\left(\frac{1}{2}\right)^n u(n)$$

Prob. 12 : Find IZT of the following

$$X(Z) = \frac{2Z^2+3Z}{Z^3+Z+1}; \quad \text{if x(n) is causal}$$

Soln. : The given function is already in terms of positive powers of Z.

Step I : First we will obtain $\frac{X(Z)}{Z}$

$$\therefore \quad X(Z) = \frac{Z(2Z+3)}{Z^2+Z+1}$$

$$\therefore \quad \frac{X(Z)}{Z} = \frac{2Z+3}{Z^2+Z+1} \qquad ...(1)$$

Step II : Now we will obtain the roots of denominator.

$$P_1, P_2 = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2}$$

$$\therefore \quad P_1 = -\frac{1}{2} + j\frac{\sqrt{3}}{2} \quad \text{and} \quad P_2 = -\frac{1}{2} - j\frac{\sqrt{3}}{2} \ldots. \text{ [as } \sqrt{(-1)} = j \text{]}$$

Step III : Thus Equation (1) can be written as :

$$\frac{X(Z)}{Z} = \frac{2Z+3}{(Z-P_1)(Z-P_2)} = \frac{2Z+3}{\left(Z+\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)\left(Z+\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)} \quad ...(2)$$

Step IV : In the partial expansion form we can write,

$$\frac{X(Z)}{Z} = \frac{A_1}{\left(Z+\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)} + \frac{A_2}{\left(Z+\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)} \quad ...(3)$$

$$\text{Now} \quad A_1 = (Z-P_1)\frac{X(Z)}{Z}\Bigg|_{Z=P_1}$$

$$\therefore \quad A_1 = \frac{\left(Z+\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)(2Z+3)}{\left(Z+\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)\left(Z+\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)}\Bigg|_{Z=-\frac{1}{2}+j\frac{\sqrt{3}}{2}}$$

$$\therefore \quad A_1 = \frac{-1+j\sqrt{3}+3}{-\frac{1}{2}+j\frac{\sqrt{3}}{2}+\frac{1}{2}+j\frac{\sqrt{3}}{2}} = \frac{2+j\sqrt{3}}{j\sqrt{3}}$$

$$\text{Similarly,} \quad A_2 = (Z-P_2)\frac{X(Z)}{Z}\Bigg|_{Z=P_2}$$

$$\therefore \quad A_2 = \frac{\left(Z+\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)(2Z+3)}{\left(Z+\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)\left(Z+\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)}\Bigg|_{Z=-\frac{1}{2}-j\frac{\sqrt{3}}{2}}$$

$$\therefore \quad A_2 = \frac{-1-j\sqrt{3}+3}{-\frac{1}{2}-j\frac{\sqrt{3}}{2}+\frac{1}{2}-j\frac{\sqrt{3}}{2}} = \frac{2-j\sqrt{3}}{-j\sqrt{3}}$$

Step V : Putting values of A_1 and A_2 in Equation (3) we get,

$$\frac{X(Z)}{Z} = \frac{\frac{(2+j\sqrt{3})}{j\sqrt{3}}}{Z+\frac{1}{2}-j\frac{\sqrt{3}}{2}} + \frac{\frac{(2-j\sqrt{3})}{-j\sqrt{3}}}{Z+\frac{1}{2}+j\frac{\sqrt{3}}{2}}$$

$$\therefore \quad \frac{X(Z)}{Z} = \frac{1}{j\sqrt{3}}\left[\frac{2+j\sqrt{3}}{Z+\frac{1}{2}-j\frac{\sqrt{3}}{2}} - \frac{2-j\sqrt{3}\,Z}{Z+\frac{1}{2}+j\frac{\sqrt{3}}{2}}\right]$$

Transferring Z to R.H.S. we get,

$$\therefore \quad X(Z) = \frac{1}{j\sqrt{3}}\left[\frac{(2+j\sqrt{3}\,Z)}{Z-\left(-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)} - \frac{(2-j\sqrt{3})Z}{Z-\left(-\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)}\right] \qquad ...(4)$$

Step VI : The given sequence is causal; so we can use the formula,

$$x(n) = IZT\left\{\frac{Z}{Z-P_K}\right\} = (P_K)^n u(n)$$

$$\therefore \quad x(n) = \frac{1}{j\sqrt{3}}\left[(2+j\sqrt{3})\cdot\left(-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)^n u(n) - \left(2-j\sqrt{3}\right)\left(-\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)^n u(n)\right]$$

$$\therefore \quad x(n) = \left[\left(\frac{2+j\sqrt{3}}{j\sqrt{3}}\right)\cdot\left(-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)^n u(n) + \left(\frac{-2+j\sqrt{3}}{j\sqrt{3}}\right)\left(-\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)^n u(n)\right] \qquad ...(5)$$

Converting from rectangular to polar form we get,

$$x(n) = \left[\left(\frac{2.64\angle 40.89^\circ}{1.73\angle 90^\circ}\right)(1\angle 120^\circ)^n u(n) + \left(\frac{2.64\angle 139.1^\circ}{1.73\angle 90^\circ}\right)\left(1\angle -120^\circ\right)^n u(n)\right]$$

$$\therefore \quad x(n) = \left[(1.52\angle -49.11^\circ)\,(1\angle 120^\circ)\,u(n) + (1.52\angle 49.11^\circ)(1\angle -120^\circ)^n u(n)\right]$$

Now the term Magnitude $\angle$ Angle$^\circ$ can be written as Magnitude $\cdot\, e^{j\,\text{Angle}}$

$$\therefore \quad x(n) = \left[(1.52\,e^{-j\,49.11})\,(1\,e^{j\,120})^n u(n) + (1.52\,e^{+j\,49.11})(1\,e^{j(-120)})^n u(n)\right]$$

$$x(n) = \left[(1.52\,e^{-j\,49.11})\,(1^n\cdot e^{j\,120\,n})\,u(n) + (1.52\,e^{+j\,49.11})(1^n\cdot e^{-j\,120\,n})\,u(n)\right]$$

$$\therefore \quad x(n) = 1.52\,(1)^n\left[e^{-j\,49.11}\cdot e^{j\,120\,n} + e^{+j\,49.11}\cdot e^{-j\,120\,n}\right]u(n)$$

$$\therefore \quad x(n) = 1.52\,(1)^n\left[e^{-j\,(49.11-120\,n)} + e^{j\,(49.11-120\,n)}\right]u(n)$$

Now we have $e^{j\theta} + e^{-j\theta} = \cos 2\theta$

$$\therefore \quad x(n) = 1.52\,(1)^n\left[\cos 2\,(49.11^\circ - 120^\circ n)\right]u(n)$$

$$\therefore \quad x(n) = 1.52\,[\cos 2\,(49.11^\circ - 120\,n)]\,u(n)$$

Prob. 13 : Find the values of $x(0)$ and $x(\infty)$ for function $x(n)$ if $X(Z) = \dfrac{Z^2}{Z^2 - 1.2Z + 0.2}$.

Soln. : First we will obtain $x(n)$ by using P.F.E.

We have, $$X(Z) = \frac{Z^2}{Z^2 - 1.2Z + 0.2}$$

$$\therefore \frac{X(Z)}{Z} = \frac{Z}{Z^2 - 1.2Z + 0.2} \quad \ldots(1)$$

We will obtain the roots of denominator term.

Here $a = 1$, $b = -1.2$ and $c = 0.2$

$$\therefore P_1, P_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1.2 \pm \sqrt{(1.2)^2 - 4 \times 0.2}}{2}$$

$$\therefore P_1, P_2 = \frac{1.2 \pm 0.8}{2}$$

$$\therefore P_1 = 1 \text{ and } P_2 = 0.2$$

$$\therefore \frac{X(Z)}{Z} = \frac{Z}{(Z-1)(Z-0.2)} \quad \ldots(2)$$

In the P.F.E. form Equation (2) can be expressed as,

$$\frac{X(Z)}{Z} = \frac{A_1}{Z-1} + \frac{A_2}{Z-0.2} \quad \ldots(3)$$

We will calculate the values of A_1 and A_2 as follows :

$$A_1 = (Z - P_1)\frac{X(Z)}{Z}\Big|_{Z = P_1}$$

$$\therefore A_1 = (Z-1)\cdot\frac{Z}{(Z-1)(Z-0.2)}\Big|_{Z = 1}$$

$$\therefore A_1 = \frac{1}{1-0.2} = \frac{1}{0.8}$$

$$\therefore A_1 = 1.25$$

and $$A_2 = (Z - P_2)\frac{X(Z)}{Z}\Big|_{Z = P_2}$$

$$\therefore A_2 = (Z-0.2)\frac{Z}{(Z-1)(Z-0.2)}\Big|_{Z = 0.2}$$

$$\therefore \quad A_2 = \frac{0.2}{0.2-1}$$

$$\therefore A_2 = -0.25$$

Putting these values in Equation (3) we get,

$$\frac{X(Z)}{Z} = \frac{1.25}{Z-1} - \frac{0.25}{Z-0.2}$$

$$\therefore \quad X(Z) = \frac{1.25\,Z}{Z-1} - \frac{0.25\,Z}{Z-0.2} \qquad ...(4)$$

IZT of Equation (4) can be written as,

$$x(n) = 1.25\,(1)^n u(n) - 0.25\,(0.2)^n u(n)$$

Now from initial value theorem,

$$x(0) = 1.25\,(1)^0 u(0) - 0.25\,(0.2)^0 u(0)$$

$$\therefore x(0) = 1$$

From final value theorem,

$$x(\infty) = \lim_{Z \to 1} (1 - Z^{-1})\, X(Z)$$

$$\therefore \quad x(\infty) = \lim_{Z \to 1} (1 - Z^{-1}) \left[\frac{1.25\,Z}{Z-1} - \frac{0.25\,Z}{Z-0.2} \right]$$

$$= \lim_{Z \to 1} (Z-1) \left[\frac{1.25}{Z-1} - \frac{0.25}{Z-0.2} \right]$$

$$= \lim_{Z \to 1} 1.25 - \frac{0.25\,(Z-1)}{Z-0.2}$$

$$\therefore x(\infty) = 1.25$$

1.2.4 IZT using Cauchy's Residue Theorem :

Let $X(Z)$ be the function of complex variable "Z" and 'C' be the closed path in the Z plane. If the derivative $\frac{dX(Z)}{dZ}$ exists on and inside the contour "C" and if $X(Z)$ has no poles at $Z = Z_0$ then,

$$\frac{1}{2\pi j} \oint_C \frac{X(Z)}{Z - Z_0} dZ = \begin{cases} X(Z_0) & \text{if } Z_0 \text{ is inside 'C'} \\ 0 & \text{if } Z_0 \text{ is outside 'C'} \end{cases} \qquad ...(1)$$

Now if the (K + 1) order derivative of X (Z) exists and if X (Z) has no poles at $Z = Z_0$ then,

$$\frac{1}{2\pi j}\oint_C \frac{X(Z)}{(Z-Z_0)^K}dZ = \begin{cases} \left.\frac{1}{(K+1)!}\frac{d^{K-1}}{dZ^{K-1}}X(Z)\right|_{Z=Z_0} & \text{if } Z_0 \text{ is inside 'C'} \\ 0 & \text{if } Z_0 \text{ is outside 'C'} \end{cases}$$

These values on the R.H.S. are called as residues of the pole at $Z = Z_0$. Here residues are the values of the function at the particular pole.

Now we will discuss how to obtain IZT using residue theorems.

Steps to be followed :

(1) Multiply given X (Z) by Z^{n-1}

(2) Obtain the residue of $X(Z)\cdot Z^{n-1}$ using the equation,

$$R_{Z=a} = \left.\frac{d^{m-1}}{dZ^{m-1}}\left\{\frac{(Z-a)^m}{(m-1)!}X(Z)\cdot Z^{n-1}\right\}\right|_{Z=a} \quad \text{for } m \geq 1$$

$$R_{Z=a} = \left.(Z-a)^2\,X(Z)\cdot Z^{n-1}\right|_{Z=a} \quad \text{for } m = 1$$

$$R_{Z=a} = \left.\frac{d}{dZ}\left[(Z-a)^2 X(Z)\cdot Z^{n-1}\right]\right|_{Z=a} \quad \text{for } m = 2$$

(3) Inverse Z of X (Z) is obtained by adding the residues in Step (2).

Solved Problems :

Prob. 14 : Find x (n) for the given X (Z)

$$X(Z) = \frac{10Z}{(Z-1)(Z-2)}$$

Soln. : The given equation is,

$$X(Z) = \frac{10Z}{(Z-1)(Z-2)} \qquad ...(1)$$

Step I : Multiply both sides of equation (1) by Z^{n-1},

$$\therefore\ X(Z)\times Z^{n-1} = \frac{10Z^1\times Z^{n-1}}{(Z-1)(Z-2)}$$

$$\therefore\ X(Z)\times Z^{n-1} = \frac{10Z^n}{(Z-1)(Z-2)} \qquad ...(2)$$

Step II : Now the residues of $X(Z) \times Z^{n-1}$ are obtained as follows :

Note that the poles of Equation (2) are $P_1 = 1$ and $P_2 = 2$.

Thus residue at $Z = 1$ is,

$$\therefore R_{Z=1} = (Z-1) \times \frac{10 Z^n}{(Z-1)(Z-2)} \Bigg|_{Z=1}$$

$$= \frac{10 Z^n}{Z-2} \Bigg|_{Z=1} = \frac{10 \times (1)^n}{1-2}$$

$$\therefore R_{Z=1} = -10(1)^n \quad \ldots(3)$$

Now residue at $Z = 2$ is,

$$R_{Z=2} = (Z-2) \frac{10 Z^n}{(Z-1)(Z-2)} \Bigg|_{Z=2}$$

$$\therefore R_{Z=2} = \frac{10 Z^n}{(Z-1)} \Bigg|_{Z=2} = \frac{10(2)^n}{(2-1)}$$

$$\therefore R_{Z=2} = 10(2)^n \quad \ldots(4)$$

Step III : The sequence x(n) is obtained by adding Equations (3) and (4).

$$\therefore x(n) = -10(1)^n + 10(2)^n \quad \text{for } n \geq 0$$

Prob. 15 : Obtain inverse Z-transform (IZT) using residue method where $X(Z) = \frac{1}{(Z-1)(Z-3)}$

Soln. :

Step I : Multiply both sides of given equation by Z^{n-1}

$$\therefore X(Z) \times Z^{n-1} = \frac{Z^{n-1}}{(Z-1)(Z-3)} \quad \ldots(1)$$

Step II : The poles are at $Z = 1$ and $Z = 3$.

The residue at $Z = 1$ is,

$$R_{Z=1} = (Z-1) \times \frac{Z^{n-1}}{(Z-1)(Z-3)} \Bigg|_{Z=1}$$

$$= \frac{Z^{n-1}}{Z-3} \Bigg|_{Z=1} = \frac{(1)^{n-1}}{(1-3)} = \frac{(1)^n \times (1)^{-1}}{-2} = \frac{(1)^n}{-2}$$

$$\therefore \quad R_{Z=1} = -0.5\,(1)^n \qquad \ldots(2)$$

Residue at Z = 3 is,

$$R_{Z=3} = \left.\frac{(Z-3)Z^{n-1}}{(Z-1)(Z-3)}\right|_{Z=3}$$

$$= \left.\frac{Z^{n-1}}{(Z-1)}\right|_{Z=3} = \frac{(3)^{n-1}}{(3-1)} = \frac{(3)^n \times (3)^{-1}}{2}$$

$$\therefore \quad R_{Z=3} = 0.17\,(3)^n \qquad \ldots(3)$$

Step III : The sequence x (n) is,

$$x(n) = -0.5\,(1)^n + 0.17\,(3)^n \quad n \geq 0$$

Prob. 16 : Use residue method to obtain x (n) from :

$$X(Z) = \frac{Z(Z+1)}{(Z-1)^2}$$

Soln. : This is the case of repeated poles.

Step I : Multiply both sides of given equation by Z^{n-1}

$$\therefore \quad X(Z)Z^{n-1} = \frac{Z(Z+1)\times Z^{n-1}}{(Z-1)^2} = \frac{Z^1(Z+1)Z^n \times Z^{-1}}{(Z-1)^2} = \frac{(Z+1)Z^n}{(Z-1)^2}$$

$$\therefore \quad X(Z)\times Z^{n-1} = \frac{Z^n + Z^{n+1}}{(Z-1)^2} \qquad \ldots(1)$$

Step II : Now the given function has second order pole so the corresponding residue is given by,

$$R_{Z=1} = \left.\frac{d}{dZ}\left[(Z-1)^2 \times \left(\frac{Z^n + Z^{n+1}}{(Z-1)^2}\right)\right]\right|_{Z=1}$$

$$= \left.\frac{d}{dZ}\left[Z^n + Z^{n+1}\right]\right|_{Z=1} = \left. n Z^{n-1} + (n+1)Z^n \right|_{Z=1}$$

$$\therefore \quad R_{Z=1} = n\,(1)^{n-1} + (n+1)\,1^n = n\,(1)^n\,(1)^{-1} + (n+1)\,(1)^n$$

$$\therefore \quad R_{Z=1} = (2n+1)\,(1)^n$$

Step III : Thus the sequence x(n) is,

$$x(n) = (2n+1)\,(1)^n \quad n \geq 0$$

1.3 LTI System Analysis :

In this section, we will study the use of z transform for analysis of LTI system. We will disucss, how to decide the stability of system using a pole zero plot.

1.3.1 Pole Zero Plot :

Consider an LTI system with input x (n) and output y (n). In terms of z transform; we can define system transfer function [H (Z)] as the ratio of output and input.

$$\text{Thus} \quad H(Z) = \frac{Y(Z)}{X(Z)}$$

We have discussed that the Z-transform can be expressed as the ratio of two polynomials. It is called as rational Z-transform. Thus,

$$X(Z) = \frac{N(Z)}{D(Z)}$$

$$= \frac{b_0 + b_1 Z^{-1} + b_2 Z^{-2} + ... + b_M Z^{-M}}{a_0 + a_1 Z^{-1} + a_2 Z^{-2} + ... + a_N Z^{-N}} \quad ...(1)$$

(1) Poles are the values of Z for which X (Z) is ∞.

(2) Zeros are the values of Z for which X (Z) is 0.

Thus the poles are not present in the ROC of X (Z) but the zeros can be obtained in the ROC. In terms of summation, Equation (1) can be written as,

$$X(Z) = \frac{\sum_{K=0}^{M} b_K Z^{-K}}{\sum_{K=0}^{N} a_K Z^{-K}} \quad ...(2)$$

Rearranging Equation (1) we get,

$$X(Z) = \frac{Z^{-M}\left[b_0 Z^M + b_1 Z^{M-1} + b_2 Z^{M-2} + ... + b_M \right]}{Z^{-N}\left[a_0 Z^N + a_1 Z^{N-1} + a_2 Z^{N-2} + ... + a_N \right]} \quad ...(3)$$

Thus the polynomials are covered into positive powers of Z-Equation (3) can be further rearranged as,

$$X(Z) = \frac{Z^{-M} b_0 \left[Z^M + \frac{b_1}{b_0} Z^{M-1} + \frac{b_2}{b_0} Z^{M-2} + ... + \frac{b_M}{b_0} \right]}{Z^{-N} a_0 \left[Z^N + \frac{a_1}{a_0} Z^{N-1} + \frac{a_2}{a_0} Z^{N-2} + ... + \frac{a_N}{a_0} \right]} \quad ...(4)$$

Here numerator and denominator are polynomials, so they can be expressed in factored form as :

$$X(Z) = \frac{b_0}{a_0} Z^{N-M} \frac{(Z-Z_1)(Z-Z_2)....(Z-Z_M)}{(Z-P_1)(Z-P_2)....(Z-P_N)} \quad ...(5)$$

Here Z_1, Z_2 .... Z_M are zeros and P_1, P_2 ..... P_N are the poles. From Equation (5) we can conclude that,

(i) $X(Z) = 0$ for $Z = Z_1, Z_2 ..... Z_M$ (ii) $X(Z) = \infty$ for $Z = P_1, P_2 .... P_N$.

1.3.1.1 Solved Problems on Pole-zero Plot :

Prob. 1 : How the characteristic behaviour of causal discrete time signals depends on pole-zero location with respect to unit circle ? Explain with examples.

Soln. : A causal LSI system is a system present at positive values of Z. All the characteristics of such system depends on pole-zero location with respect to unit circle. We will consider few examples.

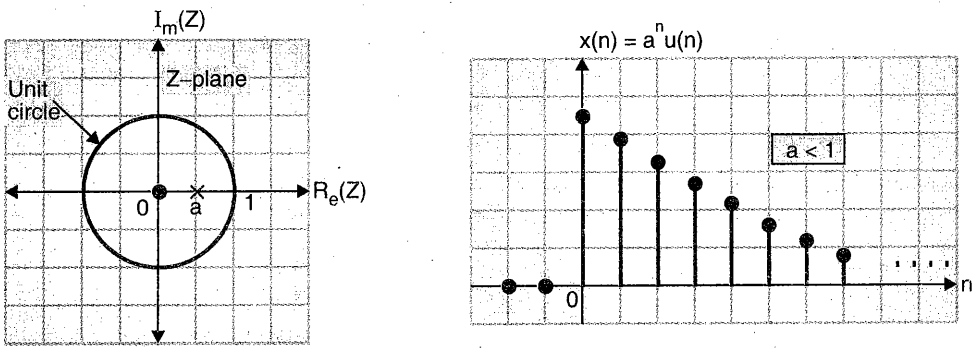


Fig. P-1(a) : Pole-zero plot of $a^n u(n)$

(i) Let $x(n) = a^n u(n)$

If $a < 1$, we will get decaying exponential. Its Z-transform is,

$$X(Z) = \frac{Z}{Z-a} = \frac{(Z-0)}{Z-a} \quad ...(1)$$

Thus zero is at $Z = 0$ (origin) and poles is at $Z = a$. It is shown in Fig. P-1(a).

Here zero is marked by '0' and pole is marked by 'x'. As shown in Fig. P-1(a); the pole is inside unit circle.

(ii) Let $x(n) = u(n)$

Then $$X(Z) = \frac{Z}{Z-1} = \frac{(Z-0)}{(Z-1)} \quad ...(2)$$

Thus zero is at $Z = 0$ (origin) and pole is at $Z = 1$ (on the unit circle.) It is shown in, Fig. P-1(b).

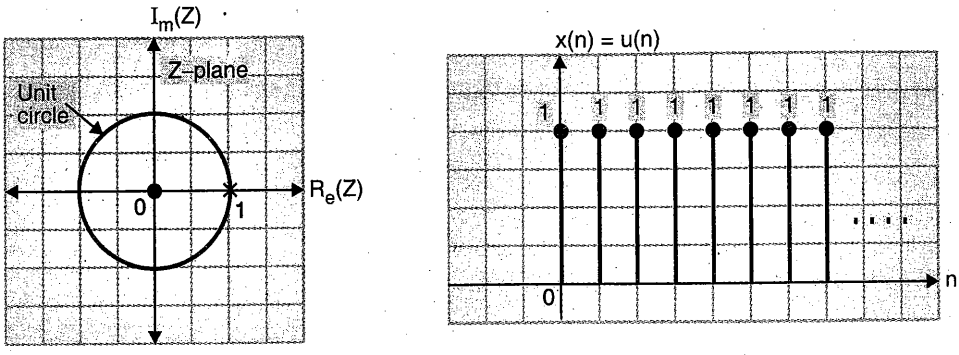


Fig. P-1(b) : Pole-zero plot of u(n)

(iii) Let $x(n) = a^n u(n)$ and $a > 1$ then we will get rising exponential.

$$\text{Now} \quad X(Z) = \frac{Z}{Z-a} = \frac{Z-0}{Z-a} \qquad \ldots(3)$$

Thus zero is at '0' (origin) and pole is at 'a'. But in this case $a > 1$. So pole is outside the unit circle. It is shown in Fig. P-1(c).

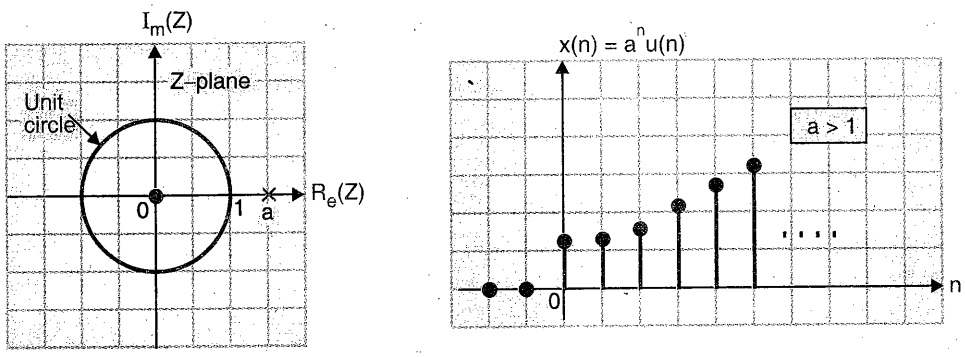


Fig. P-1(c)

(iv) Let $x(n) = -a^n u(n)$ and $a < 1$

then $$X(Z) = -\frac{Z}{Z-a} = \frac{Z}{-Z+a} \qquad \ldots(4)$$

Thus zero is at $Z = 0$' (origin) and pole is at $-Z+a = 0$ that means $Z = -a$. This pole-zero plot is shown in Fig. P-1(d).

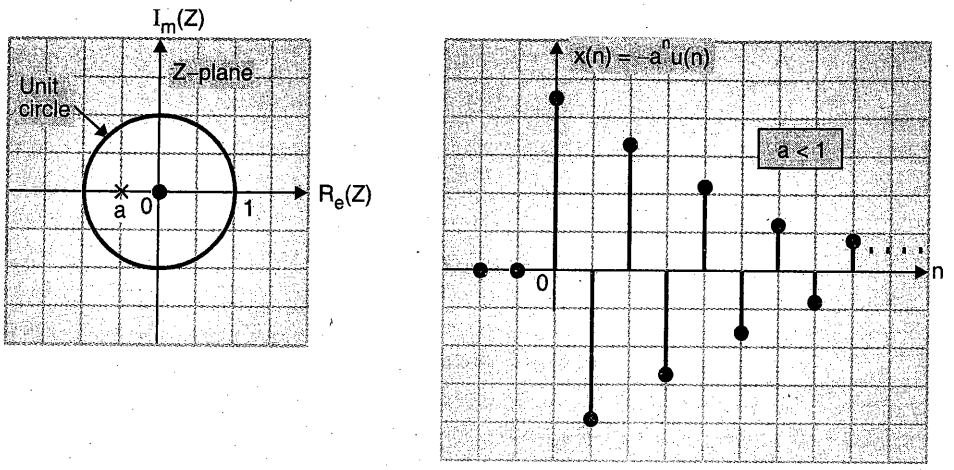


Fig. P-1(d) : $x(n) = -a^n u(n)$, if $a < 1$

(v) Let $x(n) = (-1)^n u(n)$, we have standard Z-transform pair,

$$a^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-a}$$

Let $a = -1$

$$\therefore \quad (-1)^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z+1} \qquad \ldots(5)$$

Thus zero is at Z = '0' (origin) and pole is at Z + 1 = 0 that means Z = − 1. This pole zero plot is shown in Fig. P-1(e).

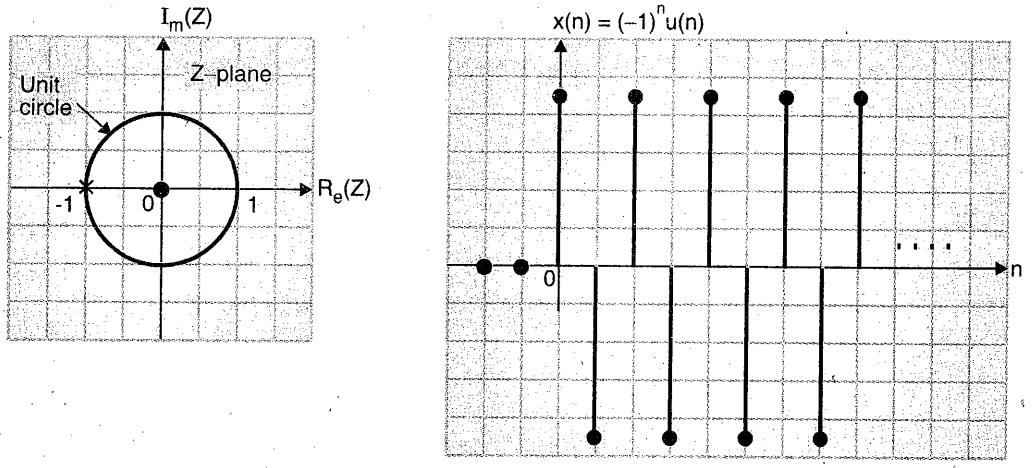


Fig. P-1(e) : $x(n) = (-1)^n u(n)$

(vi) Let $x(n) = -a^n u(n)$ and $a > 1$.

$$\therefore \quad X(Z) = -\frac{Z}{Z-a} = \frac{Z}{-Z+a} \qquad ...(6)$$

Thus zero is at $Z = 0$ (origin) and pole is at $-Z+a = 0$ that means at $Z = -a$. But in this case $a > 1$ that means $-a > -1$. This pole zero plot is shown in Fig. P-1(f).

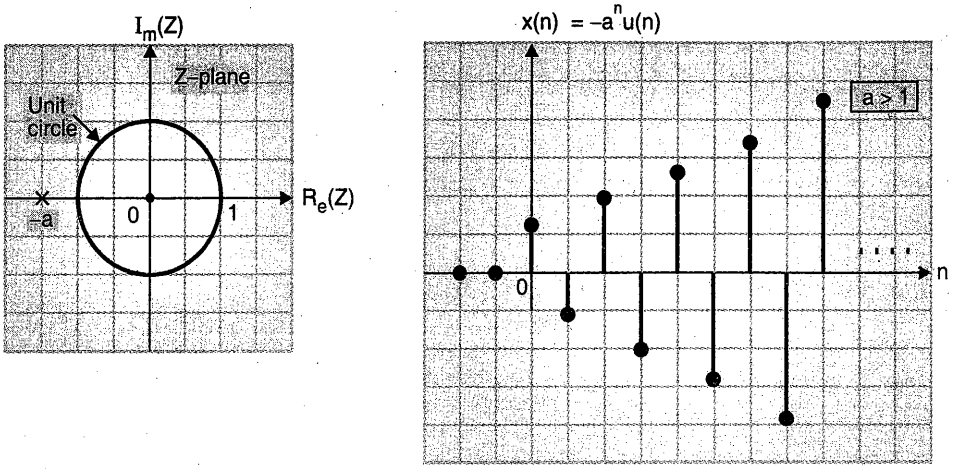


Fig. P-1(f) : $x(n) = -a^n u(n)$ if $a > 1$

From this discussion we can conclude the following points :

(1) If the pole is inside unit circle then the signal decays [Fig. P-1(a)].

(2) If the pole is on the unit circle then the signal has fixed amplitude [Fig. P-1(b)].

(3) If the pole is outside the unit circle then the amplitude of signal is increasing [Fig. P-1(c)]

(4) If there is negative pole then the signal alters in sign as shown in Figs. P-1(d), P-1(e) and P-1(f).

Also for the causal signals if the poles are outside the unit circle [Figs. P-1(c) and P-1(f)] then the signal is unstable. Because we know that for stability; ROC should include unit circle. The position of zeros also affect the behaviour of causal signal, but not as strong as the poles.

Prob. 2 : Show pole-zero diagram for a system having transfer function $H(Z) = 1 + Z^{-4}$.

Soln. : Given transfer function is,

$$H(Z) = 1 + Z^{-4} \qquad ...(1)$$

To convert it into positive powers of Z; we will divide and multiply Equation (1) by Z^4.

$$\therefore \quad H(Z) = \frac{Z^4 (1 + Z^{-4})}{Z^4}$$

$$\therefore \quad H(Z) = \frac{Z^4 + 1}{Z^4} \qquad ...(2)$$

We know that the numerator term represents zeros and denominator term represents poles of a system.

Zeros : Thus zeros are at, $Z^4 + 1 = 0$

That means $Z^4 = -1$ ...(3)

We can write $-1 = e^{j\pi}$. Because according to Euler's identity,

$$e^{j\pi} = \cos\pi + j\sin\pi;$$

where $-\cos\pi = -1$ and $\sin\pi = 0$ $\therefore e^{j\pi} = -1$

$$\therefore \quad Z^4 = e^{j\pi} \qquad ...(4)$$

Now consider the term $e^{j\pi k}$. We can write,

$$e^{j\pi k} = \cos 2\pi k + j\sin 2\pi k$$

Let 'k' be any integer. Then $\cos 2\pi k = 1$ and $\sin 2\pi k = 0$

$$\therefore \quad e^{j2\pi k} = 1$$

So we can multiply R.H.S. of Equation (4) by $e^{j2\pi k}$.

$$\therefore \quad Z^4 = e^{j\pi} \cdot e^{j2\pi k} \qquad \therefore \quad Z^4 = e^{j(2\pi k + \pi)}$$

$$\therefore \quad Z = \left[e^{j(2\pi k+\pi)}\right]^{\frac{1}{4}} \qquad \therefore \quad Z = e^{j\left(\frac{2\pi k+\pi}{4}\right)} \qquad ...(5)$$

Thus general expression for zeros can be written as,

$$Z_k = e^{j\left(\frac{2\pi k+\pi}{4}\right)} \qquad ...(6)$$

Here k is an integer. Since the power of Z in Equation (4) is '4'; the range of k is from k = 0 to k = 3.

$$\therefore \quad \text{For } k = 0 \Rightarrow Z_0 = e^{j\frac{\pi}{4}}$$

We have general expression of Z,

$$Z = r e^{j\omega}$$

Where r = Magnitude of Z

and ω = Angle

$$\therefore \quad Z_0 = e^{j\frac{\pi}{4}} \text{ represents magnitude 1 and angle } \frac{\pi}{4}.$$

$$\text{For } k = 1 \quad \Rightarrow \quad Z_1 = e^{j\left(\frac{2\pi+\pi}{4}\right)} = e^{j\left(\frac{3\pi}{4}\right)}$$

That means magnitude 1 and angle $\frac{3\pi}{4}$

$$\text{For } k = 2 \quad \Rightarrow Z_2 = e^{j\left(\frac{4\pi+\pi}{4}\right)} = e^{j\left(\frac{5\pi}{4}\right)}$$

That means magnitude 1 and angle $\frac{5\pi}{4}$.

$$\text{For } k = 3 \Rightarrow Z_3 = e^{j\left(\frac{6\pi+\pi}{4}\right)} = e^{j\left(\frac{7\pi}{4}\right)}$$

That means magnitude 1 and angle $\frac{7\pi}{4}$.

Table P-1 shows the magnitude and angles of zeros.

Table P-1

Zeros	Magnitude	Angle
Z_0	1	$\frac{\pi}{4}$
Z_1	1	$\frac{3\pi}{4}$
Z_2	1	$\frac{5\pi}{4}$
Z_3	1	$\frac{7\pi}{4}$

Note : We can also plot zeros using $e^{j\theta} = \cos\theta + j \sin\theta$.

For example $Z_0 = e^{j\frac{\pi}{4}} = \cos\frac{\pi}{4} + j \sin\frac{\pi}{4} = 0.707 + j\,0.707$.

Thus Z_0 is marked at point 0.707 and j 0.707.

Poles : Poles are obtained by equating denominator term to zero.

$$\therefore \quad Z^4 = 0$$

Thus there are four poles at origin.

The pole zero plot is as shown in Fig. P-2.

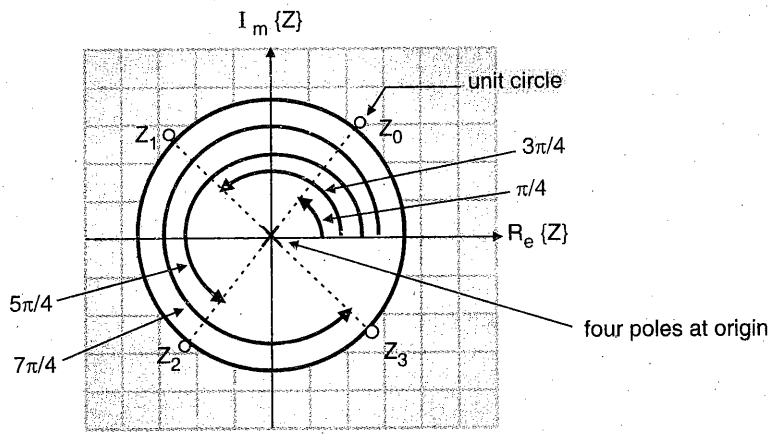


Fig. P-2

Prob. 3 : Show pole-zero diagram for a function,

$h(n) = (0.5)^n$, $0 \le n \le 7$.

Soln. : According to the definition of Z-transform,

$$H(Z) = \sum_{n=-\infty}^{\infty} h(n) Z^{-n} \quad ...(1)$$

The range of 'n' is from n = 0 to n = 7.

$$\therefore H(Z) = \sum_{n=0}^{7} h(n) Z^{-n} = \sum_{n=0}^{7} (0.5)^n \cdot Z^{-n}$$

$$H(Z) = \sum_{n=0}^{7} (0.5 Z^{-1})^n \quad ...(2)$$

We have standard summation formula,

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N & \text{for } a = 1 \\ \dfrac{1-a^N}{1-a} & \text{for } a \neq 1 \end{cases} \quad ...(3)$$

Here $a = 0.5 Z^{-1}$; that means $a \neq 1$

and comparing Equations (2) and (3);

$$N - 1 = 7 \Rightarrow N = 8$$

Thus Equation (2) becomes,

$$H(Z) = \frac{1-(0.5 Z^{-1})^8}{1-0.5 Z^{-1}} \quad ...(4)$$

Converting Equation (4) into positive powers of Z we get,

$$H(Z) = \frac{Z^8-(0.5)^8}{Z^8-0.5 Z^7} = \frac{Z^8-(0.5)^8}{Z^7(Z-0.5)} \quad ...(5)$$

Zeros : Zeros are obtained by equating numerator term to zero.

$$\therefore Z^8-(0.5)^8 = 0$$

$$\therefore Z^8 = (0.5)^8 \quad ...(6)$$

We can multiply R.H.S. by $e^{j 2\pi k}$; since $e^{j 2\pi k} = 1$.

$$\therefore Z^8 = (0.5)^8 \cdot e^{j 2\pi k} \quad ... k = 0, 1 ...7$$

$$\therefore \quad Z = \left[(0.5)^8 \cdot e^{j 2\pi k} \right]^{\frac{1}{8}}$$

$$\therefore \quad Z = 0.5\, e^{\frac{j 2\pi k}{8}} \qquad ...(7)$$

We will put values of k from k = 0 to k = 7 and we will obtain the magnitude and angles of different zeros as follows :

For k = 0 $\Rightarrow Z_0 = 0.5\, e^0 = 0.5$

$\therefore$ Magnitude = 0.5 and angle = 0

For k = 1 $\Rightarrow Z_1 = 0.5\, e^{\frac{j 2\pi}{8}} = 0.5\, e^{\frac{j\pi}{4}}$

$\therefore$ Magnitude = 0.5 and angle = $\frac{\pi}{4}$

For k = 2 $\Rightarrow Z_2 = 0.5\, e^{\frac{j 4\pi}{8}} = 0.5\, e^{\frac{j\pi}{2}}$

$\therefore$ Magnitude = 0.5 and angle = $\frac{\pi}{2}$

For k = 3 $\Rightarrow Z_3 = 0.5\, e^{\frac{j 6\pi}{8}} = 0.5\, e^{\frac{j 3\pi}{4}}$

$\therefore$ Magnitude = 0.5 and angle = $\frac{3\pi}{2}$

For k = 4 $\Rightarrow Z_4 = 0.5\, e^{\frac{j 8\pi}{8}} = 0.5\, e^{j\pi}$

$\therefore$ Magnitude = 0.5 and angle = π

For k = 5 $\Rightarrow Z_5 = 0.5\, e^{\frac{j 10\pi}{8}} = 0.5\, e^{\frac{j 5\pi}{4}}$

$\therefore$ Magnitude = 0.5 and angle = $\frac{5\pi}{4}$

For k = 6 $\Rightarrow Z_6 = 0.5\, e^{\frac{j 12\pi}{8}} = 0.5\, e^{\frac{j 3\pi}{2}}$

$\therefore$ Magnitude = 0.5 and angle = $\frac{3\pi}{2}$

For k = 7 $\Rightarrow Z_7 = 0.5\, e^{\frac{j 14\pi}{8}} = 0.5\, e^{\frac{j 7\pi}{4}}$

$\therefore$ Magnitude = 0.5 and angle = $\frac{7\pi}{4}$

Table P-2 shows magnitude and angles of different zeros.

Table P-2

Zeros	Magnitude	Angle
Z_0	0.5	0
Z_1	0.5	$\frac{\pi}{4}$
Z_2	0.5	$\frac{\pi}{2}$
Z_3	0.5	$\frac{3\pi}{4}$
Z_4	0.5	π
Z_5	0.5	$\frac{5\pi}{4}$
Z_6	0.5	$\frac{3\pi}{2}$
Z_7	0.5	$\frac{7\pi}{4}$

Poles : Poles are obtained by equating denominator term to 0.

$$\therefore \quad Z^7 (Z - 0.5) = 0$$

Thus there are seven poles at Z = 0 (origin) and one pole at Z = 0.5. The pole at Z = 0.5 cancels with zero at Z = 0.5 (Z_0). The pole zero diagram is as shown in Fig. P-3.

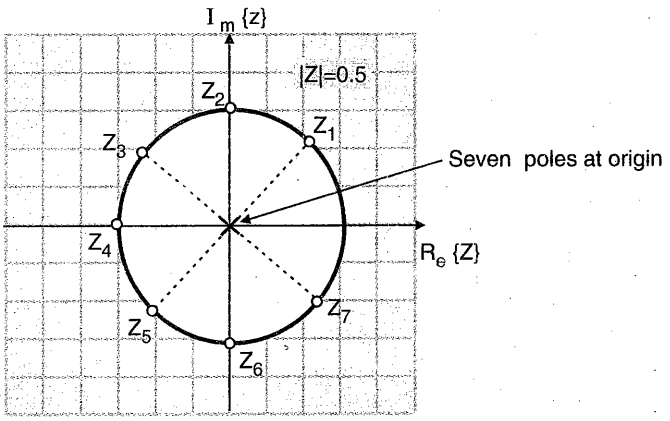


Fig. P-3

Prob. 4 : A system has impulse response h (n). Poles and zeros of system are at $Z_1 = 1$, $P_1 = -\frac{1}{2}$, $P_2 = \frac{-1}{4}$. Draw poles and zeros of a system if its impulse response is $h_1 (n) = h (n-2)$.

Soln. : The general expression of $H(Z)$ is,

$$H(Z) = \frac{(Z-Z_1)}{(Z-P_1)(Z-P_2)}$$

$$\therefore \quad H(Z) = \frac{Z-1}{\left(Z+\frac{1}{2}\right)\left(Z+\frac{1}{4}\right)} \qquad ...(1)$$

According to time shifting property,

$$x(n-k) \overset{Z}{\longleftrightarrow} Z^{-k}X(Z)$$

$$\therefore \quad h_1(n) = h(n-2) \overset{Z}{\longleftrightarrow} Z^{-2}\cdot H(Z)$$

$$\therefore \quad H_1(Z) = \frac{Z^{-2}\cdot(Z-1)}{\left(Z+\frac{1}{2}\right)\left(Z+\frac{1}{4}\right)}$$

$$\therefore \quad H_1(Z) = \frac{Z-1}{Z^2\left(Z+\frac{1}{2}\right)\left(Z+\frac{1}{4}\right)} \qquad ...(2)$$

Zeros at $Z_1 = 1$

Poles are at, $P_1, P_2 = 0$, $P_3 = \frac{-1}{2}$ and $P_4 = -1/4$.

The pole-zero diagram is shown in Fig. P-4.

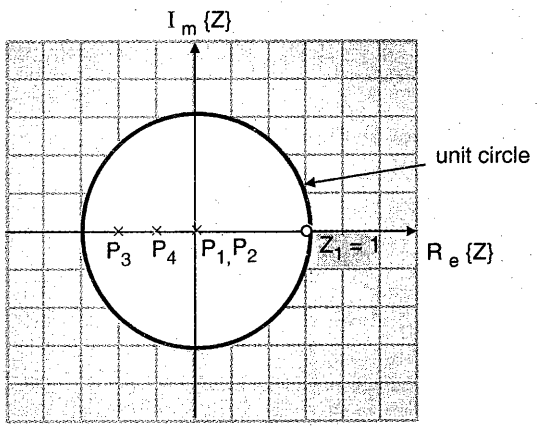


Fig. P-4

Prob. 5 : Impulse response of system is given by,

$h(n) = (0.5)^n \cos\frac{\pi}{4}n$ show pole-zero diagram.

Soln. : Given : $h(n) = (0.5)^n \cos \frac{\pi}{4} n$

We have standard Z-transform pair,

$$a^n \cos \omega_o n \quad \overset{Z}{\longleftrightarrow} \quad \frac{Z^2 - aZ \cos \omega_o}{Z^2 - 2aZ \cos \omega_o + a^2}$$

Here $a = 0.5, \quad \omega_o = \frac{\pi}{4}$

$$\therefore \quad H(Z) = \frac{Z^2 - 0.5 Z \cos\left(\frac{\pi}{4}\right)}{Z^2 - 2(0.5) Z \cos\left(\frac{\pi}{4}\right) + (0.5)^2}$$

$$\therefore \quad H(Z) = \frac{Z^2 - 0.35 Z}{Z^2 - 0.707 Z + 0.25}$$

We will obtain roots of denominator,

$$\text{roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{+0.707 \pm \sqrt{(0.707)^2 - 4(0.25)}}{2}$$

$$\therefore \quad \text{roots} = \frac{0.707 \pm \sqrt{-0.5}}{2} = \frac{0.707 \pm j\,0.707}{2}$$

$$\therefore \quad \text{roots} = 0.35 + j\,0.35 \text{ and } 0.35 - j\,0.35$$

$$\therefore \quad H(Z) = \frac{Z^2 - 0.35 Z}{(Z - 0.35 - j\,0.35)(Z - 0.35 + j\,0.35)}$$

$$\therefore \quad H(Z) = \frac{Z(Z - 0.35)}{(Z - 0.35 - j\,0.35)(Z - 0.35 + j\,0.35)}$$

Zeros are at $Z_1 = 0$ and $Z_2 = 0.35$. While poles are at $P_1 = 0.35 + j\,0.35$ and $P_2 = 0.35 - j\,0.35$.

The pole zero diagram is shown in Fig. P-5.

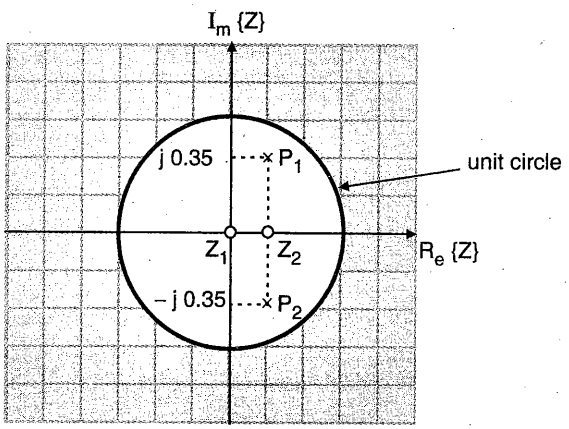


Fig. P-5

Prob. 6 : A DT system has transfer function

$H_1(Z)$ has pole at 0.6 and zero at 0.

$H_2(2)$ zero at -2 and pole at 0.

$H_3(Z) = Z^{-1}$ and $H_4(Z) = 2$.

Find overall impulse response.

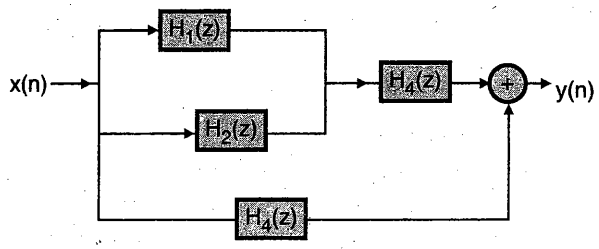


Fig. P-6

Soln. : As shown in Fig. P-6 $H_1(Z)$ and $H_2(Z)$ are in parallel and this parallel combination is in series with $H_3(Z)$. While $H_4(Z)$ is parallel to this combination. Thus overall transfer function is,

$$H(Z) = \left\{\left[H_1(Z) + H_2(Z)\right] \cdot H_3(Z)\right\} + H_4(Z) \quad \ldots(1)$$

$H_1(Z)$ has pole at 0.6 and zero at 0.

$$\therefore \quad H_1(Z) = \frac{Z}{Z - 0.6} \quad \ldots(2)$$

$H_2(Z)$ has pole at origin and zero at -2.

$$\therefore \quad H_2(Z) = \frac{Z+2}{Z} \quad \ldots(3)$$

$$H_3(Z) = Z^{-1} \text{ and } H_4(Z) = 2.$$

Putting these values in Equation (1).

$$H(Z) = \left[\left(\frac{Z}{Z-0.6} + \frac{Z+2}{Z}\right) \cdot Z^{-1}\right] + 2$$

$$\therefore \quad H(Z) = \frac{1}{Z-0.6} + \frac{1+2Z^{-1}}{Z} + 2$$

$$\therefore \quad H(Z) = Z^{-1}\left(\frac{Z}{Z-0.6}\right) + \frac{1}{Z} + \frac{2}{Z^2} + 2$$

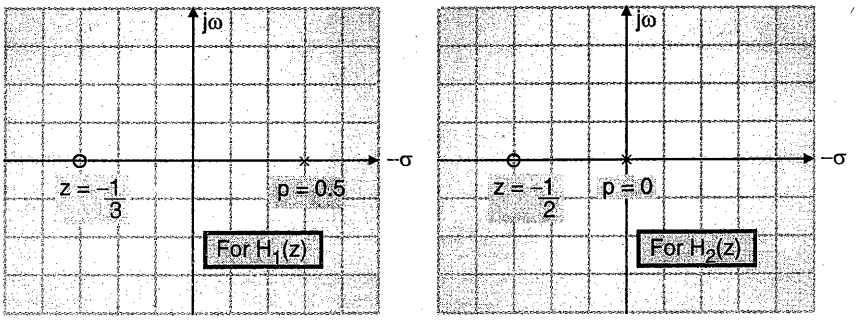
$$\therefore \quad H(Z) = Z^{-1}\left(\frac{Z}{Z-0.6}\right) + Z^{-1} + Z^{-2} \cdot 2 + 2$$

We have standard Z-transform pairs,

$$\alpha^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha} \quad \text{and} \quad \delta(n) \overset{Z}{\longleftrightarrow} 1$$

$$\therefore \quad h(n) = (0.6)^{n-1} \cdot u(n-1) + \delta(n-1) + 2\,\delta(n-2) + 2\,\delta(n)$$

Prob. 7 : A causal DT system has transfer function $H(Z)$ such that $H(Z) = H_1(Z) \cdot H_2(Z)$
The pole-zero diagram is as follows :



(a) **(b)**

Fig. P-7

(i) Find transfer function of total system.
(ii) Find difference equation of system.
(iii) Find response of system to input $x(n) = \left(-\frac{1}{2}\right)^n u(n)$

Soln. :

(i) $H_1(Z)$ has pole at $Z = 0.5$ and zero at $Z = -\frac{1}{3}$.

We have, $H_1(Z) = \frac{Z - Z_1}{Z - P_1}$

$$\therefore \quad H_1(Z) = \frac{Z + 1/3}{Z - 0.5} \qquad \ldots(1)$$

$H_2(Z)$ has zero at $Z = -1/2$ and pole at $Z = 0$.

$$\therefore \quad H_2(Z) = \frac{Z - Z_1}{Z - P_1}$$

$$\therefore \quad H_2(Z) = \frac{Z + 1/2}{Z} \qquad \ldots(2)$$

Given $H(Z) = H_1(Z) \cdot H_2(Z)$

$$\therefore \quad H(Z) = \frac{Z + \frac{1}{3}}{Z - 0.5} \cdot \frac{Z + \frac{1}{2}}{Z} \qquad \ldots(3)$$

$$\therefore \quad H(Z) = \frac{Z^2 + \frac{1}{2}Z + \frac{1}{3}Z + \frac{1}{6}}{Z(Z - 0.5)}$$

$$\therefore H(Z) = \frac{Z^2+\frac{5}{6}Z+\frac{1}{6}}{Z\left(Z-\frac{1}{2}\right)}$$

This equation gives transfer function of total system.

(ii) We have $H(Z) = \frac{Y(Z)}{X(Z)}$

$$\therefore \frac{Y(Z)}{X(Z)} = \frac{Z^2+\frac{5}{6}Z+\frac{1}{6}}{Z\left(Z-\frac{1}{2}\right)} \quad ...(4)$$

To convert R.H.S. into negative powers of Z; we will multiply numerator and denominator by Z^{-2}.

$$\therefore \frac{Y(Z)}{X(Z)} = \frac{1+\frac{5}{6}Z^{-1}+\frac{1}{6}Z^{-2}}{Z^{-1}\left(Z-\frac{1}{2}\right)} = \frac{1+\frac{5}{6}Z^{-1}+\frac{1}{6}Z^{-2}}{1-\frac{1}{2}Z^{-1}}$$

$$\therefore Y(Z)\left[1-\frac{1}{2}Z^{-1}\right] = X(Z)\left[1+\frac{5}{6}Z^{-1}+\frac{1}{6}Z^{-2}\right]$$

$$\therefore Y(Z)-\frac{1}{2}Z^{-1}Y(Z) = X(Z)+\frac{5}{6}Z^{-1}X(Z)+\frac{1}{6}Z^{-2}X(Z)$$

Taking IZT of both sides we get,

$$y(n)-\frac{1}{2}y(n-1) = x(n)+\frac{5}{6}x(n-1)+\frac{1}{6}x(n-2) \quad ...(5)$$

This is the difference equation of a system.

(iii) Given input is,

$$x(n) = \left(-\frac{1}{2}\right)^n u(n)$$

We have standard Z-transform pair.

$$a^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-a}$$

Here $a = -\frac{1}{2}$

$$\therefore X(Z) = \frac{Z}{Z+\frac{1}{2}} \quad ...(6)$$

Now $H(Z) = \frac{Y(Z)}{X(Z)}$

$$\therefore Y(Z) = H(Z)\cdot X(Z) \quad ...(7)$$

Putting Equations (3) and (6) in Equation (7)

$$Y(Z) = \left[\frac{Z+\frac{1}{3}}{Z-0.5}\cdot\frac{Z+\frac{1}{2}}{Z}\right]\cdot\frac{Z}{Z+\frac{1}{2}}$$

$$\therefore \quad Y(Z) = \frac{Z+\frac{1}{3}}{Z-0.5} \qquad \therefore \quad Y(Z) = \frac{Z+\frac{1}{3}}{Z-\frac{1}{2}}$$

We will obtain IZT using P.F.E.

$$\frac{Y(Z)}{Z} = \frac{Z+\frac{1}{3}}{Z\left(Z-\frac{1}{2}\right)} \qquad ...(8)$$

In PFE form Equation (8) can be written as,

$$\frac{Y(Z)}{Z} = \frac{A_1}{Z}+\frac{A_2}{Z-\frac{1}{2}} \qquad ...(9)$$

$$A_1 = (Z-0)\cdot\frac{Z+\frac{1}{3}}{Z\left(Z-\frac{1}{2}\right)}\Bigg|_{Z=0} = \frac{1/3}{-1/2} = -\frac{2}{3}$$

$$\text{and} \quad A_2 = \left(Z-\frac{1}{2}\right)\cdot\frac{Z+\frac{1}{3}}{Z\left(Z-\frac{1}{2}\right)}\Bigg|_{Z=\frac{1}{2}}$$

$$\therefore \quad A_2 = \frac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{2}} = \frac{5}{6}\times 2 = \frac{5}{3}$$

Thus Equation (9) becomes,

$$\frac{Y(Z)}{Z} = \frac{-2/3}{Z}+\frac{5/3}{Z-\frac{1}{2}}$$

$$\therefore \quad Y(Z) = -\frac{2}{3}+\frac{5}{3}\cdot\frac{Z}{Z-\frac{1}{2}} \qquad ...(10)$$

IZT of Equation (10) can be written as,

$$y(n) = -\frac{2}{3}\delta(n)+\frac{5}{3}\left(\frac{1}{2}\right)^n u(n)$$

This is the response of system.

1.3.2 System Transfer Function :

The linear convolution of two sequences x(n) and h(n) can be expressed as,

$$y(n) = x(n) * h(n) \quad \ldots(1)$$

We have discussed that if we take Z-transform of Equation (1) then the convolution becomes multiplication. Thus,

$$Y(Z) = X(Z) \cdot H(Z) \quad \ldots(2)$$

$$\therefore H(Z) = \frac{Y(Z)}{X(Z)} \quad \ldots(3)$$

Here H(Z) is k nown as system transfer function. Note that H(Z) is the Z-transform of h(n); where h(n) is an impulse response of the system. Thus the system is characterized by H(Z) in the Z domain. H(Z) can be obtained from its difference equation. The LSI system has a rational system function.

1.3.2.1 Solved Problems on Transfer Function :

Prob. 1 : A discrete time system has two zeros, one at Z = 0 and at Z = $\frac{1}{2}$. It has complex conjugate poles, $P_1, P_2 = 0.3\, e^{\pm j\pi/4}$. Find transfer function of system.

Soln. : The general equation of transfer function is given by,

$$H(Z) = \frac{(Z - Z_1)(Z - Z_2)}{(Z - P_1)(Z - P_2)} \quad \ldots(1)$$

Here, $Z_1 = 0$ and $Z_2 = \frac{1}{2}$

$$P_1 = 0.3\, e^{j\frac{\pi}{4}} = 0.3 \angle \frac{\pi}{4} = 0.212 + j\,0.212$$

$$P_2 = 0.3\, e^{-j\frac{\pi}{4}} = 0.3 \angle -\frac{\pi}{4} = 0.212 - j\,0.212.$$

Putting these values in Equation (1) we get,

$$H(Z) = \frac{Z\left(Z - \frac{1}{2}\right)}{(Z - 0.212 - j\,0.212)(Z - 0.212 + j\,0.212)}$$

This is the transfer function of given system.

Prob. 2 : A causal system has difference equation given by,

$$y(n) = 0.5\, y(n-1) - 0.25\, y(n-2) + x(n)$$

What is the ROC of transfer function of system.

Soln. : The given equation is,

$$y(n) = 0.5\, y(n-1) - 0.25\, y(n-2) + x(n) \quad \ldots(1)$$

Taking Z-transform of both sides,

$$Y(Z) = 0.5Z^{-1}Y(Z) - 0.25Z^{-2}Y(Z) + X(Z)$$

$$\therefore \quad Y(Z) - 0.5Z^{-1}Y(Z) + 0.25Z^{-2}Y(Z) = X(Z)$$

$$\therefore \quad Y(Z)[1 - 0.5Z^{-1} + 0.25Z^{-2}] = X(Z)$$

$$\therefore \quad \frac{Y(Z)}{X(Z)} = \frac{1}{1 - 0.5Z^{-1} + 0.25Z^{-2}} \qquad ...(2)$$

$$\text{But} \quad \frac{Y(Z)}{X(Z)} = H(Z)$$

$$\therefore \quad H(Z) = \frac{1}{1 - 0.5Z^{-1} + 0.25Z^{-2}} \qquad ..(3)$$

To convert Equation (3) into positive powers of Z; multiply numerator and denominator by Z^2.

$$\therefore \quad H(Z) = \frac{Z^2}{Z^2 - 0.5Z + 0.25} \qquad ...(4)$$

We will obtain the poles of a system by calculating roots of denominator term.

Here a = 1, b = – 0.5, c = 0.25

$$\therefore \quad P_1, P_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore \quad P_1, P_2 = \frac{0.5 \pm \sqrt{(0.5)^2 - 4 \times 0.25}}{2} = \frac{0.5 \pm \sqrt{-0.75}}{2}$$

$$\therefore \quad P_1 = 0.25 + j\,0.433 \text{ and } P_2 = 0.25 - j\,0.433$$

Now Equation (4) can be written as,

$$H(Z) = \frac{Z^2}{(Z - P_1)(Z - P_2)}$$

$$\therefore \quad H(Z) = \frac{Z^2}{(Z - 0.25 - j\,0.433)(Z - 0.25 + j\,0.433)} \qquad ...(5)$$

The given system is causal and for causal system ROC should be $|Z| > |P_1|$ or $|P_2|$.

Here $|P_1|$ or $|P_2| = \sqrt{(-0.25)^2 + (0.433)^2} = 0.499 \approx 0.5$

Thus ROC is $|Z| > 0.5$

Prob. 3 : A stable DT system has following difference equation :

$y(n) = x(n) + 0.5x(n-1) + 1.5y(n-1) + y(n-2)$

What is the ROC of transfer function of system ?

Soln. : The given function is,

$$y(n) = x(n) + 0.5x(n-1) + 1.5y(n-1) + y(n-2) \qquad ...(1)$$

Taking Z-transform of both sides we get,

$$Y(Z) = X(Z) + 0.5Z^{-1}X(Z) + 1.5Z^{-1}Y(Z) + Z^{-2}Y(Z)$$

$$\therefore \quad Y(Z) - 1.5Z^{-1}Y(Z) - Z^{-2}Y(Z) = X(Z) + 0.5Z^{-1}X(Z)$$

$$\therefore \quad Y(Z)[1 - 1.5Z^{-1} - Z^{-2}] = X(Z)[1 + 0.5Z^{-1}]$$

$$\therefore \quad \frac{Y(Z)}{X(Z)} = H(Z) = \frac{1 + 0.5Z^{-1}}{1 - 1.5Z^{-1} - Z^{-2}} \qquad ...(2)$$

To convert Equation (2) into positive powers of Z; multiply and divide by Z^2.

$$\therefore \quad H(Z) = \frac{Z^2 + 0.5Z}{Z^2 - 1.5Z - 1} \qquad ...(3)$$

To decide the location of poles we will obtain roots of denominator term. Here a = 1, b = – 1.5 and c = – 1.

$$\therefore \quad P_1, P_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1.5 \pm \sqrt{(1.5)^2 + 4}}{2}$$

$$\therefore \quad P_1, P_2 = \frac{1.5 \pm 2.5}{2}$$

$$\therefore \quad P_1 = 2 \text{ and } P_2 = -0.5$$

Thus Equation (3) can be written as,

$$H(Z) = \frac{Z^2 + 0.5Z}{(Z - 2)(Z + 0.5)}$$

It is mentioned that the system is stable. That means the ROC must include unit circle. Thus the required ROC is,

$$|Z| < 2 \text{ and } |Z| > 0.5$$

$\therefore$ ROC is $0.5 < |Z| < 2$

Prob. 4 : A causal system has difference equation,

$$y(n) = x(n) + 1.2\,x(n-1) + \frac{5}{2}y(n-1) - y(n-2)$$

What is the ROC of system transfer function ?

Soln. : The given equation is,

$$y(n) = x(n) + 1.2\,x(n-1) + \frac{5}{2}y(n-1) - y(n-2) \qquad ...(1)$$

Taking Z transform of both sides we get,

$$Y(Z) = X(Z) + 1.2Z^{-1}X(Z) + \frac{5}{2}Z^{-1}Y(Z) - Z^{-2}Y(Z)$$

$$\therefore \quad Y(Z) - \frac{5}{2}Z^{-1}Y(Z) + Z^{-2}Y(Z) = X(Z) + 1.2Z^{-1}X(Z)$$

$$\therefore \quad Y(Z)\left[1 - \frac{5}{2}Z^{-1} + Z^{-2}\right] = X(Z)[1 + 1.2Z^{-1}]$$

$$\therefore \quad \frac{Y(Z)}{X(Z)} = H(Z) = \frac{1+1.2Z^{-1}}{1-\frac{5}{2}Z^{-1}+Z^{-2}} \qquad ...(2)$$

To convert Equation (2) into positive powers of Z; multiply and divide by Z^2.

$$\therefore \quad H(Z) = \frac{Z^2+1.2Z}{Z^2-\frac{5}{2}Z+1} \qquad ...(3)$$

To decide the location of poles; we will obtain the roots of denominator.

Here $a = 1, b = -\frac{5}{2}, c = 1.$

$$\therefore \quad P_1, P_2 = \frac{\frac{5}{2} \pm \sqrt{\frac{25}{4}-4}}{2} = \frac{\frac{5}{2} \pm \sqrt{\frac{9}{4}}}{2} = \frac{\frac{5}{2} \pm \frac{3}{2}}{2}$$

$$\therefore \quad P_1 = 2 \text{ and } P_2 = \frac{1}{2}$$

Thus Equation (3) can be written as,

$$H(Z) = \frac{Z^2+1.2Z}{(Z-2)\left(Z-\frac{1}{2}\right)}$$

The given system is causal, so ROC must be greater than the larger value of pole (outer pole).

$$\therefore \text{ ROC is } |Z| > 2.$$

Prob. 5 : Write difference equation of system which generates a ramp discrete time signal as an output for $n \geq 0$.

Soln. : The ramp sequence is denoted by $n\,u(n)$.

$$\therefore \quad y(n) = n\,u(n) \qquad \text{for } n \geq 0 \qquad ...(1)$$

We have standard Z-transform pair,

$$n\,u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{(Z-1)^2}$$

Taking Z-transform of Equation (1) we get,

$$Y(Z) = \frac{Z}{(Z-1)^2} \qquad \therefore \quad Y(Z) = \frac{Z}{(Z-1)(Z-1)}$$

$$\therefore \quad Y(Z)(Z-1) = \frac{Z}{Z-1}$$

$$\therefore \quad ZY(Z)-Y(Z) = \frac{Z}{Z-1} \qquad ...(2)$$

Taking IZT of Equation (2) we get,

$$y(n+1)-y(n) = u(n)$$

$$\therefore \quad y(n) = y(n+1)-u(n)$$

This is the required difference equation.

Prob. 6 : Find difference equation of system which has response $y(n) = (0.5)^n \cos\frac{\pi n}{3}$

Soln. : Here input signal $x(n)$ is not given. So assume $x(n) = \delta(n)$

$$\therefore \quad X(Z) = 1 \qquad \ldots(1)$$

Now we have, $$y(n) = (0.5)^n \cos\frac{\pi n}{3} \qquad \ldots(2)$$

We have standard Z transform pair,

$$a^n \cos\omega_0 n \overset{Z}{\longleftrightarrow} \frac{Z^2 - aZ\cos\omega_0}{Z^2 - aZ\cos\omega_0 + a^2}$$

Here $a = 0.5$ and $\omega_0 = \frac{\pi}{3}$

$$\therefore \quad Z\{y(n)\} = Y(Z) = \frac{Z^2 - 0.5\,Z\cos\left(\frac{\pi}{3}\right)}{Z^2 - 0.5\,Z\cos\left(\frac{\pi}{3}\right) + (0.5)^2}$$

$$\therefore \quad Y(Z) = \frac{Z^2 - 0.25\,Z}{Z^2 - 0.25\,Z + 0.25} \qquad \ldots(3)$$

Now transfer function $H(Z)$ is given by,

$$H(Z) = \frac{Y(Z)}{X(Z)}$$

$$\therefore \quad H(Z) = \frac{Z^2 - 0.25\,Z}{Z^2 - 0.25\,Z + 0.25}$$

Prob. 7 : Find the difference equation satisfying the input and output of an LTI system with transfer function :

$$H(Z) = \frac{\left(1 + Z^{-1}\right)^2}{\left(1 - \frac{1}{2}Z^{-1}\right)\left(1 + \frac{3}{4}Z^{-1}\right)}$$

Soln. : The different notations used for representation of LTI systems as follows :

$x(n)$ = Input to the system and $Z\{x(n)\} = X(Z)$

$h(n)$ = Impulse response of system and $Z\{h(n)\} = H(Z)$

and $y(n)$ = Output of the system and $Z\{y(n)\} = Y(Z)$.

The given equation is,

$$H(Z) = \frac{(1+Z^{-1})^2}{\left(1-\frac{1}{2}Z^{-1}\right)\left(1+\frac{3}{4}Z^{-1}\right)} \quad ...(1)$$

But $H(Z) = \frac{Y(Z)}{X(Z)}$

$$\therefore \quad \frac{Y(Z)}{X(Z)} = \frac{(1+Z^{-1})^2}{\left(1-\frac{1}{2}Z^{-1}\right)\left(1+\frac{3}{4}Z^{-1}\right)}$$

$$\therefore \quad Y(Z)\left[\left(1-\frac{1}{2}Z^{-1}\right)\left(1+\frac{3}{4}Z^{-1}\right)\right] = X(Z)\left(1+Z^{-1}\right)^2$$

$$\therefore \quad Y(Z)\left[1+\frac{3}{4}Z^{-1}-\frac{1}{2}Z^{-1}-\frac{3}{8}Z^{-2}\right] = X(Z)\left[1+2Z^{-1}+Z^{-2}\right]$$

$$\therefore Y(Z)+\frac{3}{4}Z^{-1}Y(Z)-\frac{1}{2}Z^{-1}Y(Z)-\frac{3}{8}Z^{-2}Y(Z) = X(Z)+2Z^{-1}X(Z)+Z^{-2}X(Z) \quad ...(2)$$

Taking inverse Z-transform of Equation (2) we get,

$$y(n)+\frac{3}{4}y(n-1)-\frac{1}{2}y(n-1)-\frac{3}{8}y(n-2) = x(n)+2x(n-1)+x(n-2)$$

$$\therefore \quad y(n)-\frac{1}{4}y(n-1)-\frac{3}{8}y(n-2) = x(n)+2x(n-1)+x(n-2)$$

$$\therefore y(n) = \frac{1}{4}y(n-1)+\frac{3}{8}y(n-2)+x(n)+2x(n-1)+x(n-2)$$

This is the difference equation for the given transfer function.

Prob. 8 : Consider an LTI system which is stable and for which H(Z), the Z transform of impulse response is given by,

$$H(Z) = \frac{3-7Z^{-1}+5Z^{-2}}{1-2.5Z^{-1}+Z^{-2}}$$

Suppose x(n), the input to the system is unit step. Determine the output.

Soln. :

$$\text{Given : } H(Z) = \frac{3-7Z^{-1}+5Z^{-2}}{1-2.5Z^{-1}+Z^{-2}} \quad ...(1)$$

Given input is $x(n) = u(n)$; unit step. We have,

$$Z\{x(n)\} = X(Z) = Z\{u(n)\} = \frac{Z}{Z-1} \quad ...(2)$$

$$\text{Now, } H(Z) = \frac{Y(Z)}{X(Z)}$$

$$\therefore \quad Y(Z) = H(Z)\cdot X(Z) \quad ...(3)$$

Putting Equations (1) and (2) in Equation (3) we get,

$$Y(Z) = \left[\frac{3-7Z^{-1}+5Z^{-2}}{1-2.5Z^{-1}+Z^{-2}}\right]\times\left[\frac{Z}{Z-1}\right] \quad ...(4)$$

We will convert first bracket that means $H(Z)$ in terms of positive powers of Z. So multiplying numerator and denominator of $H(Z)$ by Z^2 we get,

$$Y(Z) = \frac{(3Z^2-7Z+5)}{(Z^2-2.5Z+1)}\times\frac{Z}{(Z-1)} \quad ...(5)$$

We have to calculate output $y(n)$. So we will take IZT of $Y(Z)$ using partial fraction expansion. From Equation (5) we get,

$$\frac{Y(Z)}{Z} = \frac{3Z^2-7Z+5}{(Z^2-2.5Z+1)(Z-1)} \quad ...(6)$$

Since M < N; the equation is in proper form. To obtain the poles we will find out roots of denominator. Consider the equation $Z^2-2.5Z+1$.

Here a = 1, b = −2.5 and c = 1.

$$\text{Now} \quad P_1, P_2 = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\therefore \quad P_1, P_2 = \frac{2.5\pm\sqrt{6.25-4}}{2} = \frac{2.5\pm\sqrt{2.25}}{2}$$

$$\therefore \quad P_1 = 2 \quad \text{and} \quad P_2 = \frac{1}{2}$$

Now the second bracket in denominator is $(Z-1)$. So we can directly write, $P_3 = 1$.

Putting these values in Equation (6) we get,

$$\frac{Y(Z)}{Z} = \frac{3Z^2-7Z+5}{(Z-2)\left(Z-\frac{1}{2}\right)(Z-1)}$$

In the partial fraction expansion form we can write,

$$\frac{Y(Z)}{Z} = \frac{A_1}{(Z-2)}+\frac{A_2}{\left(Z-\frac{1}{2}\right)}+\frac{A_3}{(Z-1)} \quad ...(7)$$

We will calculate the values of A_1, A_2 and A_3.

$$A_1 = (Z-P_1)\frac{Y(Z)}{Z}\Bigg|_{Z=P_1} = (Z-2)\left[\frac{3Z^2-7Z+5}{(Z-2)\left(Z-\frac{1}{2}\right)(Z-1)}\right]\Bigg|_{Z=2}$$

$$\therefore \quad A_1 = \frac{3(2)^2-7(2)+5}{\left(2-\frac{1}{2}\right)(2-1)} \quad \Rightarrow \quad A_1 = 2.$$

$$A_2 = (Z-P_2)\frac{Y(Z)}{Z}\Big|_{Z=P_2} = \left(Z-\frac{1}{2}\right)\left[\frac{3Z^2-7Z+5}{(Z-2)\left(Z-\frac{1}{2}\right)(Z-1)}\right]\Bigg|_{Z=1/2}$$

$$\therefore \quad A_2 = \frac{3\left(\frac{1}{2}\right)^2-7\left(\frac{1}{2}\right)+5}{\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-1\right)} \quad \Rightarrow A_2 = 3$$

$$\text{and} \quad A_3 = (Z-P_3)\frac{Y(Z)}{Z}\Big|_{Z=P_3} = (Z-1)\left[\frac{3Z^2-7Z+5}{(Z-2)\left(Z-\frac{1}{2}\right)(Z-1)}\right]\Bigg|_{Z=1}$$

$$\therefore \quad A_3 = \frac{3(1)-7(1)+5}{(1-2)\left(1-\frac{1}{2}\right)} \quad \Rightarrow \quad A_3 = -2.$$

Putting these values in Equation (7) we get,

$$\frac{Y(Z)}{Z} = \frac{2}{Z-2}+\frac{3}{Z-\frac{1}{2}}-\frac{2}{Z-1}$$

$$\therefore \quad Y(Z) = \frac{2Z}{Z-2}+\frac{3Z}{Z-\frac{1}{2}}-\frac{2Z}{Z-1} \qquad ...(8)$$

In the question it is mentioned that the system is stable. So its ROC must include unit circle. The poles are at $\frac{1}{2}$, 1 and 2. Thus to include the unit circle; for these poles the ROC is,

$$\text{ROC}: \quad \frac{1}{2} < |Z| < 2 \qquad ...(9)$$

Then the first term of Equation (8) becomes anticausal. Because for this term to be causal, required ROC is $|Z| > 2$.

$$\therefore \quad \text{IZT}\left\{\frac{2Z}{Z-2}\right\} = 2\,\text{IZT}\left\{\frac{Z}{Z-2}\right\} = -2(2)^n u(-n-1)$$

The second term is causal. Because the required ROC for this term is $|Z| > \frac{1}{2}$; so Equation (9) is satisfied.

$$\therefore \quad \text{IZT}\left\{\frac{3Z}{Z-\frac{1}{2}}\right\} = 3\,\text{IZT}\left\{\frac{Z}{Z-\frac{1}{2}}\right\} = 3\left(\frac{1}{2}\right)^n u(n)$$

The third term is also causal. For this term to be causal the required ROC is $|Z| > 1$. This satisfies Equation (9)

$$\therefore \quad \text{IZT}\left\{\frac{2Z}{Z-1}\right\} = 2\,\text{IZT}\left\{\frac{Z}{Z-1}\right\} = 2(1)^n u(n)$$

Thus equation of output can be written as,

$$y(n) = -2(2)^n u(-n-1) + 3\left(\frac{1}{2}\right)^n u(n) + 2(1)^n u(n)$$

Prob. 9 : Impulse response of system is,

$$h(n) = \left(\frac{1}{3}\right)^n \qquad n = 0, 1, 2, 3$$
$$= 0 \qquad \text{otherwise}$$

Find response of system to step input and $x(n) = \cos\left(\frac{\pi n}{4}\right) u(n)$.

Soln. : First we will obtain Z-transform of $h(n)$.

We have, $h(n) = \left(\frac{1}{3}\right)^n \qquad n = 0, 1, 2, 3$ ...(1)

$$\text{For } n = 0 \in h(0) = \left(\frac{1}{3}\right)^0 = 1$$

$$\text{For } n = 1 \in h(1) = \left(\frac{1}{3}\right)^1 = \frac{1}{3}$$

$$\text{For } n = 2 \in h(2) = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$

$$\text{For } n = 3 \in h(3) = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$$

$$\therefore \quad h(n) = \{h(0), h(1), h(2), h(3)\}$$

$$\therefore \quad h(n) = \left\{1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}\right\}$$
$$\uparrow$$

According to the definition of Z-transform,

$$H(Z) = \sum_{n=-\infty}^{\infty} h(n) Z^{-n}$$

Here limits of summation are from n = 0 to n = 3.

$$\therefore \quad H(Z) = \sum_{n=0}^{3} h(n) Z^{-n}$$

$$\therefore \quad H(Z) = h(0)Z^0 + h(1)Z^{-1} + h(2)Z^{-2} + h(3)Z^{-3}$$

$$\therefore \quad H(Z) = 1 + \frac{1}{3}Z^{-1} + \frac{1}{9}Z^{-2} + \frac{1}{27}Z^{-3} \qquad \text{...(2)}$$

Part A : Calculation for step input :

When step input is applied,

$$x(n) = u(n)$$

$$\therefore \quad X(Z) = \frac{Z}{Z-1} \qquad \ldots(3)$$

We have $H(Z) = \dfrac{Y(Z)}{X(Z)}$

$$\therefore \quad Y(Z) = X(Z) \cdot H(Z)$$

$$\therefore \quad Y(Z) = \left(\frac{Z}{Z-1}\right)\left(1+\frac{1}{3}Z^{-1}+\frac{1}{9}Z^{-2}+\frac{1}{27}Z^{-3}\right)$$

$$\therefore \quad Y(Z) = \frac{Z}{Z-1}+\frac{1}{3}\cdot Z^{-1}\cdot\left(\frac{Z}{Z-1}\right)+\frac{1}{9}Z^{-2}\left(\frac{Z}{Z-1}\right)+\frac{1}{27}Z^{-3}\left(\frac{Z}{Z-1}\right) \qquad \ldots(4)$$

Taking IZT of both sides,

$$y(n) = u(n)+\frac{1}{3}u(n-1)+\frac{1}{9}u(n-2)+\frac{1}{27}u(n-3)$$

This is the response of system to the step input.

Part B : Calculation for $x(n) = \cos\left(\frac{\pi n}{4}\right)u(n)$.

We have standard Z-transform pair,

$$\cos \omega_n u(n) \overset{Z}{\longleftrightarrow} \frac{Z^2 - Z\cos\omega_o}{Z^2 - 2Z\cos\omega_o + 1}$$

Here $\omega_o = \dfrac{\pi n}{4}$

$$\therefore \quad X(Z) = \frac{Z^2 - Z\cos\left(\frac{\pi n}{4}\right)}{Z^2 - 2Z\cos\left(\frac{\pi n}{4}\right)+1} \qquad \ldots(5)$$

But $Y(Z) = X(Z)\cdot H(Z)$

$$\therefore \quad Y(Z) = \frac{Z^2 - Z\cos\left(\frac{\pi n}{4}\right)}{Z^2 - 2Z\cos\left(\frac{\pi n}{4}\right)+1}\cdot\left[1+\frac{1}{3}Z^{-1}+\frac{1}{9}Z^{-2}+\frac{1}{27}Z^{-3}\right]$$

$$\therefore \quad Y(Z) = \frac{Z^2 - Z\cos\left(\frac{\pi n}{4}\right)}{Z^2 - 2Z\cos\left(\frac{\pi n}{4}\right) + 1} + \frac{1}{3}Z^{-1}\left[\frac{Z^2 - \cos\left(\frac{\pi n}{4}\right)}{Z^2 - 2Z\cos\left(\frac{\pi n}{4}\right) + 1}\right]$$

$$+\frac{1}{9}Z^{-2}\left[\frac{Z^2 - \cos\left(\frac{\pi n}{4}\right)}{Z^2 - 2Z\cos\left(\frac{\pi n}{4}\right) + 1}\right] + \frac{1}{27}Z^{-3}\left[\frac{Z^2 - \cos\left(\frac{\pi n}{4}\right)}{Z^2 - 2Z\cos\left(\frac{\pi n}{4}\right) + 1}\right] \quad ...(6)$$

Using time shifting property IZT of Equation (6) can be written as,

$$y(n) = \cos\left(\frac{\pi n}{4}\right)u(n) + \frac{1}{3}\cos\left[(n-1)\frac{\pi}{4}\right]u(n-1)$$

$$+\frac{1}{9}\cos\left[(n-2)\frac{\pi}{4}\right]u(n-2) + \frac{1}{27}\cos\left[(n-3)\frac{\pi}{4}\right]u(n-3)$$

This is the required response of system.

Prob. 10 : Consider a system with impulse response

$$h(n) = \left(\frac{1}{2}\right)^n \quad \text{for } 0 \le n \le 4$$

$$= 0 \quad \text{Otherwise}$$

Determine input $x(n)$ for $0 \le n \le 8$ that will generate output sequence

$y(n) = \{1, 2, 2.5, 3, 3, 3, 2, 1, 0 \ldots\}$

Soln. :

Step I : We have,

$$h(n) = \left(\frac{1}{2}\right)^n \quad \text{for } 0 \le n \le 4 \quad ...(1)$$

For $n = 0 \in h(n) = \left(\frac{1}{2}\right)^0 = 1$

For $n = 1 \in h(n) = \left(\frac{1}{2}\right)^1 = \frac{1}{2}$

For $n = 2 \in h(n) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

For $n = 3 \in h(n) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

For $n = 4 \in h(n) = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$

$$\therefore \quad h(n) = \left\{ \underset{\uparrow}{1}, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \right\} \qquad \ldots(2)$$

According to definition of Z-transform we have,

$$H(Z) = \sum_{n=-\infty}^{\infty} h(n) Z^{-n}$$

Here limits of summation are from n = 0 to n = 4

$$\therefore \quad H(Z) = \sum_{n=0}^{4} h(n) Z^{-n}$$

$$\therefore \quad H(Z) = h(0)Z^{0} + h(1)Z^{-1} + h(2)Z^{-2} + h(3)Z^{-3} + h(4)Z^{-4}$$

$$\therefore \quad H(Z) = 1 + \frac{1}{2}Z^{-1} + \frac{1}{4}Z^{-2} + \frac{1}{8}Z^{-3} + \frac{1}{16}Z^{-4} \qquad \ldots(3)$$

Step II : Sequence y (n) is,

$$y(n) = \left\{ \underset{\uparrow}{1}, 2, 2.5, 3\,3\,3, 2, 1, 0, --- \right\}$$

According to definition of Z-transform

$$Y(Z) = \sum_{n=-\infty}^{\infty} y(n) Z^{-n}$$

Here limits of summation are n = 0 to n = 8.

$$\therefore \quad Y(Z) = \sum_{n=0}^{8} y(n) Z^{-n}$$

$$\therefore \quad Y(Z) = Y(0)Z^{0} + Y(2)Z^{-2} + Y(3)Z^{-3} + Y(4)Z^{-4} + Y(5)Z^{-5}$$

$$+ Y(6)Z^{-6} + Y(7)Z^{-7} + Y(8)Z^{-8}$$

$$\therefore \quad Y(Z) = 1 + 2Z^{-1} + 2.5Z^{-2} + 3Z^{-3} + 3Z^{-4} + 3Z^{-5} + 2Z^{-6} + Z^{-7} \qquad \ldots(4)$$

Step III : Now we have,

$$H(Z) = \frac{Y(Z)}{X(Z)}$$

$$\therefore \quad X(Z) = \frac{Y(Z)}{H(Z)} = \frac{1 + 2Z^{-1} + 2.5Z^{-2} + 3Z^{-3} + 3Z^{-4} + 3Z^{-5} + 2Z^{-6} + Z^{-7}}{1 + \frac{1}{2}Z^{-1} + \frac{1}{4}Z^{-2} + \frac{1}{8}Z^{-3} + \frac{1}{16}Z^{-4}}$$

We will perform long division as follows :

$$1+\frac{1}{2}Z^{-1}+\frac{1}{4}Z^{-2}+\frac{1}{8}Z^{-3}+\frac{1}{16}Z^{-4}\;\overline{)\;1+2Z^{-1}+2.5Z^{-2}+3Z^{-3}+3Z^{-4}+3Z^{-5}+2Z^{-6}+Z^{-7}}\;\;1+Z^{-1}+1.5Z^{-2}+1.85Z^{-3}+\ldots.$$

$$1+\frac{1}{2}Z^{-1}+\frac{1}{4}Z^{-2}+\frac{1}{8}Z^{-3}+\frac{1}{16}Z^{-4}$$

$$Z^{-1}+2Z^{-2}+2.875Z^{-3}+2.93Z^{-4}+3Z^{-5}+2Z^{-6}+Z^{-7}$$

$$Z^{-1}+\frac{1}{2}Z^{-2}+\frac{1}{4}Z^{-3}+\frac{1}{8}Z^{-4}+\frac{1}{16}Z^{-5}$$

$$1.5Z^{-2}+2.6Z^{-3}+2.8Z^{-4}+2.93Z^{-5}+2Z^{-6}+Z^{-7}$$

$$1.5Z^{-2}+0.75Z^{-3}+0.375Z^{-4}+0.18Z^{-5}+0.093Z^{-6}$$

$$1.85Z^{-3}+2.4Z^{-4}+2.75Z^{-5}+1.9Z^{-6}+Z^{-7}$$

$$1.85Z^{-3}+0.9Z^{-4}+0.46Z^{-5}+0.23Z^{-6}+0.11Z^{-7}$$

$$1.5Z^{-4}+2.29Z^{-5}+1.67Z^{-6}+0.89Z^{-7}$$

Thus $X(Z)$ can be approximately written as,

$$X(Z) = 1+Z^{-1}+1.5Z^{-2}+1.85Z^{-3}+\ldots \qquad \ldots(5)$$

We have $X(Z) = \sum_{n=0}^{3} x(n)Z^{-n}$

$$\therefore\; X(Z) = x(0)+x(1)Z^{-1}+x(2)Z^{-2}+x(3)Z^{-3} \qquad \ldots(6)$$

Comparing Equations (5) and (6),

$x(0) = 1$, $x(1)$, $x(2) = 1.5$, $x(3) = 1.85$.

$$\therefore\; x(n) = \{1, 1, 1.5, 1.85, \ldots\}$$
$$\uparrow$$

Prob. 11 : Determine the response of system characterized by impulse response $h(n) = (0.5)^n u(n)$ to input signals.

(i) $x(n) = 2^n u(n)$ (ii) $x(n) = u(-n)$

Soln. :

(i) We have, $x(n) = 2^n u(n)$

Use standard Z transform pair,

$$a^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-a}, \qquad \text{ROC} : |Z| > |a|$$

$\therefore$ Here $a = 2$ $\quad\therefore\quad X(Z) = \dfrac{Z}{Z-2}, \qquad \text{ROC} : |Z| > 2 \qquad \ldots(1)$

Given : $h(n) = (0.5)^n u(n)$

Here a = 0.5 $\therefore H(Z) = \frac{Z}{Z-0.5}$, ROC : $|Z| > 0.5$ ...(2)

Now $H(Z) = \frac{Y(Z)}{X(Z)}$

$\therefore Y(Z) = H(Z) \cdot X(Z)$ $\therefore Y(Z) = \left(\frac{Z}{Z-0.5}\right)\left(\frac{Z}{Z-2}\right)$

$\therefore Y(Z) = \frac{Z^2}{(Z-0.5)(Z-2)}$ ...(3)

To obtain the response of system we will take IZT of Y(Z). From Equation (3) we get,

$$\frac{Y(Z)}{Z} = \frac{Z}{(Z-0.5)(Z-2)} \quad ...(4)$$

In PFE form Equation (4) can be written as,

$$\frac{Y(Z)}{Z} = \frac{A_1}{Z-0.5} + \frac{A_2}{Z-2} \quad ...(5)$$

The coefficients A_1 and A_2 are calculated as follows :

$$A_1 = (Z-P_1)\frac{Y(Z)}{Z}\Big|_{Z=P_1}$$

$$\therefore A_1 = (Z-0.5)\cdot\frac{Z}{(Z-0.5)(Z-2)}\Big|_{Z=0.5}$$

$$\therefore A_1 = \frac{0.5}{0.5-2} = \frac{0.5}{-1.5}$$

$$\therefore A_1 = -0.33$$

Similarly $A_2 = (Z-P_2)\frac{Y(Z)}{Z}\Big|_{Z=P_2}$

$$\therefore A_2 = (Z-2)\cdot\frac{Z}{(Z-0.5)(Z-2)}\Big|_{Z=2}$$

$$\therefore A_2 = \frac{2}{2-0.5} = \frac{2}{1.5}$$

$$\therefore A_2 = 1.33$$

Putting these values in Equation (5) we get,

$$\frac{Y(Z)}{Z} = -\frac{0.33}{Z-0.5} + \frac{1.33}{Z-2}$$

$$\therefore \quad Y(Z) = -0.33\frac{Z}{Z-0.5} + 1.33\frac{Z}{Z-2} \qquad ...(6)$$

Taking IZT of Equation (6) we get,

$$\therefore \quad y(n) = -0.33\,(0.5)^n u(n) + 1.33\,(2)^n u(n)$$

(ii) We have $x(n) = u(-n)$

$$\therefore \quad X(Z) = \frac{1}{1-Z} \qquad ...(7)$$

Since $H(Z) = \dfrac{Y(Z)}{X(Z)}$

$$\therefore \quad Y(Z) = H(Z)\cdot X(Z) \qquad \therefore \quad Y(Z) = \left(\frac{Z}{Z-0.5}\right)\left(\frac{1}{1-Z}\right)$$

$$\therefore \quad Y(Z) = \frac{Z}{(Z-0.5)(1-Z)} \qquad \therefore \quad Y(Z) = -\frac{Z}{(Z-0.5)(Z-1)}$$

$$\therefore \quad \frac{Y(Z)}{Z} = -\frac{1}{(Z-0.5)(Z-1)} \qquad ..(8)$$

In PFE form Equation (8) can be written as,

$$\frac{Y(Z)}{Z} = \frac{A_1}{Z-0.5} + \frac{A_2}{Z-1} \qquad ...(9)$$

The values of A_1 and A_2 are obtained as follows :

$$A_1 = (Z-P_1)\frac{Y(Z)}{Z}\Big|_{Z=P_1}$$

$$\therefore \quad A_1 = (Z-0.5)\cdot\left[\frac{-1}{(Z-0.5)(Z-1)}\right]\Bigg|_{Z=0.5}$$

$$\therefore \quad A_1 = -\frac{1}{0.5-1} = -\frac{1}{-0.5}$$

$$\therefore \quad A_1 = 2$$

Similarly,

$$A_2 = (Z-P_2)\frac{Y(Z)}{Z}\Big|_{Z=P_2}$$

$$\therefore \quad A_2 = (Z-1)\cdot\left[\frac{-1}{(Z-0.5)(Z-1)}\right]\Bigg|_{Z=1}$$

$$\therefore \quad A_2 = \frac{-1}{1-0.5} = \frac{-1}{0.5}$$

$$\therefore \quad A_2 = -2$$

Putting these values in Equation (9) we get,

$$\frac{Y(Z)}{Z} = \frac{2}{Z-0.5} - \frac{2}{Z-1}$$

$$\therefore \quad Y(Z) = \frac{2Z}{Z-0.5} - \frac{2Z}{Z-1} \qquad ...(10)$$

IZT of Equation (10) can be written as,

$$y(n) = 2(0.5)^n u(n) - 2(1)^n u(n)$$

Prob. 12 : If the step response of system is given by,

$$s(n) = \left(\frac{1}{2}\right)^n u(n)$$

Find impulse response.

Soln. : Step response means, the output of a system when input $x(n)$ is $u(n)$. Thus,

$$y(n) = s(n) = \left(\frac{1}{2}\right)^n u(n) \qquad ...(1)$$

and $$x(n) = u(n) \qquad ...(2)$$

Now, we will obtain Z-transform of $y(n)$ and $x(n)$.

We have standard Z-transform pairs,

$$a^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-a}, \qquad ROC\ |Z| > |a|$$

and $$u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-1}, \qquad ROC\ |Z| > 1$$

$$\therefore \quad Y(Z) = \frac{Z}{Z-\frac{1}{2}} \qquad \text{and} \qquad X(Z) = \frac{Z}{Z-1}$$

Now, we have $$H(Z) = \frac{Y(Z)}{X(Z)}$$

$$\therefore \quad H(Z) = \frac{Z}{Z-\frac{1}{2}} \cdot \frac{Z-1}{Z} = \frac{Z-1}{Z-\frac{1}{2}} \qquad ...(3)$$

To get equation of $h(n)$; we will obtain IZT of Equation (3)

$$\therefore \quad \frac{H(Z)}{Z} = \frac{Z-1}{Z\left(Z-\frac{1}{2}\right)} \qquad ...(4)$$

In PFE form, Equation (4) can be written as :

$$\frac{H(Z)}{Z} = \frac{A_1}{Z} + \frac{A_2}{Z-\frac{1}{2}} \qquad ...(5)$$

The values of A_1 and A_2 are calculated as follows :

$$A_1 = (Z-0)\frac{H(Z)}{Z}\Big|_{Z=0}$$

$$\therefore A_1 = Z\cdot\frac{Z-1}{Z\left(Z-\frac{1}{2}\right)}\Big|_{Z=0} = \frac{0-1}{0-\frac{1}{2}}$$

$$\therefore A_1 = 2$$

and $$A_2 = \left(Z-\frac{1}{2}\right)\cdot\frac{H(Z)}{Z}\Big|_{Z=\frac{1}{2}}$$

$$\therefore A_2 = \left(Z-\frac{1}{2}\right)\frac{Z-1}{Z\left(Z-\frac{1}{2}\right)}\Big|_{Z=\frac{1}{2}} = \frac{\frac{1}{2}-1}{\frac{1}{2}} = \frac{-\frac{1}{2}}{\frac{1}{2}}$$

$$\therefore A_2 = -1$$

Putting these values in Equation (5) we get,

$$\frac{H(Z)}{Z} = \frac{2}{Z} - \frac{1}{Z-\frac{1}{2}}$$

$$\therefore H(Z) = 2 - \frac{Z}{Z-\frac{1}{2}} \qquad ...(6)$$

Taking IZT of Equation (6) we get,

$$h(n) = 2\delta(n) - \left(\frac{1}{2}\right)^n u(n)$$

This is the required impulse response.

Prob. 13 : Sketch the impulse response h (n) w.r.t. n for a system having transfer function,

$$H(Z) = \frac{1}{1-2Z^{-1}} \qquad \text{for } |Z| < 2.$$

Soln. : The given ROC is $|Z| < 2$ that means h (n) is anticausal. The expression of H (Z) is,

$$H(Z) = \frac{1}{1-2Z^{-1}} = \frac{Z}{Z-2} \qquad ...(1)$$

For anticausal signal we have standard Z transform pair,

$$-\alpha^n u(-n-1) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha}, \qquad \text{ROC} : |Z| < |\alpha|$$

$$\therefore \quad h(n) = -(2)^n u(-n-1) \qquad \ldots(2)$$

Here multiplication by $u(-n-1)$ indicates that the sequence is in the range, $n = -1$ to $n = -\infty$.

For $n = -1$, $h(-1) = -(2)^{-1} = -\frac{1}{2}$

For $n = -2$, $h(-2) = -(2)^{-2} = -\frac{1}{4}$

For $n = -3$, $h(-3) = -(2)^{-3} = -\frac{1}{8}$

For $n = -4$, $h(-4) = -(2)^{-4} = -\frac{1}{16}$

It is exponential left handed sequence. The plot of $h(n)$ w.r.t. n is shown in Fig. P-8.

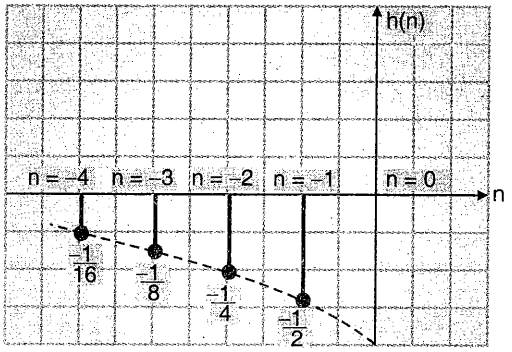


Fig. P-8

Prob. 14 : Design a system which can generate $Y(n) = \cos \omega_o n$.

Soln.:

Here $y(n) = \cos \omega_o n$

Let input applied to the system be $x(n) = \delta(n)$.

$$\therefore \quad X(Z) = 1. \qquad \text{and} \qquad Y(Z) = \frac{Z^2 - Z\cos\omega_o}{Z^2 - 2Z\cos\omega_o + 1}$$

$$\text{Now} \quad H(Z) = \frac{Y(Z)}{X(Z)} \qquad \therefore \quad H(Z) = \frac{Z^2 - Z\cos\omega_o}{Z^2 - 2Z\cos\omega_o + 1}$$

This is the required transfer function of a system.

Prob. 15 : Consider a system given by difference equation :

$$y(n) = x(n) + x(n-1) + \frac{1}{2} y(n-1)$$

This system is represented by cascaded blocks such that,

$$H(Z) = H_1(Z) \cdot H_2(Z).$$

Find $h_1(n)$ and $h_2(n)$

Soln. : Given difference equation is,

$$y(n) = x(n)+x(n-1)+\frac{1}{2}y(n-1) \quad ...(1)$$

Taking Z-transform of both sides,

$$Y(Z) = X(Z)+Z^{-1}X(Z)+\frac{1}{2}Z^{-1}Y(Z)$$

$$\therefore \quad Y(Z)-\frac{1}{2}Z^{-1}Y(Z) = X(Z)+Z^{-1}X(Z)$$

$$\therefore \quad Y(Z)\left[1-\frac{1}{2}Z^{-1}\right] = X(Z)[1+Z^{-1}]$$

$$\therefore \quad \frac{Y(Z)}{X(Z)} = H(Z) = \frac{1+Z^{-1}}{1-\frac{1}{2}Z^{-1}}$$

Multiply numerator and denominator by Z.

$$\therefore \quad H(Z) = \frac{Z+1}{Z-\frac{1}{2}}$$

$$\therefore \quad \frac{H(Z)}{Z} = \frac{Z+1}{Z\left(Z-\frac{1}{2}\right)} = \frac{Z+1}{Z}\cdot\frac{1}{Z-\frac{1}{2}}$$

$$\therefore \quad H(Z) = (Z+1)\cdot\left(\frac{Z}{Z-\frac{1}{2}}\right) \quad ...(2)$$

Let $H_1(Z) = (Z+1)$ and $H_2(Z) = \dfrac{Z}{Z-\frac{1}{2}}$

$$\therefore \quad h_1(n) = \delta(n+1)+\delta(n) \text{ and } h_2(n) = \left(\frac{1}{2}\right)^n u(n)$$

Prob. 16 : Find impulse response of system if input $x(n)$ produces output $y(n)$ where,

$x(n) = \{3, 2, 1, 4\}$

$y(n) = \{1, 1, 2, 2, 3, 4\}$.

Soln. : According to definition of Z-transform,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n)Z^{-n} \quad ...(1)$$

Here $x(n) = \underset{\uparrow}{\{3, 2, 1, 4\}}$. Thus limits of summation are from n = 0 to n = 3.

$$\therefore \quad X(Z) = \sum_{n=0}^{3} x(n)z^{-n}$$

$$\therefore \quad X(Z) = x(0)+x(1)Z^{-1}+x(2)Z^{-2}+x(3)Z^{-3}$$

$$\therefore \quad X(Z) = 3+2Z^{-1}+Z^{-2}+4Z^{-3} \qquad ...(2)$$

Given $y(n) = \{\underset{\uparrow}{1}, 1, 2, 2, 3, 4\}$. So we can write,

$$Y(Z) = \sum_{n=0}^{5} y(n)Z^{-n}$$

$$\therefore \quad Y(Z) = y(0)+y(1)Z^{-1}+y(2)Z^{-2}+y(3)Z^{-3}+y(4)Z^{-4}+y(5)Z^{-5}$$

$$\therefore \quad Y(Z) = 1+Z^{-1}+2Z^{-2}+2Z^{-3}+3Z^{-4}+4Z^{-5} \qquad ...(3)$$

We have, $$H(Z) = \frac{Y(Z)}{X(Z)} = \frac{1+Z^{-1}+2Z^{-2}+2Z^{-3}+3Z^{-4}+4Z^{-5}}{3+2Z^{-1}+Z^{-2}+4Z^{-3}}$$

We will perform long division as follows :

$$\begin{array}{r}
\frac{1}{3}+\frac{1}{9}Z^{-1}+\frac{13}{7}Z^{-2}+\ldots \\
3+2Z^{-1}+Z^{-2}+4Z^{-3} \;\overline{)}\; 1+Z^{-1}+2Z^{-2}+2Z^{-3}+3Z^{-4}+4Z^{-5} \\
1+\frac{2}{3}Z^{-1}+\frac{1}{3}Z^{-2}+\frac{4}{3}Z^{-3} \\
\hline
\frac{1}{3}Z^{-1}+\frac{5}{3}Z^{-2}+\frac{2}{3}Z^{-3}+3Z^{-4}+4Z^{-5} \\
\frac{1}{3}Z^{-1}+\frac{2}{9}Z^{-2}+\frac{1}{9}Z^{-3}+\frac{4}{9}Z^{-4} \\
\hline
\frac{13}{9}Z^{-2}+\frac{5}{9}Z^{-3}+\frac{23}{9}Z^{-4}+4Z^{-5} \\
\frac{13}{9}Z^{-2}+\frac{26}{27}Z^{-3}+\frac{13}{27}Z^{-4}+\frac{52}{27}Z^{-5} \\
\hline
-\frac{11}{27}Z^{-3}+\frac{56}{27}Z^{-4}+\frac{56}{27}Z^{-5}+\ldots
\end{array}$$

Thus approximately we can write,

$$H(Z) \approx \frac{1}{3}+\frac{1}{9}Z^{-1}+\frac{13}{27}Z^{-2}+\ldots \qquad ...(4)$$

Comparing Equation (4) with the definition of Z transform,

$$\therefore h(n) = \left\{\frac{1}{3}, \frac{1}{9}, \frac{13}{27}, \ldots\right\}$$

Prob. 17 : Using Z-transform method obtain impulse response of a system described by :

$$y(n) = 2.5\,y(n-1)+x(n)$$

Soln. : Given equation is,

$$y(n) = 2.5\,y(n-1)+x(n) \qquad ...(1)$$

Taking Z-transform of Equation (1),

$$Y(Z) = 2.5Z^{-1}Y(Z)+X(Z)$$

$$\therefore \quad Y(Z)-2.5Z^{-1}Y(Z) = X(X)$$

$$\therefore \quad Y(Z)\left[1-2.5Z^{-1}\right] = X(Z) \qquad \therefore \quad \frac{Y(Z)}{X(Z)} = \frac{1}{1-2.5Z^{-1}}$$

$$\text{But} \quad \frac{Y(Z)}{X(Z)} = H(Z) \qquad \therefore \quad H(Z) = \frac{1}{1-2.5Z^{-1}} = \frac{Z}{Z-2.5}$$

Now, impulse response $h(n)$ is obtained by taking IZT of $H(Z)$

$$\therefore \quad h(n) = IZT\left\{\frac{Z}{Z-2.5}\right\}$$

$$\therefore \quad h(n) = (2.5)^n u(n)$$

Prob. 18 : Find magnitude and phase response of given system at frequency $\omega = 0$ and $\omega = \pi$

$$H(Z) = \frac{1}{1-0.5Z^{-1}}$$

Soln. : The magnitude response is obtained by putting $Z = e^{+j\omega}$

$$\therefore \quad H(e^{j\omega}) = \frac{1}{1-0.5\,e^{-j\omega}} \qquad ...(1)$$

$$\text{But} \quad e^{-j\omega} = \cos\omega - j\sin\omega$$

$$\therefore \quad H(e^{j\omega}) = \frac{1}{1-0.5(\cos\omega - j\sin\omega)} = \frac{1}{1-0.5\cos\omega+0.5\,j\sin\omega} \qquad ...(2)$$

$$\therefore \quad |H(e^{j\omega})| = \frac{1}{\sqrt{(1-0.5\cos\omega)^2+(0.5\sin\omega)^2}}$$

$$\therefore \quad |H(e^{j\omega})| = \frac{1}{\sqrt{1-\cos\omega+(0.5)^2\cos^2\omega+(0.5)^2\sin^2\omega}}$$

$$= \frac{1}{\sqrt{1-\cos\omega+(0.5)^2\cdot(\sin^2\omega+\cos^2\omega)}}$$

$$|H(e^{j\omega})| = \frac{1}{\sqrt{1+0.25-\cos\omega}} \qquad ...(3)$$

$$\therefore \quad \text{At } \omega = 0 \in |H(e^{j\omega})| = \frac{1}{\sqrt{1+0.25-1}} = 2$$

$$\text{At } \omega = \pi \in |H(e^{j\omega})| = \frac{1}{\sqrt{1+0.25-\cos\pi}} = 0.66.$$

From Equation (2), the phase response is given by,

$$\angle H(e^{j\omega}) = \tan^{-1} 0 - \tan^{-1} \frac{0.5 \sin \omega}{1 - 0.5 \cos\omega}$$

$$\text{At } \omega = 0 \in \cap H(e^{j\omega}) = \tan^{-1} 0 - \tan^{-1} \frac{0.5 \sin 0}{1 - 0.5 \cos 0} = 0$$

$$\text{At } \omega = \pi \in H(e^{j\omega}) = \tan^{-1} 0 - \tan^{-1} \frac{0.5 \sin \pi}{1 - 0.5 \cos\pi} = 0.$$

Prob. 19 : Find the magnitude response and impulse response of a system described by the difference equation.

$$y(n) - \frac{1}{2} y(n-1) = x(n) - \frac{1}{4} x(n-1).$$

Soln. :

Calculation of impulse response : The given difference equation is,

$$y(n) - \frac{1}{2} y(n-1) = x(n) - \frac{1}{4} x(n-1) \qquad \ldots(1)$$

Taking Z-transform

$$Y(Z) - \frac{1}{2} Z^{-1} Y(Z) = X(Z) - \frac{1}{4} Z^{-1} X(Z)$$

$$\therefore \quad Y(Z)\left[1 - \frac{1}{2} Z^{-1}\right] = X(Z)\left[1 - \frac{1}{4} Z^{-1}\right]$$

$$\therefore \quad \frac{Y(Z)}{X(Z)} = H(Z) = \frac{1 - \frac{1}{4} Z^{-1}}{1 - \frac{1}{2} Z^{-1}} \qquad \ldots(2)$$

We will obtain impulse response by taking IZT of Equation (2). This is simple equation; so just by rearranging Equation (2) we can obtain IZT as follows :

$$H(Z) = \frac{1}{1 - \frac{1}{2} Z^{-1}} - \frac{\frac{1}{4} Z^{-1}}{1 - \frac{1}{2} Z^{-1}}$$

$$\therefore \quad H(Z) = \frac{Z}{Z - \frac{1}{2}} - \frac{1}{4} Z^{-1} \cdot \frac{Z}{Z - \frac{1}{2}} \qquad \ldots(3)$$

Taking IZT of Equation (3),

$$h(n) = \left(\frac{1}{2}\right)^n u(n) - \frac{1}{4}\left(\frac{1}{2}\right)^{n-1} u(n-1)$$

Calculation of magnitude response :

The frequency response is obtained by putting $Z = e^{j\omega}$ in the equation of $H(Z)$. The equation of $H(Z)$ is,

$$H(Z) = \frac{1-\frac{1}{4}Z^{-1}}{1-\frac{1}{2}Z^{-1}}$$

$$\therefore \quad H(\omega) = \frac{1-\frac{1}{4}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}} \qquad ...(4)$$

We will obtain different values of $H(\omega)$ by putting values of 'ω' in Equation (4) as follows :

$$\text{At } \omega = 0 \Rightarrow \quad H(\omega) = \frac{1-\frac{1}{4}e^{0}}{1-\frac{1}{2}e^{0}} = \frac{1-\frac{1}{4}}{1-\frac{1}{2}} = 1.5$$

$$\text{At } \omega = \pi \Rightarrow \quad H(\omega) = \frac{1-\frac{1}{4}e^{-j\pi}}{1-\frac{1}{2}e^{-j\pi}} = \frac{1-\frac{1}{4}(\cos\pi - j\sin\pi)}{1-\frac{1}{2}(\cos\pi - j\sin\pi)} = 0.833$$

$$\text{At } \omega = 2\pi \Rightarrow \quad H(\omega) = \frac{1-\frac{1}{4}e^{-j2\pi}}{1-\frac{1}{2}e^{-j2\pi}}$$

$$= \frac{1-\frac{1}{4}(\cos 2\pi - j\sin 2\pi)}{1-\frac{1}{2}(\cos 2\pi - j\sin 2\pi)}$$

$$= 1.5$$

The magnitude response is a plot of ω vs. $H(\omega)$. It is shown in Fig. P-9.

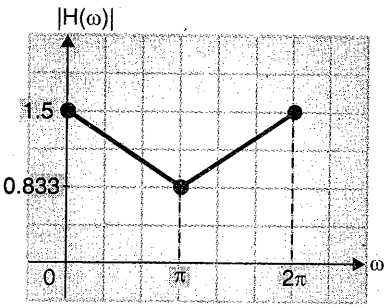


Fig. P-9 : Magnitude response

Prob. 20 : Find the magnitude response and impulse response of an LTI system whose input and output satisfy the difference equation

$$y(n) - \frac{1}{2}y(n-1) = x(n) + 2x(n-1) + x(n-2)$$

Soln. :

Calculation of impulse response :

The given difference equation is,

$$y(n) - \frac{1}{2}y(n-1) = x(n) + 2x(n-1) + x(n-2) \qquad ...(1)$$

Taking Z-transform,

$$Y(Z)-\frac{1}{2}Z^{-1}Y(Z) = X(Z)+2Z^{-1}X(Z)+Z^{-2}X(Z)$$

$$\therefore \quad Y(Z)\left[1-\frac{1}{2}Z^{-1}\right] = X(Z)\left[1+2Z^{-1}+Z^{-2}\right]$$

$$\therefore \quad \frac{Y(Z)}{X(Z)} = H(Z) = \frac{1+2Z^{-1}+Z^{-2}}{1-\frac{1}{2}Z^{-1}} \qquad ...(2)$$

We will obtain h (n) by taking IZT of Equation (3).

Here M = 2 and N = 1. So the function is not in the proper form. So taking long division we get,

$$\begin{array}{r|l} & -2Z^{-1} \\ \hline -\frac{1}{2}Z^{-1}+1 & Z^{-2}+2Z^{-1}+1 \\ & Z^{-2}-2Z^{-1} \\ & - \quad + \\ \hline & 0+4Z^{-1}+1 \end{array}$$

$$\therefore \quad H(Z) = -2Z^{-1}+\frac{4Z^{-1}+1}{-\frac{1}{2}Z^{-1}+1} \qquad ...(3)$$

Let $H_1(Z) = -2Z^{-1}$ and $H_2(Z) = \dfrac{4Z^{-1}+1}{-\frac{1}{2}Z^{-1}+1}$

$$\therefore \quad h_1(n) = -2\delta(n-1) \qquad ...(4)$$

$$\text{Now} \quad H_2(Z) = \frac{4Z^{-1}+1}{-\frac{1}{2}Z^{-1}+1} \qquad ...(5)$$

Converting Equation (5) in terms of positive powers of Z;

$$H_2(Z) = \frac{4+Z}{-\frac{1}{2}+Z} \qquad ...(6)$$

We will take partial fraction expansion of Equation (6).

$$\therefore \quad \frac{H_2(Z)}{Z} = \frac{4+Z}{Z\left(-\frac{1}{2}+Z\right)}$$

$$\therefore \quad \frac{H_2(Z)}{Z} = \frac{4+Z}{Z\left(Z-\frac{1}{2}\right)} \qquad ...(7)$$

Thus poles are at $P_1 = 0$ and $P_2 = \frac{1}{2}$

$$\therefore \quad \frac{H_2(Z)}{Z} = \frac{A_1}{Z} + \frac{A_2}{Z-\frac{1}{2}} \qquad ...(8)$$

$$A_1 = (Z-P_1)\cdot\frac{H_2(Z)}{Z}\Bigg|_{Z=P_1} = Z\left[\frac{4+Z}{Z\left(Z-\frac{1}{2}\right)}\right]\Bigg|_{Z=0}$$

$$\therefore \quad A_1 = \frac{4+Z}{Z-\frac{1}{2}}\Bigg|_{Z=0} = \frac{4+0}{0-\frac{1}{2}} = -8$$

$$\text{and} \quad A_2 = (Z-P_2)\frac{H_2(Z)}{Z}\Bigg|_{Z=P_2} = \left(Z-\frac{1}{2}\right)\left[\frac{4+Z}{Z\left(Z-\frac{1}{2}\right)}\right]\Bigg|_{Z=\frac{1}{2}}$$

$$\therefore \quad A_2 = \left[\frac{4+Z}{Z}\right]\Bigg|_{Z=\frac{1}{2}} = \frac{4+\frac{1}{2}}{\frac{1}{2}} = 9$$

Putting values of A_1 and A_2 in Equation (8) we get,

$$\frac{H_2(Z)}{Z} = -\frac{8}{Z} + \frac{9}{Z-\frac{1}{2}}$$

$$\therefore \quad H_2(Z) = -8 + \frac{9Z}{Z-\frac{1}{2}} \qquad ...(9)$$

Taking IZT of Equation (9) we get,

$$h_2(n) = -8\,\delta(n) + 9\left(\frac{1}{2}\right)^n u(n)$$

Thus the total impulse response is,

$$h(n) = h_1(n) + h_2(n)$$

$$\therefore \quad h(n) = -2\,\delta(n-1) - 8\,\delta(n) + 9\left(\frac{1}{2}\right)^n u(n)$$

Calculation of magnitude response :

The equation of H(Z) is,

$$H(Z) = -2Z^{-1} + \frac{4Z^{-1}+1}{-\frac{1}{2}Z^{-1}+1} \qquad ...(10)$$

$H(\omega)$ is obtained by putting $Z = e^{j\omega}$ in Equation (10)

$$\therefore \quad H(\omega) = -2e^{-j\omega} + \frac{4e^{-j\omega}+1}{-\frac{1}{2}e^{-j\omega}+1}$$

$$\text{at } \omega = 0, \Rightarrow \quad H(\omega) = -2e^{0} + \frac{4e^{0}+1}{-\frac{1}{2}e^{0}+1} = 8$$

$$\text{at } \omega = \pi, \Rightarrow \quad -2e^{-j\pi} + \frac{4e^{-j\pi}+1}{-\frac{1}{2}e^{-j\pi}+1} = -0.67$$

$$\text{at } \omega = 2\pi \Rightarrow \quad -2e^{-j2\pi} + \frac{4e^{-j2\pi}+1}{-\frac{1}{2}e^{-j2\pi}+1} = 8$$

The magnitude response is plotted as shown in Fig. P-10.

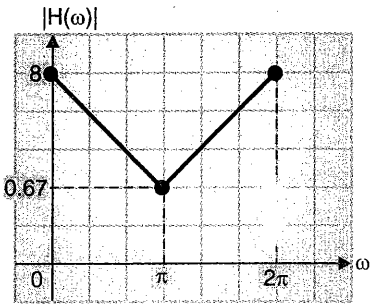


Fig. P-10 : Magnitude response

Prob. 21 : LTI system is described by $h(n) = (0.9)^n u(n)$. Plot magnitude response of system using analytical method.

Soln. :

Given $h(n) = (0.9)^n u(n)$

We have standard Z-transform pair

$$a^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-a}, \qquad \text{ROC} : |Z| > |a|$$

Here $a = 0.9$ $\quad \therefore \quad H(Z) = \dfrac{Z}{Z-0.9}$

$$\therefore \quad H(Z) = \frac{1}{1-0.9Z^{-1}} \qquad \ldots(1)$$

$H(\omega)$ is obtained by putting $Z = e^{j\omega}$ in Equation (1)

$$\therefore \quad H(\omega) = \frac{1}{1-0.9e^{-j\omega}}$$

$$\therefore \quad H(\omega) = \frac{1}{1-0.9(\cos\omega - j\sin\omega)}$$

$$\therefore \quad H(\omega) = \frac{1}{1-0.9\cos\omega + j\,0.9\sin\omega}$$

$$\therefore \quad |H(\omega)| = \frac{1}{\sqrt{(1-0.9\cos\omega)^2 + (0.9\sin\omega)^2}}$$

$$\therefore \quad |H(\omega)| = \frac{1}{\sqrt{1 - 1.8\cos\omega + 0.81\cos^2\omega + 0.81\sin^2\omega}}$$

$$\therefore \quad |H(\omega)| = \frac{1}{\sqrt{1 - 1.8\cos\omega + 0.81(\cos^2\omega + \sin^2\omega)}}$$

$$\therefore \quad |H(\omega)| = \frac{1}{\sqrt{1 - 1.8\cos\omega + 0.81}} \qquad \ldots(2)$$

For $\omega = -\pi$ $\quad |H(\omega)| = \dfrac{1}{\sqrt{1 - 1.8\cos(-\pi) + 0.81}} = 0.52$

For $\omega = -\dfrac{\pi}{2}$, $\quad |H(\omega)| = \dfrac{1}{\sqrt{1 - 1.8\cos\left(-\dfrac{\pi}{2}\right) + 0.81}} = 0.743$

For $\omega = 0$, $\quad |H(\omega)| = \dfrac{1}{\sqrt{1 - 1.8\cos(0) + 0.81}} = 10$

For $\omega = \dfrac{\pi}{2}$, $\quad |H(\omega)| = \dfrac{1}{\sqrt{1 - 1.8\cos\left(\dfrac{\pi}{2}\right) + 0.81}} = 0.743$

For $\omega = \pi$, $\quad H(\omega) = \dfrac{1}{\sqrt{1 - 1.8\cos(\pi) + 0.81}} = 0.52.$

The magnitude response is as shown in Fig. P-11.

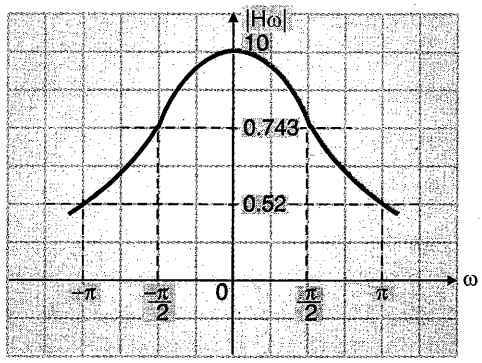


Fig. P-11

Prob. 22 : Obtain the system function for the following :

(i) $y(n) = \frac{x(n)+x(n-1)}{2}$

(ii) $y(n) = 0.8\,y(n-1)+x(n)$

(iii) $y(n) = x(n)-x(n-1)$

Soln. :

(i) $y(n) = \frac{x(n)+x(n-1)}{2}$

Taking Z-transform of both sides,

$$Y(Z) = Z\left\{\frac{x(n)}{2}+\frac{x(n-1)}{2}\right\}$$

$$\therefore \quad Y(Z) = Z\left\{\frac{x(n)}{2}\right\}+Z\left\{\frac{x(n-1)}{2}\right\}$$

$$\therefore \quad Y(Z) = \frac{1}{2}X(Z)+\frac{1}{2}Z^{-1}X(Z)$$

$$\therefore \quad Y(Z) = X(Z)\left[\frac{1}{2}+\frac{1}{2}Z^{-1}\right]$$

$$\therefore \quad H(Z) = \frac{Y(Z)}{X(Z)} = \frac{1}{2}+\frac{1}{2}Z^{-1}$$

(ii) $y(n) = 0.8\,y(n-1)+x(n)$

Taking Z-transform of both sides

$$Y(Z) = 0.8\,Z^{-1}Y(Z)+X(Z)$$

$$\therefore \quad Y(Z)-0.8\,Z^{-1}Y(Z) = X(Z)$$

$$\therefore \quad Y(Z)\left[1-0.8\,Z^{-1}\right] = X(Z)$$

$$\therefore \quad H(Z) = \frac{Y(Z)}{X(Z)} = \frac{1}{1-0.8\,Z^{-1}}$$

$$\therefore \quad H(Z) = \frac{Z}{Z-0.8}$$

(iii) $y(n) = x(n)-x(n-1)$

Taking Z-transform of both sides,

$$Y(Z) = X(Z)-Z^{-1}X(Z)$$

$$\therefore \quad Y(Z) = X(Z)[1-Z^{-1}]$$

$$\therefore \quad H(Z) = \frac{Y(Z)}{X(Z)} = 1-Z^{-1}$$

1.3.3 Conditions for Causality and Stability in terms of Z- Transform :

We know that the condition for causality in terms of h(n) is,

$$h(n) = 0 \quad n < 0 \qquad \ldots(1)$$

Here the condition $n < 0$ indicates that value of $h(n)$ for negative 'n' is zero. That h(n) is causal sequence. In Z-transforms we know that if a sequence is causal then the ROC is exterior part of circle having radius 'r'. Now the system transfer function is,

$$H(Z) = Z\{h(n)\} \qquad \ldots(2)$$

Thus the condition for causality is that the ROC of system transfer function should be exterior part of circle having radius 'r' and $r < \infty$

We have already studied the condition for stability. It is,

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \qquad \ldots(3)$$

Now according to the definition of Z-transform; $Z\{h(n)\}$ can be written as,

$$Z\{h(n)\} = H(Z) = \sum_{n=-\infty}^{\infty} h(n) Z^{-n} \qquad \ldots(4)$$

Taking absolute value on both the sides,

$$|H(Z)| = \left| \sum_{n=-\infty}^{\infty} h(n) Z^{-n} \right| \qquad \ldots(5)$$

Now the absolute value on a sum is always less than the sum of absolute value.

$$\therefore \quad |H(Z)| < \sum_{n=-\infty}^{\infty} |h(n)|\,|Z^{-n}| \qquad \ldots(6)$$

If we evaluate the Z-transform on the unit circle that means $|Z^{-n}| = |Z| = 1$ then Equation (6) becomes,

$$|H(Z)| < \sum_{n=-\infty}^{\infty} |h(n)| \qquad \ldots(7)$$

According to Equation (3), if the system is BIBO stable then $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$. Thus the condition of stability is Z-domain is,

$$|H(Z)| < \infty \text{ when evaluated on unit circle.} \qquad \ldots(8)$$

That means the system transfer function should be finite if evaluated on unit circle.

Equation (6) gives the condition of stability in Z domain. *This condition requires that, unit circle must be present in the ROC of H(Z).* Otherwise we cannot find H(Z) on unit circle at all.

Combined condition :

(1) For a causal LSI system we know that ROC is exterior part of circle; that means $|Z| > |r|$.

(2) For the stable LSI system, ROC of H(Z) must include unit circle. Here unit circle is a circle having radius 1. Now consider we have a causal system having ROC as exterior part of circle with radius 'r'. Then for this system to be stable; it should include unit circle as shown in Fig. P-12.

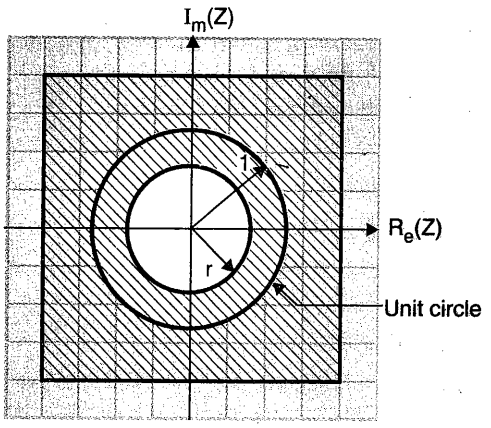


Fig. P-12 : ROC including unit circle

From Fig. P-12 we can say that,

$$r < 1$$

Thus the combined condition for causal and stable LSI system is,

$$|Z| > r < 1 \qquad ...(9)$$

We know that,

(i) ROC of H(Z) does not contain any poles.

(ii) ROC of stable and causal LSI system includes the unit circle.

Thus we can state that *all the poles of H(Z) of a causal and stable system are inside the unit circle.*

1.3.3.1 Solved Problems on Stability and Causality :

Prob. 1 : Consider the system

$$H(Z) = \frac{1 - 2Z^{-1} + 2Z^{-2} - Z^{-3}}{(1 - Z^{-1})(1 - 0.5Z^{-1})(1 - 0.2Z^{-1})}$$

with ROC : $0.5 < |Z| < 1$.

(i) Sketch the pole-zero pattern. Is the system stable ?

(ii) Determine the impulse response of system.

Soln. :

(i) Pole-zero pattern : The given function is,

$$H(Z) = \frac{1-2Z^{-1}+2Z^{-2}-Z^{-3}}{(1-Z^{-1})(1-0.5Z^{-1})(1-0.2Z^{-1})} \quad ...(1)$$

To convert it into positive powers of Z we will multiply numerator and denominator by Z^3.

$$\therefore \quad H(Z) = \frac{Z^3(1-2Z^{-1}+2Z^{-2}-Z^{-3})}{Z(1-Z^{-1})\,Z(1-0.5Z^{-1})\,Z(1-0.2Z^{-1})}$$

$$\therefore \quad H(Z) = \frac{Z^3-2Z^2+2Z-1}{(Z-1)(Z-0.5)(Z-0.2)} \quad ...(2)$$

To obtain zeros, consider the numerator polynomial. We will obtain the roots of Z^3-2Z^2+2Z-1. Let us perform synthetic division.

$1-2+2-1=0$. Thus $(Z-1)$ exists. Now dividing Z^3-2Z^2+2Z-1 by $(Z-1)$ we get,

$$Z^3-2Z^2+2Z-1 = (Z-1)(Z^2-Z+1) \quad ...(3)$$

Now we will obtain roots of Z^2-Z+1

$$Z^2-Z+1 = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{1\pm\sqrt{1-4}}{2} = \frac{1}{2}\pm j\frac{\sqrt{3}}{2}$$

$$\therefore \quad Z^2-Z+1 = \left(\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)$$

Thus Equation (1) becomes,

$$Z^3-2Z^2+2Z-1 = (Z-1)\left(Z-\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)\left(Z-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right) \quad ...(4)$$

Putting Equation (4) in Equation (2) we get,

$$H(Z) = \frac{(Z-1)\left(Z-\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)\left(Z-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)}{(Z-1)(Z-0.5)(Z-0.2)}$$

$$\therefore \quad H(Z) = \frac{\left(Z-\frac{1}{2}-\frac{j\sqrt{3}}{2}\right)\left(Z-\frac{1}{2}+\frac{j\sqrt{3}}{2}\right)}{(Z-0.5)\,(Z-0.2)} \quad ...(5)$$

We have the standard equation of $H(Z)$ in terms of poles and zeros,

$$H(Z) = \frac{(Z-Z_1)(Z-Z_2)}{(Z-P_1)(Z-P_2)} \quad ...(6)$$

Comparing Equations (5) and (6) we can write,

$$\text{Zeros} \Rightarrow Z_1 = \frac{1}{2}+j\frac{\sqrt{3}}{2} \quad \text{and} \quad Z_2 = \frac{1}{2}-j\frac{\sqrt{3}}{2}$$

$$\text{and} \quad \text{Poles} \Rightarrow P_1 = 0.5 \quad \text{and} \quad P_2 = 0.2.$$

The pole-zero plot is shown in Fig. P-13.

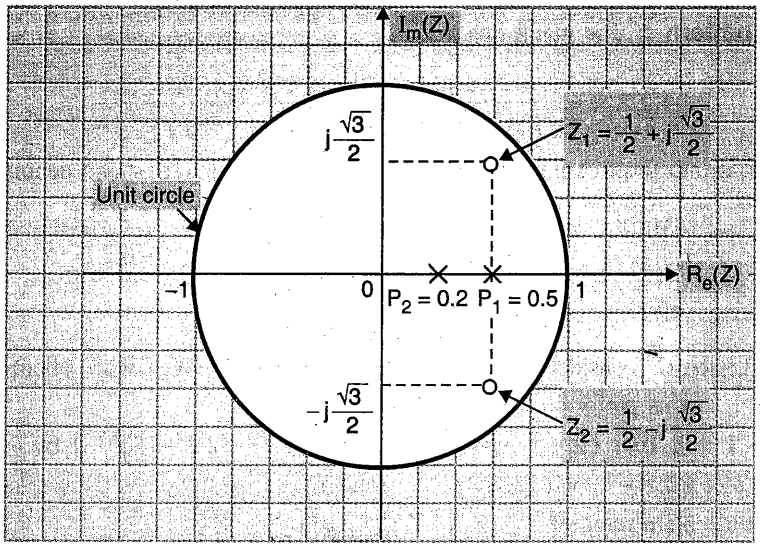


Fig. P-13 : Pole-zero plot

(ii) Impulse response :

Impulse response is denoted by h (n). It is obtained by taking IZT of H (Z).

$$\text{We have,} \quad H(Z) = \frac{\left(Z - \frac{1}{2} - j\frac{\sqrt{3}}{2}\right)\left(Z - \frac{1}{2} + j\frac{\sqrt{3}}{2}\right)}{(Z - 0.5)(Z - 0.2)}$$

$$\therefore \quad H(Z) = \frac{Z^2 - Z + 1}{Z^2 - 0.7Z + 0.1} \qquad \ldots(7)$$

Here M = N; we want M < N. Thus multiplying and dividing by Z,

$$H(Z) = \frac{Z(Z^2 - Z + 1)}{Z(Z^2 - 0.7Z + 0.1)}$$

$$\therefore \quad \frac{H(Z)}{Z} = \frac{Z^2 - Z + 1}{Z(Z^2 - 0.7Z + 0.1)} \qquad \ldots(8)$$

Now the degree of numerator (M) is less than the degree of denominator (N). So the function is in proper form.

$$\therefore \quad \frac{H(Z)}{Z} = \frac{Z^2 - Z + 1}{Z(Z - 0.5)(Z - 0.2)} \qquad \ldots(9)$$

Here poles are $P_1 = 0$, $P_2 = 0.5$ and $P_3 = 0.2$.

We will write Equation (9) in partial fraction expansion form.

$$\therefore \quad \frac{H(Z)}{Z} = \frac{A_1}{Z} + \frac{A_2}{Z - 0.5} + \frac{A_3}{Z - 0.2} \qquad \ldots(10)$$

$$\text{Now } A_1 = (Z-P_1)\frac{H(Z)}{Z}\Big|_{Z=P_1} = \frac{Z(Z^2-Z+1)}{Z(Z-0.5)(Z-0.2)}\Big|_{Z=0}$$

$$\therefore A_1 = \frac{0+0+1}{(0-0.5)(0-0.2)}$$

$$\therefore A_1 = 10.$$

$$A_2 = (Z-P_2)\frac{H(Z)}{Z}\Big|_{Z=P_2} = \frac{(Z-0.5)(Z^2-Z+1)}{Z(Z-0.5)(Z-0.2)}\Big|_{Z=0.5}$$

$$\therefore A_2 = \frac{Z^2-Z+1}{Z(Z-0.2)}\Big|_{Z=0.5} = \frac{(0.5)^2-0.5+1}{0.5(0.5-0.2)}$$

$$\therefore A_2 = 5$$

$$\text{and } A_3 = (Z-P_3)\frac{H(Z)}{Z}\Big|_{Z=P_3} = \frac{(Z-0.2)(Z^2-Z+1)}{Z(Z-0.5)(Z-0.2)}\Big|_{Z=0.2}$$

$$\therefore A_3 = \frac{Z^2-Z+1}{Z(Z-0.5)}\Big|_{Z=0.2} = \frac{(0.2)^2-0.2+1}{0.2(0.2-0.5)}$$

$$\therefore A_3 = -14$$

Putting these values in Equation (10) we get,

$$\frac{H(Z)}{Z} = \frac{10}{Z} + \frac{5}{Z-0.5} + \frac{(-14)}{Z-0.2}$$

$$\therefore H(Z) = 10 + \frac{5Z}{Z-0.5} - \frac{14Z}{Z-0.2} \qquad \ldots(11)$$

The given ROC is $0.5 < |Z| < 1$. So all the terms are causal.

$$\therefore \text{IZT}\{10\} = 10\,\text{IZT}\{1\} = 10\,\delta(n)$$

$$\text{IZT}\left\{\frac{5Z}{Z-0.5}\right\} = 5\,\text{IZT}\left\{\frac{Z}{Z-0.5}\right\} = 5(0.5)^n u(n)$$

$$\text{and } \text{IZT}\left\{\frac{-14Z}{Z-0.2}\right\} = -14\,\text{IZT}\left\{\frac{Z}{Z-0.2}\right\} = -14(0.2)^n u(n).$$

$$\therefore h(n) = 10\,\delta(n) + 5(0.5)^n u(n) - 14(0.2)^n u(n)$$

Prob. 2 : A causal LTI system has transfer function,

$$H(Z) = \frac{(1-0.5Z^{-1})(1-Z^{-1})}{(1+0.2Z^{-1})(1+0.8Z^{-1})(1-0.8Z^{-1})}$$

(i) Give ROC condition. (ii) Show pole zero diagram of system.
(iii) Find impulse response of system. (iv) Comment on system stability.

Soln. :

(i) The given transfer function is,

$$H(Z) = \frac{(1-0.5Z^{-1})(1-Z^{-1})}{(1+0.2Z^{-1})(1+0.8Z^{-1})(1-0.8Z^{-1})} \quad ...(1)$$

To convert it into positive powers of 'Z' we will multiply numerator and denominator by Z^3.

$$\therefore \quad H(Z) = \frac{Z \cdot Z(1-0.5Z^{-1}) \cdot Z(1-Z^{-1})}{Z(1+0.2Z^{-1}) \cdot Z(1+0.8Z^{-1}) \cdot Z(1-0.8Z^{-1})}$$

$$\therefore \quad H(Z) = \frac{Z(Z-0.5)\cdot(Z-1)}{(Z+0.2)(Z+0.8)(Z-0.8)} \quad ...(2)$$

Here, the poles are at -0.2, -0.8 and $+0.8$.
The given system is causal. So ROC should be greater than the poles having larger value.

∴ Required ROC is $|Z| > 0.8$.

(ii) Similar to Equation (2); we can write generalised expression in terms of poles and zeros as,

$$H(Z) = \frac{(Z-Z_1)(Z-Z_2)(Z-Z_3)}{(Z-P_1)(Z-P_2)(Z-P_3)} \quad ...(3)$$

Comparing Equations (2) and (3);

Zeros $\Rightarrow Z_1 = 0$, $Z_2 = 0.5$ and $Z_3 = 1$.

Poles $\Rightarrow P_1 = -0.2$, $P_2 = -0.8$ and $P_3 = 0.8$

The pole-zero diagram is as shown in Fig. P-14.

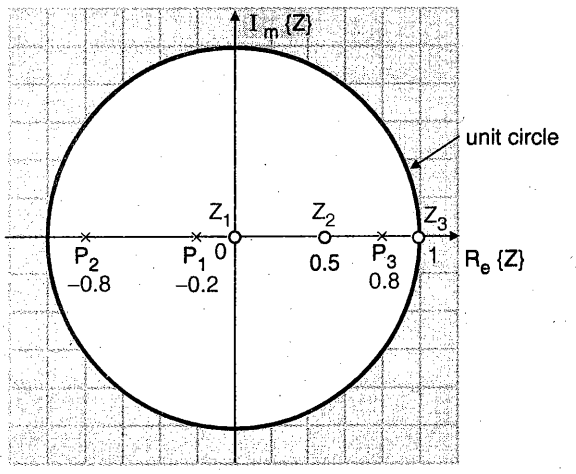


Fig. P-14

(iii) To calculate impulse response h (n); we have to obtain IZT of H (Z). From Equation (2) we get,

$$\frac{H(Z)}{Z} = \frac{(Z-0.5)(Z-1)}{(Z+0.2)(Z+0.8)(Z-0.8)} \quad ...(4)$$

In P.F.E. form Equation (4) can be written as,

$$\frac{H(Z)}{Z} = \frac{A_1}{Z+0.2} + \frac{A_2}{Z+0.8} + \frac{A_3}{Z-0.8} \quad ...(5)$$

The values of A_1, A_2 and A_3 are calculated as follows :

$$A_1 = (Z-P_1)\frac{H(Z)}{Z}\Big|_{Z=P_1}$$

$$\therefore A_1 = (Z+0.2)\cdot\frac{(Z-0.5)(Z-1)}{(Z+0.2)(Z+0.8)(Z-0.8)}\Big|_{Z=-0.2}$$

$$\therefore A_1 = \frac{(-0.2-0.5)(-0.2-1)}{(-0.2+0.8)(-0.2-0.8)} = \frac{(-0.7)(-1.2)}{(0.6)(-1)}$$

$$\therefore A_1 = -1.4$$

$$\therefore A_2 = (Z-P_2)\frac{H(Z)}{Z}\Big|_{Z=P_2}$$

$$\therefore A_2 = (Z+0.8)\frac{(Z-0.5)(Z-1)}{(Z+0.2)(Z+0.8)(Z-0.8)}\Big|_{Z=-0.8}$$

$$\therefore A_2 = \frac{(-0.8-0.5)(-0.8-1)}{(-0.8+0.2)(-0.8-0.8)} = \frac{(-1.3)(-1.8)}{(-0.6)(-1.6)}$$

$$\therefore A_2 = 2.44$$

$$A_3 = (Z-P_3)\frac{H(Z)}{Z}\Big|_{Z=P_3}$$

$$\therefore A_3 = (Z-0.8)\cdot\frac{(Z-0.5)(Z-1)}{(Z+0.2)(Z+0.8)(Z-0.8)}\Big|_{Z=0.8}$$

$$\therefore A_3 = \frac{(0.8-0.5)(0.8-1)}{(0.8+0.2)(0.8+0.8)} = \frac{(0.3)(-0.2)}{1(1.6)}$$

$$\therefore A_3 = -0.0375$$

Putting these values in Equation (5) we get,

$$\frac{H(Z)}{Z} = \frac{-1.4}{Z+0.2} + \frac{2.44}{Z+0.8} - \frac{0.0375}{Z-0.8}$$

$$\therefore \quad H(Z) = \frac{-1.4\,Z}{Z+0.2} + \frac{2.44\,Z}{Z+0.8} - \frac{0.0375\,Z}{Z-0.8} \qquad ...(6)$$

Since the given function is causal; its IZT can be written as,

$$h(n) = -1.4(-0.2)^n u(n) + 2.44(-0.8)^n u(n) - 0.0375(0.8)^n u(n)$$

(iv) Since all the poles lie inside the unit circle; the given system is stable.

Prob. 3 : A causal LTI filter has transfer function

$$H(Z) = \frac{1}{1 - 1.5\,Z^{-1} - Z^{-2}}$$

What is the ROC of transfer function ? Is the system stable.

Soln. :

Given : $$H(Z) = \frac{1}{1 - 1.5\,Z^{-1} - Z^{-2}}$$

Multiply numerator and denominator by Z^2

$$\therefore \quad H(Z) = \frac{Z^2}{Z^2 - 1.5\,Z - 1} \qquad ...(1)$$

We will obtain the roots of denominator term $Z^2 - 1.5\,Z - 1$

Here a = 1, b = – 1.5 and c = – 1

$$\text{roots} = \frac{-b \pm \sqrt{b^2 - 4\,ac}}{2\,a} = \frac{1.5 \pm \sqrt{(1.5)^2 - 4(1)(-1)}}{2}$$

$$\therefore \quad \text{roots} = \frac{1.5 \pm 2.5}{2} = 2 \quad \text{and} \quad -\frac{1}{2}$$

$$\therefore \quad H(Z) = \frac{Z^2}{(Z-2)\left(Z+\frac{1}{2}\right)}$$

$$\therefore \quad \frac{H(Z)}{Z} = \frac{Z}{(Z-2)\left(Z+\frac{1}{2}\right)} \qquad ...(2)$$

In P.F.E. form, Equation (2) can be written as,

$$\frac{H(Z)}{Z} = \frac{A_1}{Z-2} + \frac{A_2}{Z+\frac{1}{2}} \qquad ...(3)$$

Given system is causal, so for first term ROC is $|Z| > 2$ and for second term ROC is $|Z| > 2$. Since this ROC does not include unit circle; the system is unstable.

Prob. 4 : If $$H(Z) = \frac{Z^2 - 6Z + 8}{\left(Z - \frac{1}{3}\right)(Z-4)(Z+3)}$$

What is the ROC condition to make the system stable ?

Soln. :

Given : $$H(Z) = \frac{Z^2 - 6Z + 8}{\left(Z - \frac{1}{3}\right)(Z-4)(Z+3)}$$

For the system to be stable; ROC must include unit circle.

For given H (Z) ROC should be,

$|Z| > \frac{1}{3}$, $|Z| < 4$ and $|Z| < 3$

Thus the required ROC is,

$$\frac{1}{3} < |Z| < 3$$

Prob. 5 : A system transfer function is given by,

$$H(Z) = \frac{3 - 4Z^{-1}}{1 - 3.5Z^{-1} + 1.5Z^{-2}}$$

Find ROC of the system if system is stable.

Soln. :

$$\text{Given :} \quad H(Z) = \frac{3 - 4Z^{-1}}{1 - 3.5Z^{-1} + 1.5Z^{-2}}$$

Multiply numerator and denominator by Z^2

$$\therefore \quad H(Z) = \frac{3Z^2 - 4Z}{Z^2 - 3.5Z + 1.5} \qquad ...(1)$$

We will obtain the roots of denominator term $Z^2 - 3.5Z + 1.5$

Here a = 1, b = – 3.5 and c = 1.5.

$$\text{roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{3.5 \pm \sqrt{(3.5)^2 - (4 \times 1.5)}}{2}$$

$$\therefore \quad \text{roots} = \frac{3.5 \pm 2.5}{2} = 3 \text{ and } 0.5$$

For the system to be stable; ROC must include unit circle. Thus ROC is $|Z| < 3$ and $|Z| > 0.5$.

Thus required ROC is, $0.5 < |Z| < 3$

Prob. 6 : An LTI system is characterized by the transfer function,

$$H(Z) = \frac{3 - 4Z^{-1}}{1 - 3.5Z^{-1} + 1.5Z^{-2}}$$

Determine h (n) for the following conditions :

(i) The system is stable.

(ii) The system is causal.

(iii) The system is non-causal.

Soln. : The given Equation is,

$$H(Z) = \frac{3-4Z^{-1}}{1-3.5Z^{-1}+1.5Z^{-2}} \quad \ldots(1)$$

To convert it into positive powers of Z; multiply and divide Equation (1) by Z^2

$$\therefore \quad H(Z) = \frac{Z^2(3-4Z^{-1})}{Z^2(1-3.5Z^{-1}+1.5Z^{-2})}$$

$$\therefore \quad H(Z) = \frac{3Z^2-4Z}{Z^2-3.5Z+1.5} = \frac{Z(3Z-4)}{Z^2-3.5Z+1.5}$$

$$\therefore \quad \frac{H(Z)}{Z} = \frac{3Z-4}{Z^2-3.5Z+1.5} \quad \ldots(2)$$

Since M < N; this function is in the proper form. Now we will obtain roots (poles) of denominator.

$$P_1, P_2 = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{3.5 \pm \sqrt{(3.5)^2-(4\times 1.5)}}{2}$$

$$\therefore \quad P_1, P_2 = \frac{3.5 \pm 2.5}{2} \qquad \therefore \quad P_1 = 3 \quad \text{and} \quad P_2 = 0.5$$

Thus Equation (2) becomes,

$$\frac{H(Z)}{Z} = \frac{3Z-4}{(Z-3)(Z-0.5)} \quad \ldots(3)$$

In the partial expansion form; Equation (3) can be written as,

$$\frac{H(Z)}{Z} = \frac{A_1}{Z-3} + \frac{A_2}{Z-0.5} \quad \ldots(4)$$

$$\text{Here} \quad A_1 = (Z-P_1)\frac{H(Z)}{Z}\Big|_{Z=P_1} = \frac{(Z-3)(3Z-4)}{(Z-3)(Z-0.5)}\Big|_{Z=3}$$

$$\therefore \quad A_1 = \frac{3Z-4}{Z-0.5}\Big|_{Z=3} = \frac{9-4}{3-0.5}$$

$$\therefore \quad A_1 = 2$$

$$\text{and} \quad A_2 = (Z-P_2)\frac{H(Z)}{Z}\Big|_{Z=P_2} = \frac{(Z-0.5)(3Z-4)}{(Z-3)(Z-0.5)}\Big|_{Z=0.5}$$

$$\therefore \quad A_2 = \frac{3Z-4}{Z-3}\Big|_{Z=0.5}$$

$$= \frac{3(0.5)-4}{0.5-3}$$

$$\therefore \quad A_2 = 1$$

Putting values of A_1 and A_2 in Equation (4) we get,

$$\frac{H(Z)}{Z} = \frac{2}{(Z-3)} + \frac{1}{(Z-0.5)}$$

$$\therefore \quad H(Z) = \frac{2Z}{Z-3} + \frac{Z}{Z-0.5} \qquad ...(5)$$

Now we will determine h (n) for the given conditions.

(i) h (n) for stable system :

Here the poles are at $P_1 = 3$ and $P_2 = 0.5$. We know that for the system to be stable; ROC of H (Z) must include unit circle. Thus considering given poles; ROC can be written as

$$\text{ROC} : \quad 0.5 < |Z| < 3 \qquad ...(6)$$

Now for first term in Equation (5) to be causal ROC should be greater than 3. But from Equation (6) it is less than 3. So this term is anticausal.

$$\therefore \quad \text{IZT}\left\{\frac{2Z}{Z-3}\right\} = 2\,\text{IZT}\left\{\frac{Z}{Z-3}\right\} = 2\left(-(3)^n u(-n-1)\right) \qquad ...(7)$$

For the second term of Equation (5) to be causal; the ROC should be greater than 0.5. According to Equation (6) it is greater than 0.5. So this term is causal.

$$\therefore \quad \text{IZT}\left\{\frac{Z}{Z-0.5}\right\} = (0.5)^n u(n) \qquad ...(8)$$

Impulse response h (n) is obtained by adding Equations (7) and (8).

$$\therefore \quad h(n) = (0.5)^n u(n) - 2(3)^n u(-n-1)$$

(ii) h (n) for causal system :

For the first term to be causal ROC should be greater than 3 and for the second term to be causal ROC should be greater than 0.5. Now if we will say that ROC is greater than 3 then by default it is greater than 0.5. Thus for system to be causal ROC must be greater than 3. So for causal system

$$\therefore \quad h(n) = \text{IZT}\left\{\frac{2Z}{Z-3}\right\} + \text{IZT}\left\{\frac{Z}{Z-0.5}\right\}$$

$$\therefore \quad h(n) = 2(3)^n u(n) + (0.5)^n u(n)$$

(iii) h (n) for anticausal system :

The first term will be anticausal if ROC is $|Z| < 3$ and the second term will be anticausal if ROC is $|Z| < 0.5$. Now for total system, if we will say ROC is less than 0.5 then by default it is less than 3. Thus for this system to be anticausal ROC is, $|Z| < 0.5$.

$$\therefore \quad h(n) = \text{IZT}\left\{\frac{2Z}{Z-3}\right\} + \text{IZT}\left\{\frac{Z}{Z-0.5}\right\}$$

$$\therefore \quad h(n) = -2(3)^n u(-n-1) - (0.5)^n u(-n-1)$$

Note : Here ROC is $|Z| < 0.5$. That means it is interior part of circle having radius 0.5. Thus it will not include unit circle. So the system is unstable.

Prob. 7 : Find impulse response corresponding to each of the following system transfer function and comment on the stability and causality of each.

(i) $H(Z) = \dfrac{Z^2}{Z-1}$

(ii) $H(Z) = \dfrac{Z^3+Z^2+Z^{-1}+Z^{-2}}{Z^2}$

Soln. :

(i) Given $H(Z) = \dfrac{Z^2}{Z-1}$ ...(1)

We will perform the long division as follows :

$$
\begin{array}{r|l}
 & Z+1+Z^{-1}+Z^{-2}+Z^{-3} \\
\hline
Z-1 & Z^2 \\
 & Z^2-Z \\
 & - \quad + \\
\hline
 & Z \\
 & Z-1 \\
 & - \quad + \\
\hline
 & 1 \\
 & 1-Z^{-1} \\
 & - \quad + \\
\hline
 & Z^{-1} \\
 & Z^{-1}-Z^{-2} \\
 & - \quad + \\
\hline
 & Z^{-2} \\
 & Z^{-2}-Z^{-3} \\
 & - \quad + \\
\hline
 & Z^{-3} \ldots
\end{array}
$$

Thus approximately we can write,

$$H(Z) = Z+1+Z^{-1}+Z^{-2}+Z^{-3}+\ldots \quad \ldots(2)$$

According to the definition of Z transform,

$$H(Z) = \sum_{n=-\infty}^{\infty} h(n)Z^{-n}$$

Expanding the summation we get,

$$H(Z) = \ldots + h(-1)Z^{+1} + h(0) + h(1)Z^{-1} + h(2)Z^{-2} + h(3)Z^{-3} + \ldots \quad \ldots(3)$$

Comparing Equations (2) and (3) we get,

$$h(-1) = 1, h(0) = 1, h(1) = 1, h(2) = 1, h(3) = 1 \ldots$$

$$\therefore h(n) = \{1, 1, 1, 1, \ldots\}$$
$$\uparrow$$

...(4)

This is the required impulse response. Since it contains negative samples; it is non-causal system. It is infinite duration sequence, so for every bounded input; unbounded output is obtained. Thus system is unstable.

(ii) Given $H(Z) = \dfrac{Z^3 + Z^2 + Z^{-1} + Z^{-2}}{Z^2}$

$$\therefore H(Z) = Z + 1 + Z^{-1} + Z^{-2} \quad ..(5)$$

Comparing Equations (5) and (3) we get,

$$h(-1) = 1, h(0) = 1, h(1) = 1 \text{ and } h(2) = 1.$$

$$\therefore h(n) = \{1, 1, 1, 1\}$$
$$\uparrow$$

...(6)

This sequence contains negative samples; so it is non-causal. But this sequence contains finite number of samples. So every bounded input produces bounded output. Thus this system is stable.

Prob. 8 : Find ROC of system, which is known to be stable and has transfer function.

$$H(Z) = \frac{1 - 2Z^{-1}}{1 + 3Z^{-1} + 9Z^{-2}}$$

Soln. :

Given : $H(Z) = \dfrac{1 - 2Z^{-1}}{1 + 3Z^{-1} + 9Z^{-2}}$

Multiply numerator and denominator by Z^2

$$\therefore H(Z) = \frac{Z^2 - 2Z}{Z^2 + 3Z + 9}$$

We will obtain roots of denominator term;

$$\text{roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-3 \pm \sqrt{9 - 36}}{2} = \frac{-3 \pm \sqrt{-27}}{2}$$

$$\therefore \text{roots} = \frac{-3 \pm j\sqrt{27}}{2}$$

$$= -1.5 + j2.6, -1.5 - j2.6$$

That means poles are at,

$$P_1 = -1.5 + j\,2.6 \quad \text{and} \quad P_2 = -1.5 - j\,2.6$$

We will obtain magnitudes of poles,

$$|P_1| = \sqrt{(-1.5)^2 + (2.6)^2} = 3$$

and $$|P_2| = \sqrt{(-1.5)^2 + (-2.6)^2} = 3$$

For the system to be stable, ROC must include unit circle.
Thus ROC is $|Z| < 3$

Prob. 9 : A DT system has transfer function,

$$H(Z) = \frac{1 + 2Z^{-1}}{(1 + 0.5Z^{-1})(1 + 3Z^{-1})}$$

Give all possible ROCs of this system. State in each case whether the system is stable.

Soln. :

Given : $$H(Z) = \frac{1 + 2Z^{-1}}{(1 + 0.5Z^{-1})(1 + 3Z^{-1})}$$

Multiply numerator and denominator by Z^2

$$\therefore \quad H(Z) = \frac{Z^2(1 + 2Z^{-1})}{Z(1 + 0.5Z^{-1})Z(1 + 3Z^{-1})}$$

$$\therefore \quad H(Z) = \frac{Z^2 + 2Z}{(Z + 0.5)(Z + 3)}$$

Here poles are at $P_1 = -0.5$ and $P_2 = -3$. The possible ROCs are as follows :

i) When the system is causal,

ROC is $|Z| > 3$.

This ROC does not include unit circle; so system is unstable.

ii) When system is anticausal,

ROC is $|Z| < 0.5$

This ROC doesnot include unit circle; so system is unstable.

iii) When system is both sided.

ROC is $|Z| > 0.5$ and $|Z| < 3$

That means ROC is $0.5 < |Z| < 3$.
This ROC includes unit circle; so system is stable.

Prob. 10 : When input to causal LTI system is,

$$x(n) = -\frac{1}{3}\left(\frac{1}{2}\right)^n u(n) - \frac{4}{3}2^n u(-n-1)$$

Z-transform of output is,

$$Y(Z) = \frac{1+Z^{-1}}{(1-Z^{-1})\left(1+\frac{1}{2}Z^{-1}\right)(1-2Z^{-1})}$$

a) Find Z-transform of x (n).
b) What is region of convergence of Y (Z).
c) Find impulse response of system.
d) Is the system stable.

Soln. :

a) Given : $$x(n) = -\frac{1}{3}\left(\frac{1}{2}\right)^n u(n) - \frac{4}{3}2^n u(-n-1) \quad ...(1)$$

Let $x_1(n) = -\frac{1}{3}\left(\frac{1}{2}\right)^n u(n)$ and $x_2(n) = -\frac{4}{3}2^n u(-n-1)$

First term of Equation (1) is causal; since it is multiplied by U (n). Use standard Z-transform pair.

$$\alpha^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha}, \qquad ROC : |Z| > |\alpha|$$

$$\therefore \quad X_1(Z) = -\frac{1}{3}\left[\frac{Z}{Z-\frac{1}{2}}\right], \qquad ROC : |Z| > \frac{1}{2}$$

Second term is anticausal, since it is multiplied by u (− n − 1)
Use standard Z-transform pair.

$$-\alpha^n u(-n-1) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha}, \qquad ROC : |Z| < |\alpha|$$

$$\therefore \quad X_2(Z) = +\frac{4}{3}\cdot\frac{Z}{Z-2}, \qquad ROC : |Z| < 2$$

Now $X(Z) = X_1(Z) + X_2(Z)$

$$\therefore X(Z) = -\frac{1}{3}\left[\frac{Z}{Z-\frac{1}{2}}\right] + \frac{4}{3}\left[\frac{Z}{Z-2}\right] \quad ...(2)$$

b) Given $$Y(Z) = \frac{1+Z^{-1}}{(1+Z^{-1})\left(1+\frac{1}{2}Z^{-1}\right)\cdot(1-2Z^{-1})}$$

Multiply numerator and denominator by Z^3

$$Y(Z) = \frac{Z^3(1+Z^{-1})}{Z\left(1-Z^{-1}\right)\left(1+\frac{1}{2}Z^{-1}\right)\left(1-2Z^{-1}\right)}$$

$$\therefore \quad Y(Z) = \frac{Z^3+Z^2}{(Z-1)\left(Z+\frac{1}{2}\right)(Z-2)} \qquad ...(3)$$

Given LTI system is causal; thus ROC is $|Z| > 2$.

c) We have,

$$H(Z) = \frac{Y(Z)}{X(Z)}$$

$$\therefore \quad H(Z) = \left[\frac{Z^3+Z^2}{(Z-1)\left(Z+\frac{1}{2}\right)(Z-2)}\right]\cdot\left[\frac{1}{-\frac{1}{3}\left(\frac{Z}{Z-\frac{1}{2}}\right)+\frac{4}{3}\left(\frac{Z}{Z-2}\right)}\right]$$

We will simplify second bracket as follows :

$$\therefore \quad H(Z) = \left[\frac{Z^3+Z^2}{(Z-1)\left(Z+\frac{1}{2}\right)(Z-2)}\right]\left[\frac{3}{\frac{-Z}{Z-\frac{1}{2}}+\frac{4Z}{Z-2}}\right]$$

$$\therefore \quad H(Z) = \left[\frac{Z^3+Z^2}{(Z-1)\left(Z+\frac{1}{2}\right)(Z-2)}\right]\left[\frac{3}{\frac{-Z^2+2Z+4Z^2-2Z}{\left(Z-\frac{1}{2}\right)(Z-2)}}\right]$$

$$\therefore \quad H(Z) = \left[\frac{Z^3+Z^2}{(Z-1)\left(Z+\frac{1}{2}\right)(Z-2)}\right]\left[\frac{3\left(Z-\frac{1}{2}\right)(Z-2)}{3Z^2}\right]$$

$$\therefore \quad H(Z) = \left[\frac{Z^2(Z+1)}{(Z-1)\left(Z+\frac{1}{2}\right)(Z-2)}\right]\left[\frac{\left(Z-\frac{1}{2}\right)(Z-2)}{Z^2}\right]$$

$$\therefore \quad H(Z) = \frac{(Z+1)\left(Z-\frac{1}{2}\right)}{(Z-1)\left(Z+\frac{1}{2}\right)}$$

$$\therefore \quad \frac{H(Z)}{Z} = \frac{(Z+1)\left(Z-\frac{1}{2}\right)}{Z(Z-1)\left(Z+\frac{1}{2}\right)} \qquad ...(4)$$

In P.F.E. form we can write,

$$\frac{H(Z)}{Z} = \frac{A_1}{Z} + \frac{A_2}{Z-1} + \frac{A_3}{Z+\frac{1}{2}} \qquad ...(5)$$

The values of A_1, A_2 and A_3 are calculated as follows :

$$A_1 = Z \cdot \frac{H(Z)}{Z}\Big|_{Z=0}$$

$$\therefore \quad A_1 = \frac{Z(Z+1)\left(Z-\frac{1}{2}\right)}{Z(Z-1)\left(Z+\frac{1}{2}\right)}\Bigg|_{Z=0} \qquad \therefore \quad A_1 = \frac{(1)\left(-\frac{1}{2}\right)}{(-1)\left(\frac{1}{2}\right)} = 1$$

$$A_2 = \frac{(Z-1)\cdot(Z+1)\left(Z-\frac{1}{2}\right)}{Z(Z-1)\left(Z+\frac{1}{2}\right)}\Bigg|_{Z=1}$$

$$\therefore \quad A_2 = \frac{(1+1)\left(1-\frac{1}{2}\right)}{1\left(1+\frac{1}{2}\right)} = \frac{1}{3/2} = \frac{2}{3}$$

$$A_3 = \frac{\left(Z+\frac{1}{2}\right)\cdot(Z+1)\left(Z-\frac{1}{2}\right)}{Z(Z-1)\left(Z+\frac{1}{2}\right)}\Bigg|_{Z=\frac{-1}{2}}$$

$$\therefore \quad A_3 = \frac{\left(-\frac{1}{2}+1\right)\left(-\frac{1}{2}-\frac{1}{2}\right)}{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)} = \frac{\left(\frac{1}{2}\right)(-1)}{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)} = -\frac{2}{3}$$

Putting these values in Equation (5) we get,

$$\frac{H(Z)}{Z} = \frac{1}{Z} + \frac{2/3}{Z-1} - \frac{2/3}{Z+\frac{1}{2}}$$

$$\therefore \quad H(Z) = 1 + \frac{2}{3}\cdot\frac{Z}{Z-1} - \frac{2}{3}\cdot\frac{Z}{Z+\frac{1}{2}}$$

Since the given system is causal; its IZT can be written as,

$$h(n) = \delta(n) + \frac{2}{3}\cdot(1)^n u(n) - \frac{2}{3}\cdot\left(-\frac{1}{2}\right)^n u(n)$$

d) ROC for this total system is $|Z| > 1$. So the system is stable.

Prob. 11 : Consider a causal LSI system function :

$$H(Z) = \frac{1 - a^{-1}Z^{-1}}{1 - aZ^{-1}} \quad \text{where 'a' is real.}$$

Determine the value of 'a' for which the system is stable. Show graphically for $0 < a < 1$; pole-zero plot and the ROC.

Soln. : The given function is,

$$H(Z) = \frac{1 - a^{-1}Z^{-1}}{1 - aZ^{-1}} \qquad ...(1)$$

To convert it into positive powers of Z; multiply numerator and denominator by Z.

$$\therefore \quad H(Z) = \frac{Z(1 - a^{-1}Z^{-1})}{Z(1 - aZ^{-1})} = \frac{Z - a^{-1}}{Z - a}$$

$$\therefore \quad H(Z) = \frac{\left(Z - \frac{1}{a}\right)}{(Z - a)} \qquad ...(2)$$

Now we have the standard equation,

$$H(Z) = \frac{Z - Z_1}{Z - P_1} \qquad ...(3)$$

Comparing Equations (2) and (3).

Zero is at, $Z_1 = \frac{1}{a}$ and pole is at $P_1 = a$.

Now in the question it is mentioned that, the system is causal.

We know that for the causal system ROC is exterior part of circle. Thus here ROC is $|Z| > |a|$. We want this system to be stable also. We know that for the system to be stable ROC should include unit circle. Thus to satisfy this condition; *value of 'a' should be less than 1.* So for a causal and stable system; we can write the ROC as,

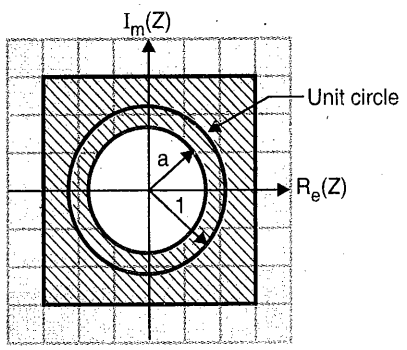


Fig. P-15(a) ROC : $|Z| > |a| < 1$

ROC : $|Z| > |a|$ and $|a| < 1$

$\therefore$ ROC : $|Z| > |a| < 1$.

This ROC is shown in Fig. P-15(a).

Zero is at $Z_1 = \frac{1}{a}$ and pole is at $P_1 = a$. Here the value of 'a' is less than 1. The pole zero plot is shown in Fig. P-15(b).

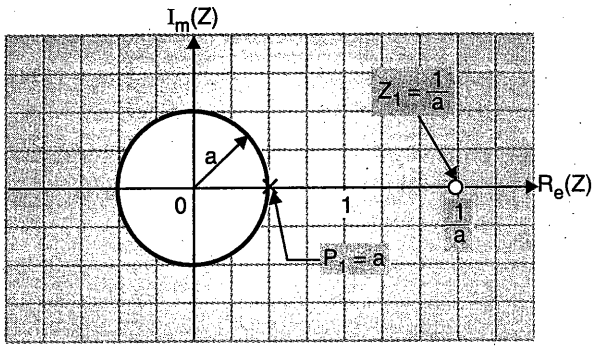


Fig. P-15(b) : Pole-zero plot

Prob. 12 : Determine whether the system is stable or not

$$H(Z) = \frac{Z(Z-0.5)}{(Z-0.6-j0.6)(Z-0.6+j0.6)}$$

Soln. : In terms of poles and zeros, equation of H(Z) can be written as,

$$H(Z) = \frac{(Z-Z_1)}{(Z-P_1)}\frac{(Z-Z_2)}{(Z-P_2)}$$

Thus zeros are at, $Z_1 = 0$ and $Z_2 = 0.5$

And poles are at, $P_1 = 0.6+j0.6$ and $P_2 = 0.6-j0.6$

The pole zero plot is as shown in Fig. P-16.

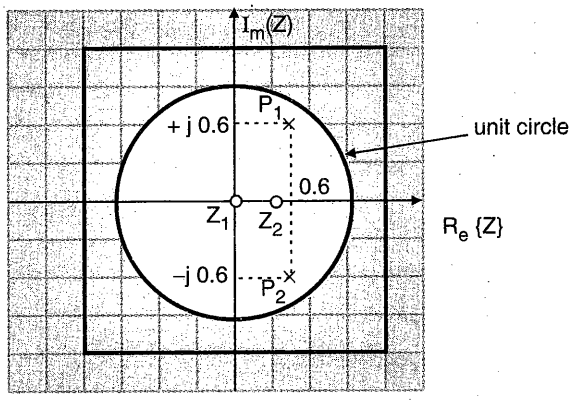


Fig. P-16

Since both the poles lie inside unit circle; the given system is stable.

Prob. 13 : Consider the purely recursive LTI second order system given by,

$$H(Z) = \frac{b_0}{1 + a_1 Z^{-1} + a_2 Z^{-2}}$$

Discuss the stability of the system and find out range of values of a_1 and a_2 to obtain the stability. Also comment upon the nature of poles for this stable system.

Soln. : The given equation is,

$$H(Z) = \frac{b_0}{1 + a_1 Z^{-1} + a_2 Z^{-2}} \quad ...(1)$$

To convert it in terms of positive powers of 'Z'; multiply numerator and denominator by Z^2.

$$\therefore \quad H(Z) = \frac{b_0 Z^2}{Z^2 + a_1 Z + a_2} \quad ...(2)$$

Now we will obtain roots of denominator (Poles).

$$\therefore \quad P_1, P_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here in Equation (2), a = 1, b = a_1 and c = a_2.

$$\therefore \quad P_1, P_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

$$\therefore \quad P_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2} \quad ...(3)$$

$$\text{and} \quad P_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2} \quad ...(4)$$

We know that for the system to be stable; poles must be inside the unit circle.

$$\therefore \quad |P_1| < 1 \quad \text{and} \quad |P_2| < 1 \quad ...(5)$$

Now from Equations (3) and (4) we can write,

$$a_1 = -(P_1 + P_2) \quad ...(6)$$

$$\text{and} \quad a_2 = P_1 \times P_2 \quad ...(7)$$

Taking magnitude of both sides of Equation (7).

$$|a_2| = |P_1 P_2| \quad ...(8)$$

Taking magnitudes of P_1 and P_2 separately we get,

$$|a_2| \leq |P_1| |P_2| \quad ...(9)$$

But from Equation (5) $|P_1| < 1$ and $|P_2| < 1$.

$$\therefore |a_2| < 1$$

This is the required range of a_2 . Therefore for the system to be stable,

$$\boxed{|a_1| = |P_1 + P_2| < 1 + a_2}$$

Prob. 14 : A causal LTI system has transfer function,

$$H(Z) = H_1(Z) \cdot H_2(Z)$$

$$H_1(Z) = \frac{1 - 0.2\,Z^{-1}}{1 + 0.5\,Z^{-1}}, \qquad H_2(Z) = \frac{1}{1 + 0.3\,Z^{-1}}$$

(i) If system is stable, give ROC condition.

(ii) Find impulse response.

(iii) Find system response if $X(Z) = \dfrac{1}{1 - 0.2\,Z^{-1}}$

(iv) Draw pole-zero diagram.

Soln. :

(i) We have,

$$H(Z) = H_1(Z) \cdot H_2(Z)$$

$$\therefore \quad H(Z) = \left(\frac{1 - 0.2\,Z^{-1}}{1 + 0.5\,Z^{-1}}\right)\left(\frac{1}{1 + 0.3\,Z^{-1}}\right)$$

To convert $H(Z)$ into positive powers of Z; we will multiply numerator and denominator by Z^2.

$$\therefore \quad H(Z) = \frac{Z^2\,(1 - 0.2\,Z^{-1})}{Z\,(1 + 0.5\,Z^{-1}) \cdot Z\,(1 + 0.3\,Z^{-1})}$$

$$\therefore \quad H(Z) = \frac{Z^2 - 0.2\,Z}{(Z + 0.5)\,(Z + 0.3)} \qquad \ldots(1)$$

Here poles are at $P_1 = -0.5$ and $P_2 = -0.3$. The given system is causal. Now for the system to be stable; ROC must include unit circle. So ROC should be $|Z| > 0.5$.

(ii) To obtain impulse response, we will take IZT of equation (1).

$$\therefore \quad \frac{H(Z)}{Z} = \frac{(Z - 0.2)}{(Z + 0.5)\,(Z + 0.3)} \qquad \ldots(2)$$

In PFE form, Equation (2) can be written as,

$$\frac{H(Z)}{Z} = \frac{A_1}{Z + 0.5} + \frac{A_2}{Z + 0.3} \qquad \ldots(3)$$

The coefficients A_1 and A_2 are calculated as follows :

$$A_1 = (Z - P_1)\,\frac{H(Z)}{Z}\bigg|_{Z = P_1} = \frac{(Z + 0.5) \cdot (Z - 0.2)}{(Z + 0.5)\,(Z + 0.3)}\bigg|_{Z = -0.5}$$

$$\therefore \quad A_1 = \frac{-0.5 - 0.2}{-0.5 + 0.3} = \frac{-0.7}{-0.2}$$

$$\boxed{\therefore A_1 = \frac{7}{2}}$$

$$\text{and} \quad A_2 = (Z-P_2)\frac{H(Z)}{Z}\Bigg|_{Z=P_2} = \frac{(Z+0.3)(Z-0.2)}{(Z+0.5)(Z+0.3)}\Bigg|_{Z=-0.3}$$

$$\therefore \quad A_2 = \frac{-0.3-0.2}{-0.3+0.5} = \frac{-0.5}{0.2}$$

$$\therefore \quad A_2 = -5/2$$

Putting these values in Equation (3) we get,

$$\frac{H(Z)}{Z} = 7/2 \cdot \frac{1}{Z+0.5} - \frac{5}{2} \cdot \frac{1}{Z+0.3}$$

$$\therefore \quad H(Z) = \frac{7}{2} \cdot \left(\frac{Z}{Z+0.5}\right) - \frac{5}{2}\left(\frac{Z}{Z+0.3}\right) \qquad \ldots(4)$$

Since the given system is causal, IZT can be written as,

$$h(n) = \frac{7}{2}(-0.5)^n u(n) - \frac{5}{2}(-0.3)^n u(n)$$

(iii) Given, $$X(Z) = \frac{1}{1-0.2Z^{-1}}$$

$$\therefore \quad X(Z) = \frac{Z}{Z-0.2} \qquad \ldots(5)$$

We have, $$H(Z) = \frac{Y(Z)}{X(Z)}$$

$$\therefore \quad Y(Z) = H(Z) \cdot X(Z) \qquad \ldots(6)$$

Putting Equations (1) and (5) in Equation (6) we get,

$$Y(Z) = \frac{Z^2-0.2Z}{(Z+0.5)(Z+0.3)} \cdot \left[\frac{Z}{Z-0.2}\right]$$

$$\therefore \quad Y(Z) = \frac{Z(Z^2-0.2Z)}{(Z+0.5)(Z+0.3)(Z-0.2)} \qquad \ldots(7)$$

To obtain system response, we will calculate IZT of Equation (7),

$$\therefore \quad \frac{Y(Z)}{Z} = \frac{Z^2-0.2Z}{(Z+0.5)(Z+0.3)(Z-0.2)} \qquad \ldots(8)$$

In PFE form Equation (8) can be written as,

$$\frac{Y(Z)}{Z} = \frac{A_1}{Z+0.5} + \frac{A_2}{Z+0.3} + \frac{A_3}{Z-0.2} \qquad \ldots(9)$$

The values of A_1, A_2 and A_3 are calculated as follows :

$$A_1 = (Z+0.5) \cdot \frac{Z^2-0.2Z}{(Z+0.5)(Z+0.3)(Z-0.2)}\Bigg|_{Z=-0.5}$$

$$\therefore \quad A_1 = \frac{Z^2-0.2Z}{(Z+0.3)(Z-0.2)}\Bigg|_{Z=-0.5}$$

$$\therefore \quad A_1 = \frac{(-0.5)^2 [(0.2)(-0.5)]}{(-0.5+0.3)(-0.5-0.2)} = \frac{0.25+0.1}{(-0.2)(-0.7)}$$

$$\therefore \quad A_1 = 2.5$$

$$A_2 = (Z+0.3) \cdot \frac{(Z^2-0.2Z)}{(Z+0.5)(Z+0.3)(Z-0.2)} \Bigg|_{Z=-0.3}$$

$$\therefore \quad A_2 = \frac{(-0.3)^2-(0.2)(-0.3)}{(-0.3+0.5)(-0.3-0.2)} = \frac{0.09+0.06}{(0.2)(-0.5)}$$

$$\therefore \quad A_2 = -1.5$$

$$A_3 = \frac{(Z-0.2)(Z^2-0.2Z)}{(Z+0.5)(Z+0.3)(Z-0.2)} \Bigg|_{Z=0.2}$$

$$\therefore \quad A_3 = \frac{(0.2)^2-(0.2)(0.2)}{(0.2+0.5)(0.2+0.5)} = \frac{0.04-0.04}{(0.7)(0.5)}$$

$$A_3 = 0$$

Putting these values in Equation (9) we get,

$$\frac{Y(Z)}{Z} = \frac{2.5}{Z+0.5} - \frac{1.5}{Z+0.3}$$

$$\therefore \quad Y(Z) = \frac{2.5Z}{Z+0.5} - \frac{1.5Z}{Z+0.3}$$

Since the system is causal; its IZT is,

$$y(n) = 2.5(-0.5)^n u(n) - 1.5(-0.3)^n u(n)$$

(iv) We have equation of $H(Z)$,

$$H(Z) = \frac{Z^2-0.2Z}{(Z+0.5)(Z+0.3)}$$

$$\therefore \quad H(Z) = \frac{Z(Z-0.2)}{(Z+0.5)(Z+0.3)}$$

Zeros are at $Z_1 = 0$ and $Z_2 = 0.2$.

Poles are at $P_1 = -0.5$ and $P_2 = -0.3$.

The pole zero diagram is shown in Fig. P-17.

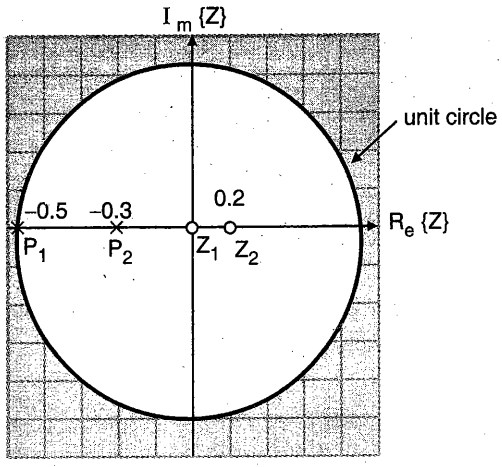


Fig. P-17

Prob. 15 : The transfer function of a causal LTI system is,

$$H(Z) = \frac{1 - Z^{-1}}{1 + \frac{3}{4} Z^{-1}}$$

(i) Find the impulse response of the system

(ii) Find the output of the system to the input

$$x(n) = \left(\frac{1}{3}\right)^{n} u(n) + u(-n-1)$$

(iii) Is the system stable ?

Soln. :

(i) Calculation of impulse response h (n) :

$$\text{Given, } H(Z) = \frac{1 - Z^{-1}}{1 + \frac{3}{4} Z^{-1}} \qquad \ldots(1)$$

We will obtain h (n) by taking IZT of H (Z) using partial fraction expansion. Multiplying numerator and denominator by Z we get,

$$H(Z) = \frac{Z - 1}{Z + \frac{3}{4}}$$

$$\frac{H(Z)}{Z} = \frac{Z - 1}{Z\left(Z + \frac{3}{4}\right)} \qquad \ldots(2)$$

The poles are at $P_1 = 0$ and $P_2 = -\frac{3}{4}$. We can write Equation (2) in the partial fraction expansion form as,

$$\frac{H(Z)}{Z} = \frac{A_1}{(Z-0)} + \frac{A_2}{\left(Z+\frac{3}{4}\right)} \qquad ...(3)$$

Here $$A_1 = (Z-P_1)\frac{H(Z)}{Z}\Big|_{Z=P_1} = \frac{(Z-0)(Z-1)}{Z\left(Z+\frac{3}{4}\right)}\Big|_{Z=0}$$

$$\therefore \quad A_1 = \frac{(0-1)}{(0+3/4)} \quad \Rightarrow \quad A_1 = -\frac{4}{3}$$

and $$A_2 = (Z-P_2)\frac{H(Z)}{Z}\Big|_{Z=P_1} = \frac{\left(Z+\frac{3}{4}\right)(Z-1)}{Z(Z+3/4)}\Big|_{Z=-\frac{3}{4}}$$

$$\therefore \quad A_2 = \frac{Z-1}{Z}\Big|_{Z=-\frac{3}{4}} = \frac{-\frac{3}{4}-1}{-\frac{3}{4}} = \frac{-\frac{7}{4}}{-\frac{3}{4}}$$

$$\therefore \quad A_2 = \frac{7}{3}$$

Putting values of A_1 and A_2 in Equation (3) we get,

$$\frac{H(Z)}{Z} = \frac{-4/3}{Z} + \frac{7/3}{Z+3/4}$$

$$\therefore \quad H(Z) = \frac{-4/3}{Z} + \frac{7/3\,Z}{Z+3/4}$$

$$\therefore \quad H(Z) = -4/3 + \frac{7/3\,Z}{Z+3/4} \qquad ...(4)$$

The system is causal. So we can obtain $h(n)$ as follows :

$$h(n) = IZT\left\{\left(-\frac{4}{3}\right)\right\} + \frac{7}{3}IZT\left\{\frac{Z}{Z+3/4}\right\}$$

$$\therefore \quad h(n) = -\frac{4}{3}\delta(n) + \frac{7}{3}\left(-\frac{3}{4}\right)^n u(n) \qquad ...(5)$$

(ii) Calculation of output $y(n)$:

$$\text{Given} : \quad x(n) = \left(\frac{1}{3}\right)^n u(n) + u(-n-1) \qquad ...(6)$$

Taking Z-transform we get,

$$X(Z) = \frac{Z}{Z-\frac{1}{3}} - \frac{Z}{Z-1} \quad \ldots\left[\text{since } Z\{u(-n-1)\} = \frac{-Z}{Z-1}\right] \qquad \ldots(7)$$

And given $H(Z)$ is, $H(Z) = \dfrac{Z-1}{Z+\frac{3}{4}}$ ...(8)

Now $H(Z) = \dfrac{Y(Z)}{X(Z)}$

$\therefore \quad Y(Z) = H(Z) \times X(Z)$

$$\therefore \quad Y(Z) = \left(\frac{Z-1}{Z+\frac{3}{4}}\right)\left(\frac{Z}{Z-\frac{1}{3}} - \frac{Z}{Z-1}\right)$$

$$\therefore \quad Y(Z) = \frac{Z(Z-1)}{\left(Z+\frac{3}{4}\right)\left(Z-\frac{1}{3}\right)} - \frac{Z(Z-1)}{\left(Z+\frac{3}{4}\right)(Z-1)}$$

$$\therefore \quad Y(Z) = \frac{Z(Z-1)}{\left(Z+\frac{3}{4}\right)\left(Z-\frac{1}{3}\right)} - \frac{Z}{Z+\frac{3}{4}} \qquad \ldots(9)$$

Consider, $Y(Z) = \dfrac{Z(Z-1)}{\left(Z+\frac{3}{4}\right)(Z-\frac{1}{3})}$ and $Y_2(Z) = \dfrac{Z}{Z+\frac{3}{4}}$

$$\therefore \quad \frac{Y_1(Z)}{Z} = \frac{Z-1}{\left(Z+\frac{3}{4}\right)\left(Z-\frac{1}{3}\right)}$$

$$\therefore \quad \frac{Y_1(Z)}{Z} = \frac{A_1}{\left(Z+\frac{3}{4}\right)} + \frac{A_2}{\left(Z-\frac{1}{3}\right)} \qquad \ldots(10)$$

The poles are at $P_1 = -\frac{3}{4}$ and $P_2 = \frac{1}{3}$

Here $A_1 = (Z-P_1)\left.\dfrac{Y_1(Z)}{Z}\right|_{Z=P_1}$

$$\therefore \quad A_1 = \left(Z+\frac{3}{4}\right)\left[\frac{(Z-1)}{\left(Z+\frac{3}{4}\right)(Z-\frac{1}{3})}\right]\Bigg|_{Z=\frac{3}{4}}$$

$$\therefore \quad A_1 = \left[\frac{Z-1}{Z-\frac{1}{3}}\right]\Bigg|_{Z=-\frac{3}{4}} = \frac{-\frac{3}{4}-1}{-\frac{3}{4}-\frac{1}{3}} = \frac{-\frac{7}{4}}{\frac{-9-4}{12}}$$

$$\therefore \quad A_1 = \frac{-7}{4}\times\left(\frac{-12}{13}\right) = +\frac{21}{13}$$

$$\text{and} \quad A_2 = (Z-P_2)\frac{Y_1(Z)}{Z}\Bigg|_{Z=P_2} = \left(Z-\frac{1}{3}\right)\left[\frac{Z-1}{\left(Z+\frac{3}{4}\right)\left(Z-\frac{1}{3}\right)}\right]\Bigg|_{Z=\frac{1}{3}}$$

$$\therefore \quad A_2 = \left[\frac{Z-1}{Z+\frac{3}{4}}\right]\Bigg|_{Z=\frac{1}{3}} = \frac{\frac{1}{3}-1}{\frac{1}{3}+\frac{3}{4}} = \frac{-\frac{2}{3}}{\frac{4+9}{12}} = \frac{-2}{3}\times\frac{12}{13}$$

$$\therefore \quad A_2 = \frac{-8}{13}$$

Putting A_1 and A_2 in Equation (10) we get,

$$\frac{Y_1(Z)}{Z} = \frac{\frac{21}{13}}{(Z+\frac{3}{4})} - \frac{\frac{8}{13}}{(Z-\frac{1}{3})} \quad \therefore \quad Y_1(Z) = \frac{\left(\frac{21}{13}\right)Z}{Z+\frac{3}{4}} - \frac{\left(\frac{8}{13}\right)Z}{Z-\frac{1}{3}}$$

Given system is causal.

$$\therefore \quad y(n) = \frac{21}{13}\left(-\frac{3}{4}\right)^n u(n) - \frac{8}{13}\left(\frac{1}{3}\right)^n u(n) \qquad \ldots(11)$$

Now consider $Y_2(Z) = \dfrac{Z}{Z+\frac{3}{4}}$

$$\therefore \quad y_2(n) = IZT\left\{\frac{Z}{Z+\frac{3}{4}}\right\} = \left(-\frac{3}{4}\right)^n u(n) \qquad \ldots(12)$$

Thus total output $y(n) = y_1(n) + y_2(n)$

$$\therefore \quad y(n) = \frac{21}{13}\left(-\frac{3}{4}\right)^n u(n) - \frac{8}{13}\left(\frac{1}{3}\right)^n u(n) + \left(-\frac{3}{4}\right)^n u(n)$$

(iii) Deciding the stability of System :

The given transfer function is,

$$H(Z) = \frac{Z-1}{Z+\frac{3}{4}} \qquad ...(13)$$

It has one pole at $P_1 = -\frac{3}{4}$. It is clear that the pole lies inside the unit circle. when the pole is inside the unit circle then the system is stable. So this is a stable system.

Prob. 16 : Plot the pole zero pattern and determine which of the system is stable.

(i) $y(n) = y(n-1) - 0.5\,y(n-2) + x(n) + x(n-1)$

(ii) $y(n) = 0.7\,y(n-1) - 0.1\,y(n-2) + 2x(n) - x(n-2)$

Soln. :

(i) Given, $y(n) = y(n-1) - 0.5\,y(n-2) + x(n) + x(n-1)$ ..(1)

Taking IZT we get,

$$Y(Z) = Z^{-1}Y(Z) - 0.5\,Z^{-2}Y(Z) + X(Z) + Z^{-1}X(Z)$$

$$\therefore \quad Y(Z) - Z^{-1}Y(Z) + 0.5\,Z^{-2}Y(Z) = X(Z)[1 + Z^{-1}]$$

$$\therefore \quad Y(Z)[1 - Z^{-1} + 0.5\,Z^{-2}] = X(Z)[1 + Z^{-1}]$$

$$\therefore \quad \frac{Y(Z)}{X(Z)} = H(Z) = \frac{1 + Z^{-1}}{1 - Z^{-1} + 0.5\,Z^{-2}}$$

Multiplying numerator and denominator by Z^2 we get,

$$H(Z) = \frac{Z^2 + Z}{Z^2 - Z + 0.5} \qquad \therefore \quad H(Z) = \frac{Z(Z+1)}{Z^2 - Z + 0.5} \qquad ...(2)$$

We will obtain the roots of denominator,

$$P_1, P_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1-2}}{2} = \frac{1 \pm \sqrt{-1}}{2}$$

$$\therefore \quad P_1, P_2 = \frac{1 \pm j\sqrt{1}}{2} = \frac{1}{2} \pm j\frac{1}{2}$$

Thus poles are at, $\therefore \quad P_1 = 0.5 + j\,0.5$ and $P_2 = 0.5 - j\,0.5$

Thus Equation (2) becomes,

$$H(Z) = \frac{(Z-0)(Z+1)}{(Z-0.5-j\,0.5)(Z-0.5+j\,0.5)}$$

Here zeros are at $Z_1 = 0$ and $Z_2 = -1$

The pole zero pole is as shown in Fig. P-18.

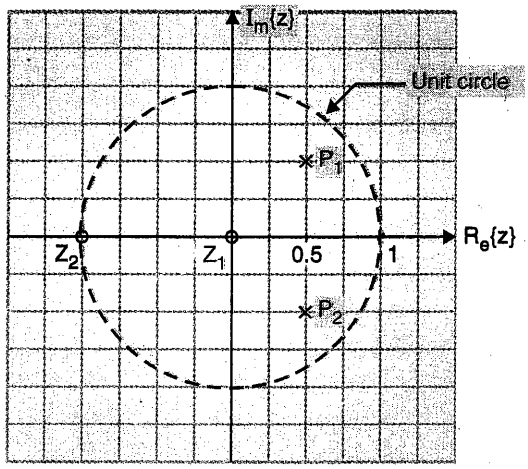


Fig. P-18 : Pole zero plot

Since both the poles are inside the unit circle; the system is stable.

(ii) Given, $y(n) = 0.7\,y(n-1) - 0.1\,y(n-2) + 2x(n) - x(n-2)$ ...(3)

Taking IZT we get, $Y(Z) = 0.7\,Z^{-1}\,Y(Z) - 0.1\,Z^{-2}\,Y(Z) + 2\,X(Z) - Z^{-2}\,X(Z)$

$\therefore \quad Y(Z) - 0.7\,Z^{-1}\,Y(Z) + 0.1\,Z^{-2}\,Y(Z) = 2X(Z) - Z^{-2}\,X(Z)$

$\therefore \quad Y(Z)[1 - 0.7\,Z^{-1} + 0.1\,Z^{-2}] = X(Z)[2 - Z^{-2}]$

$$\therefore \quad \frac{Y(Z)}{X(Z)} = H(Z) = \frac{2 - Z^{-2}}{1 - 0.7\,Z^{-1} + 0.1\,Z^{-2}}$$

Multiplying numerator and denominator by Z^2 we get,

$$H(Z) = \frac{2Z^2 - 1}{Z^2 - 0.7\,Z + 0.1} \qquad ...(4)$$

We will obtain roots of numerator and denominator,

$$Z - Z \Rightarrow Z_1, Z_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore \quad Z_1, Z_2 = \frac{0 \pm \sqrt{0 + 2}}{2} = \pm\frac{\sqrt{2}}{2} = \pm 0.707$$

$$\therefore \quad Z_1 = 0.707 \text{ and } Z_2 = -0.707$$

Now, $$P_1, P_2 = \frac{0.7 \pm \sqrt{0.49 - 0.4}}{2} = \frac{0.7 \pm 0.3}{2}$$

$$\therefore \quad P_1 = 0.5 \text{ and } P_2 = 0.2$$

Putting these values in Equation (4) we get,

$$H(Z) = \frac{(Z - 0.707)(Z + 0.707)}{(Z - 0.5)(Z - 0.2)}$$

The pole zero plot is shown in Fig. P-19.

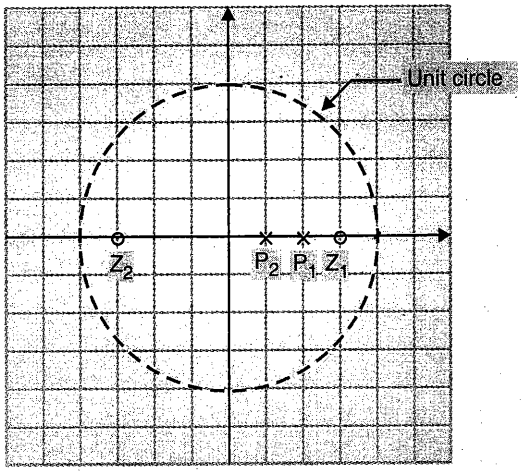


Fig. P-19 : Pole-zero plot

Since both the poles lie inside the unit circle; the system stable.

1.3.4 Inverse System :

A linear time invariant system generates output y(n) by taking convolution of input x(n) and its impulse response h(n). That means y(n) = x(n) * h(n). But in many applications it is required to generate the input signal from its output y(n). This system is called as inverse system.

Invertibility of LTI system :

A system is said to be invertible if input of a system can be recovered from the output of system. To achieve the same input consider two system connected in series; where second system is used to recover the input. The block schematic is shown in Fig. P-20.

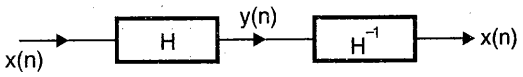


Fig. P-20

Here H represents discrete time system and produces y(n) when x(n) is the input. Then y(n) is applied to the second system. This is also discrete time system denoted by H^{-1}. Here H^{-1} denotes inverse operation.

$$\therefore \quad y(n) = H[x(n)]$$

$$\text{and} \quad H^{-1}[y(n)] = H^{-1}\{H[x(n)]\} = HH^{-1}[x(n)]$$

$$\text{But} \quad HH^{-1} = 1$$

$$\therefore \quad H^{-1}y(n) = x(n)$$

Here operator H^{-1} is the inverse operator and the associated system is called as inverse system.

Solved Problems :

Prob. 1 : Which of the following systems are invertible ? If invertible find the inverse

(i) $y(t) = x(t-n)$ (ii) $y(t) = x(2t)$ (iii) $y(n) = n\,x(n)$.

Soln. :

(i) Given equation is, $y(t) = x(t-n)$

It indicates that signal $x(t)$ is delayed by 'n' samples. So it is invertible system. We can design another system in which $x(t)$ can be made advanced by 'n' samples. Thus inverse system is,

$$Z(t) = y(t+n)$$

This is shown in Fig. P-21.

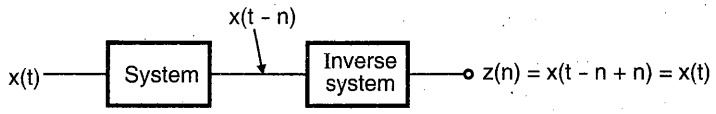


Fig. P-21

(ii) Given, $y(t) = x(2t)$

Input is $x(t)$; so this system indicates that output is obtained by compressing input by the factor 2. There can be another system which will expand input by 2. Thus it is invertible system and inverse system is,

$$Z(t) = y\left(\frac{t}{2}\right)$$

This is shown in Fig. P-22.

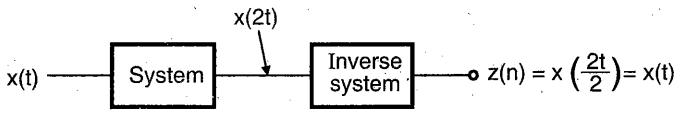


Fig. P-22

(iii) Given $y(n) = n\,x(n)$

It indicates that input $x(n)$ is multiplied by 'n'. We can design a system which will divide input by 'n'. Thus it is invertible system and inverse system is,

$$Z(n) = \frac{1}{n}y(n)$$

This is shown in Fig. P-23

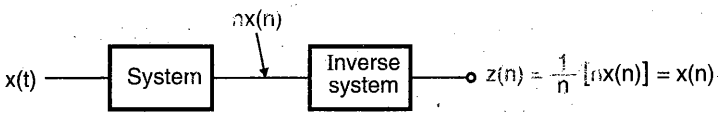


Fig. P-23

Prob. 2 : Consider an accumulator which computes the running sum of all inputs upto present time and expressed as,

$$y(n) = \sum_{m=0}^{n} x(m)$$

It is an invertible system ? If yes, find inverse of this system. If no give reasons.

Soln. : The given equation is,

$$y(n) = \sum_{m=0}^{n} x(m) \quad ...(1)$$

Expanding the summation we get,

$$y(n) = x(0)+x(1)+x(2)+...+x(n) \quad ...(2)$$

Replacing 'n' by $n-1$ in Equation (1) we get,

$$y(n-1) = \sum_{m=0}^{n-1} x(m) \quad ...(3)$$

Expanding the summation we get,

$$y(n-1) = x(0)+x(1)+x(2)+...+x(n-1) \quad ...(4)$$

Invertible system is designed to generate the original input.

If we subtract Equation (4) from Equation (3) then we can generate original input.

$$\therefore \quad x(n) = y(n)-y(n-1) \quad ...(5)$$

Thus it is invertible system. Equation (5) indicates equation of inverse system.

1.3.5 Frequency Response from Pole-Zero Plot using Simple Geometric Construction :

To sketch the frequency response functions of a system from pole-zero plot, geometric construction is very simplest method.

Principle :

Frequency response corresponds to the system function evaluated on the unit circle in the z-plane and that the complex value of each pole and zero factor can be represented by a vector in the z-plane from the pole or zero to a point on the unit circle.

Procedure :

1. Find poles and zeros of a given system function and draw pole-zero plot with a unit circle on z-plane.
2. Draw vectors from all zeros and poles to the same point on unit circle at,

   $$Z = re^{j\omega} = e^{j\omega} \ (r = 1).$$

   Vectors from zeros to the unit circle are called zero vectors and vectors from poles to the unit circle are called pole vectors.
3. Find the length of zero vectors and pole vectors. Magnitude response is the product of the lengths of the zero vectors divided by the product of the lengths of pole vectors. With fixed r, sketch magnitude response for different values of θ and ω.

4. Find, angle of each zero and pole vector with respect to positive real axis in anticlockwise direction.

Phase function is the difference between the sum of angles of zero vectors and sum of angles of the pole vectors.

With fixed r, sketch phase response for different values of θ and ω.

Examples on Geometric Construct :

Prob. 1 : The difference equation of the system is given as,

$$y(n) = 2x(n) + x(n-1)$$

Find out magnitude and phase response using geometric construction.

Soln. : Given difference equation of the system is,

$$y(n) = 2x(n) + x(n-1)$$

Taking Z-transform on both sides,

$$Y(Z) = 2X(Z) + Z^{-1}X(Z)$$

$$\therefore \quad Y(Z) = X(Z)[2 + Z^{-1}]$$

$$\therefore \quad \frac{Y(Z)}{X(Z)} = \left[2 + \frac{1}{Z}\right] = 2\left[1 + \frac{1}{2Z}\right]$$

$$\therefore \quad H(Z) = \frac{Y(Z)}{X(Z)} = 2\left[\frac{2Z+1}{2Z}\right] \qquad ...(1)$$

where, $H(Z)$ is transfer function.

Step I : From Equation (1) :

Zero is at $Z = -1/2 \Rightarrow Z_1$

and pole is at $Z = 0 \Rightarrow P_1$

The pole-zero plot is as shown in Fig. P-24.

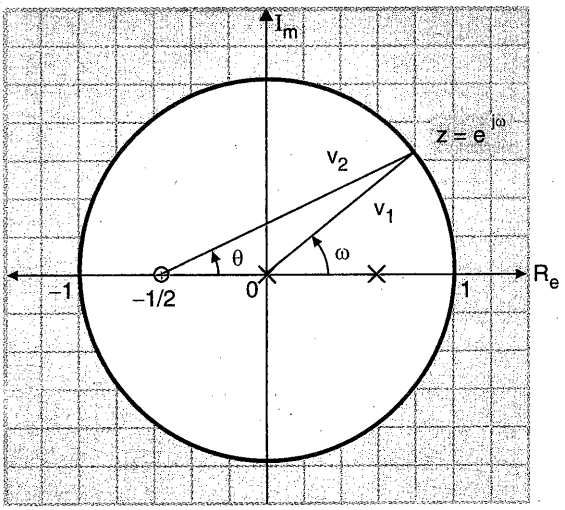


Fig. P-24

Step II : The pole vector and zero vector are shown in Fig. P-24.

As H (Z) is a function of 'ω', let us consider various values of 'ω' as follows :

Case I : When ω = 0, the pole vector and zero vector are as shown in Fig. P-25.

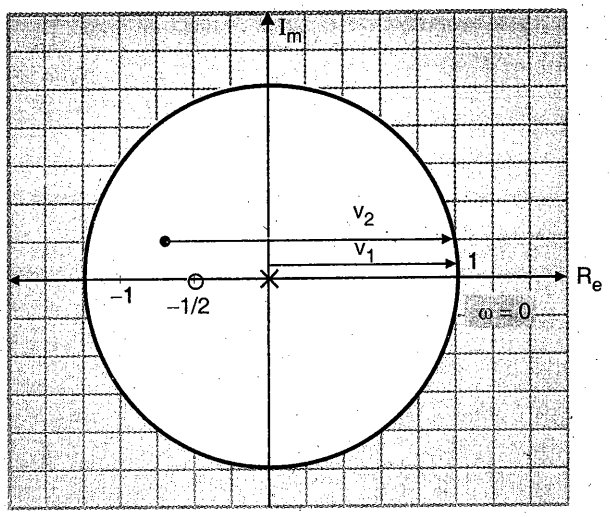


Fig. P-25

$\therefore \quad |V_1| = 1 \quad$ and $\quad |V_2| = \left|1+\frac{1}{2}\right| = \left|\frac{3}{2}\right| = \frac{3}{2}$

and $\quad \angle V_1 = 0 \quad$ and $\quad \angle V_2 = 0$

Case II : When ω = π / 2, the pole vector and zero vector are as shown in Fig. P-26.

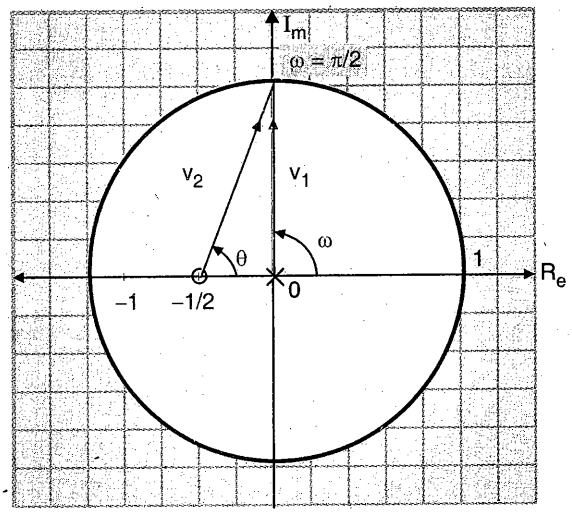


Fig. P-26

$$\therefore \quad |V_1| = 1,$$

$$|V_2| = \sqrt{1^2 + \left(\frac{1}{2}\right)^2} = \sqrt{1 + \frac{1}{4}}$$

$$\therefore |V_2| = \sqrt{5}/2$$

$$\angle V_1 = \omega = \pi/2,$$

$$\angle V_2 = \theta$$

$$= \tan^{-1}\left(\frac{1}{1/2}\right) = 63.43^\circ$$

Case III : When $\omega = \pi$, pole vector and zero vector are as shown in Fig. P-27.

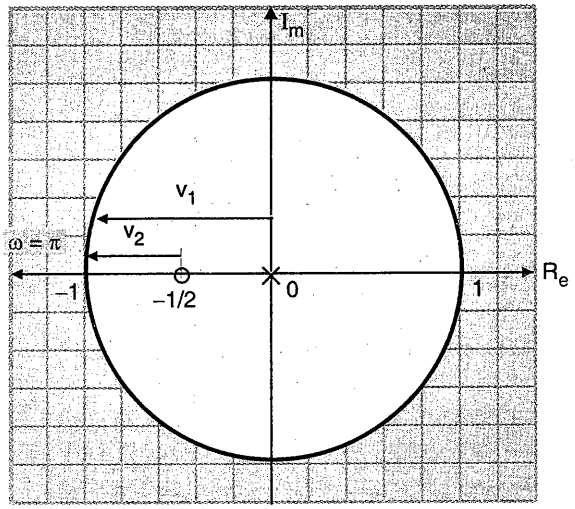


Fig. P-27

$$\therefore \quad |V_1| = 1,$$

$$|V_2| = \frac{1}{2}$$

$$\angle V_1 = \pi$$

and $$\angle V_2 = \pi$$

Case IV : When $\omega = 3\pi/2$, pole-zero vectors are as shown in Fig. P-28.

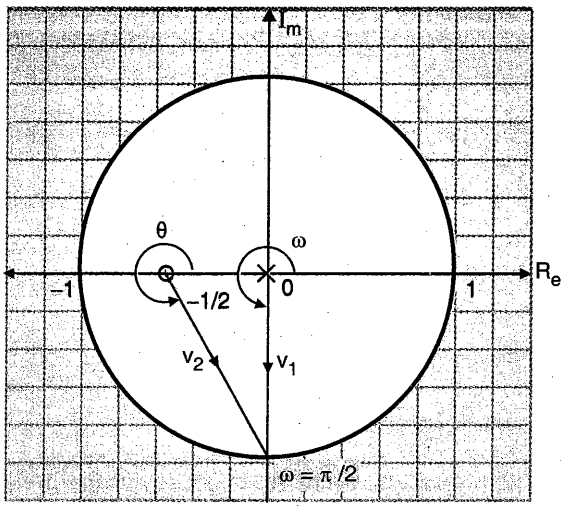


Fig. P-28

$$\therefore \quad |V_1| = 1, \quad |V_2| = \left|\sqrt{(1/2)^2+1^2}\right| = \left|\sqrt{\frac{5}{4}}\right| = \frac{\sqrt{5}}{2}$$

$$\text{and} \quad \angle V_1 = \omega = \frac{3\pi}{2} \quad \text{or} \quad \frac{-\pi}{2}$$

$$\angle V_2 = \theta = -\tan^{-1}\left(\frac{1}{1/2}\right) = -63.43^\circ$$

Step III : Magnitude response and phase response is given as,

$$\text{At } \omega = 0, \quad |H(e^{j\omega})| = 2\frac{|V_2|}{|V_1|} = 2 \cdot \frac{3/2}{1} = 3$$

$$\angle H(e^{j\omega}) = \angle V_2 - \angle V_1 = \theta - \omega = 0 - 0 = 0$$

$$\text{At } \omega = \frac{\pi}{2}, \quad |H(e^{j\omega})| = 2\frac{|V_2|}{|V_1|} = 2 \cdot \frac{\sqrt{5}/2}{1} = \sqrt{5}$$

$$\angle H(e^{j\omega}) = \angle V_2 - \angle V_1 = \theta - \omega = 63.43 - \frac{\pi}{2} = -26.57^\circ$$

$$\text{At } \omega = \pi, \quad |H(e^{j\omega})| = 2\frac{|V_2|}{|V_1|} = 2 \cdot \frac{1/2}{1} = 1$$

$$\angle H(e^{j\omega}) = \angle V_2 - \angle V_1 = \theta - \omega = \pi - \pi = 0$$

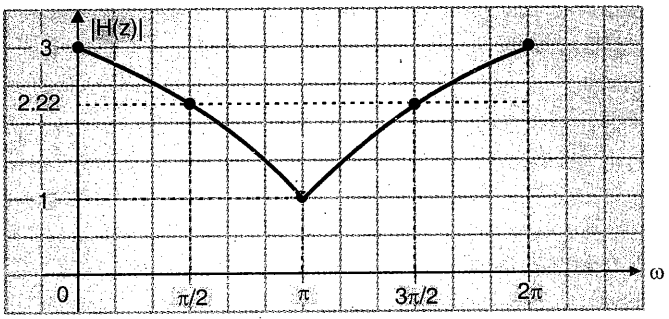
$$\text{At } \omega = 3\pi/2,$$

$$|H(e^{j\omega})| = 2\frac{|V_2|}{|V_1|} = 2 \cdot \frac{\sqrt{5}/2}{1} = \sqrt{5}$$

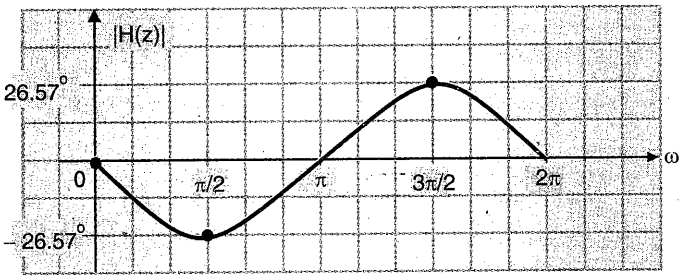
$$\therefore \angle H(e^{j\omega}) = \angle V_2 - \angle V_1 = \theta - \omega = -63.43^\circ - \left(\frac{-\pi}{2}\right) = 26.57^\circ$$

Step IV : Magnitude and phase response are as shown in Fig. P-29(a) and P-29(b) response.

ω	$\lvert H(Z)\rvert$	$\angle H(Z)$
0	3	0
$\pi/2$	$\sqrt{5}$	-26.57°
π	1	0
$3\pi/2$	$\sqrt{5}$	26.57°



(a) Magnitude Response



(b) Phase response

Fig. P-29

Prob. 2 : The transfer function of a system is given as,

$$H(Z) = \frac{Z+1}{Z(Z-1)}$$

Find and sketch magnitude, phase response.

Soln. : The given transfer function is,

$$H(Z) = \frac{Z+1}{Z(Z-1)}$$

Step I :

Zero is at $Z_1 = -1$

Poles are at $P_1 = 0$ and $P_2 = 1$

The pole-zero plot is as shown in Fig. P-30.

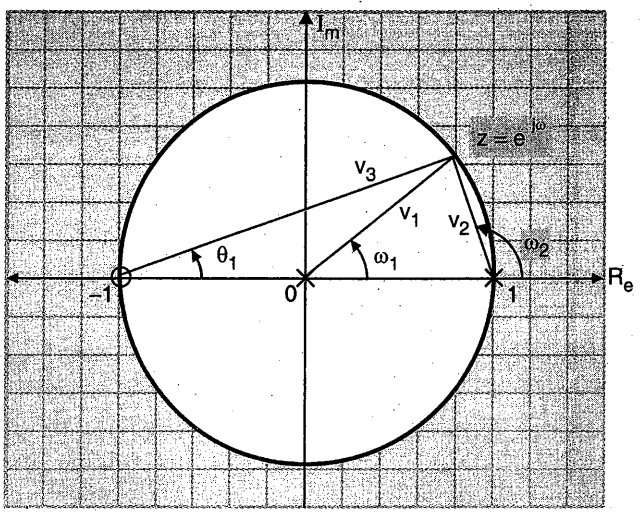


Fig. P-30

Case I : When $\omega = 0$, pole and zero vectors are as shown in Fig. P-31.

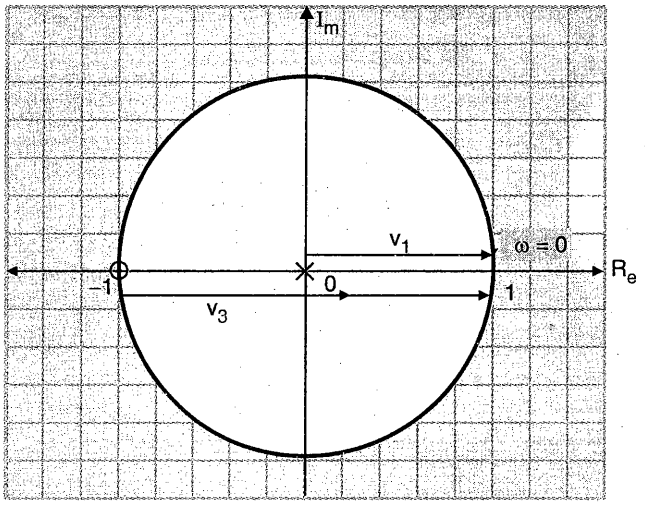


Fig. P-31

$\therefore \quad |V_1| = 1, \quad |V_2| = 0 \quad \text{and} \quad |V_3| = 1 + 1 = 2$

$\angle V_1 = \omega_1 = 0, \quad \angle V_2 = \omega_2 = \pi \quad \text{and} \quad \angle V_3 = \theta_1 = 0$

Case II : When $\omega = \pi / 2$,

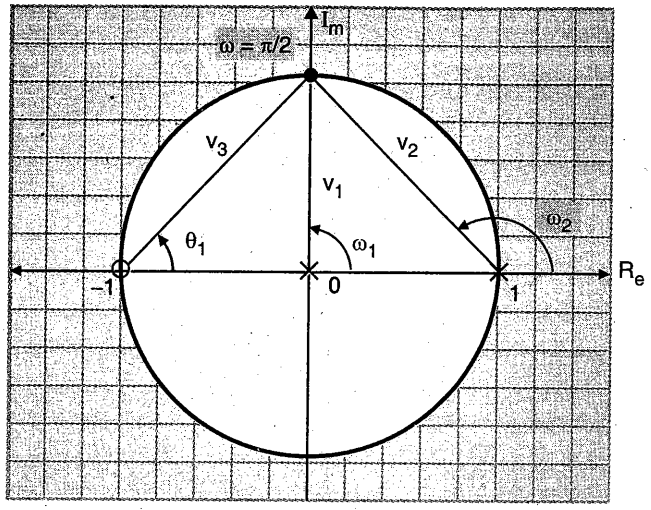


Fig. P-32

$|V_1| = 1, |V_2| = \sqrt{1^2+1^2} = \sqrt{2}, \quad |V_3| = \sqrt{1^2+1^2} = \sqrt{2}$

$\angle V_1 = \omega_1 = \pi/2, \angle V_2 = \omega_2 = \pi - \frac{\pi}{4} = \frac{3\pi}{4}, \text{ or } \frac{-\pi}{4} \qquad \angle V_3 = \theta_1 = \pi/4$

Case III : When $\omega = \pi$,

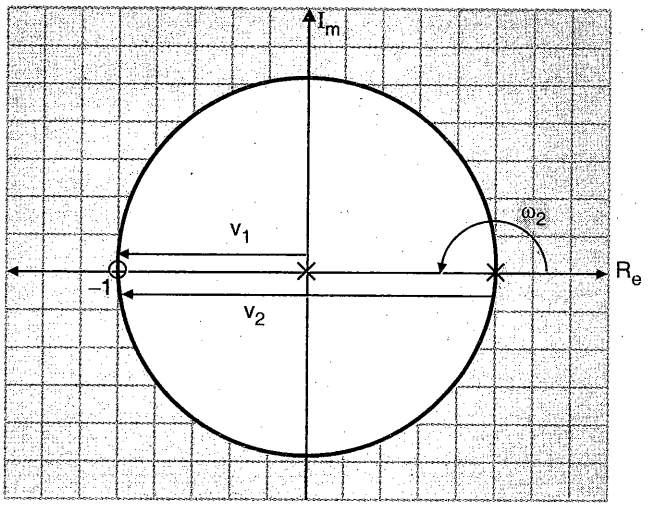


Fig. P-33

$|V_1| = 1, \quad |V_2| = 1 + 1 = 2, \quad |V_3| = 0$

$\angle V_1 = \omega_1 = \pi, \quad \angle V_2 = \omega_2 = \pi,$

$\angle V_3 = \theta_1 = \pi/2$ ...since at $\omega = \pi$,

Complete overlap occurs and the line becomes tangent. Hence θ_3 is $\pi/2$.

Case IV : When $\omega = 3\pi/2$,

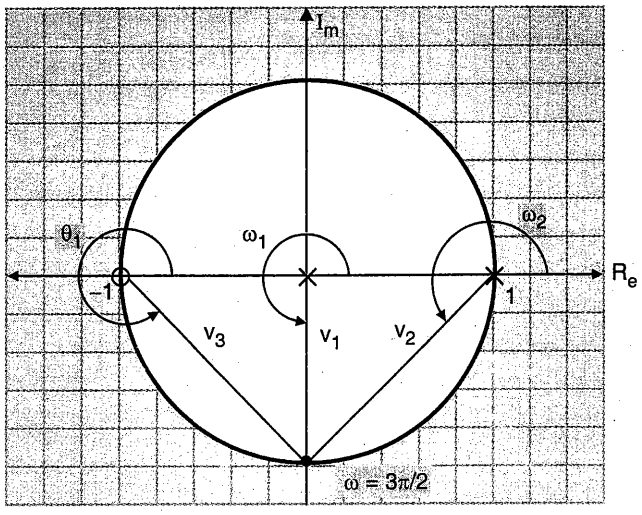


Fig. P-34

$$\therefore \quad |V_1| = 1, \; |V_2| = \sqrt{1^2+1^2} = \sqrt{2}$$

$$|V_3| = \sqrt{1^2+1^2} = \sqrt{2}$$

$$\angle V_1 = \omega_1 = 3\pi/2 \quad \text{or} \quad -\pi/2,$$

$$\angle V_2 = \omega_2 = \pi + \frac{\pi}{4} = \frac{5\pi}{4} \quad \text{or} \quad -\left(\frac{\pi}{2}+\frac{\pi}{4}\right) = \frac{-3\pi}{4}$$

$$\angle V_3 = \theta_1 = -\pi/4$$

Step III :

Magnitude and phase response are given as, at $\omega = 0$,

$$|H(Z)| = \frac{|V_3|}{|V_1| \cdot |V_2|} = \frac{2}{1 \times 0} = \infty$$

$$\angle H(Z) = \angle V_3 - (\angle V_1 + \angle V_2) = \theta_1 - (\omega_1 + \omega_2)$$

$$= 0 - (0+0) = 0$$

At $\omega = \pi/2$, $|H(Z)| = \frac{|V_3|}{|V_1| \cdot |V_2|} = \frac{\sqrt{2}}{1 \cdot \sqrt{2}} = 1$

$$\angle H(Z) = \angle V_3 - (\angle V_1 + \angle V_2) = \theta_1 - (\omega_1 + \omega_2)$$

$$= \frac{\pi}{4} - \left(\frac{\pi}{2} - \frac{\pi}{4}\right) = 0$$

$$\text{at } \omega = \pi, \quad |H(Z)| = \frac{|V_3|}{|V_1| \cdot |V_2|} = \frac{0}{1.2} = 0$$

$$\angle H(Z) = \angle V_3 - (\angle V_1 + \angle V_2) = \theta_1 - (\omega_1 + \omega_2)$$

$$= \frac{\pi}{2} - (\pi + \pi) = \frac{-3\pi}{2}$$

$$\text{at } \omega = 3\pi/2, \quad |H(Z)| = \frac{|V_3|}{|V_1| \cdot |V_2|} = \frac{\sqrt{2}}{1 \cdot \sqrt{2}} = 1$$

$$\angle H(Z) = \angle V_3 - (\angle V_1 + \angle V_2) = \theta_1 - (\omega_1 + \omega_2)$$

$$= -\frac{\pi}{4} - \left(\frac{-\pi}{2} - \frac{3\pi}{4}\right) = \frac{-3\pi}{2}$$

ω	$\lvert H(Z) \rvert$	$\angle H(Z)$
0	∞	0
$\pi/2$	1	0
π	0	$-3\pi/2$
$3\pi/2$	1	$-3\pi/2$
2π	∞	0

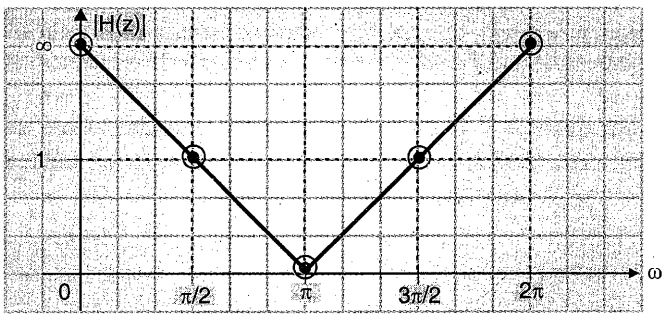


Fig. P-35(a) : Magnitude response

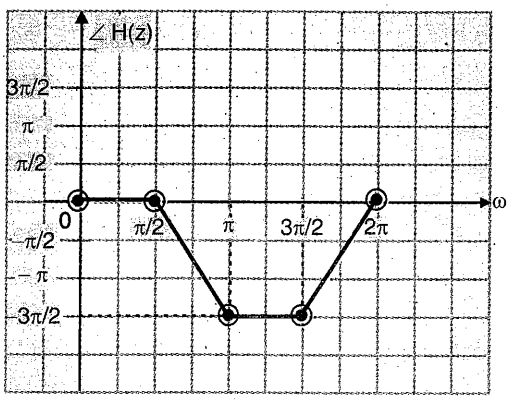


Fig. P-35(b) : Phase response

1.4 Transient and Steady State Response :

In this section we will study how to determine the transient and steady state response of LTI system using z transform.

1.4.1 Introduction to Types of Response :

The output of discrete time system can be calculated using equation of linear convolution.

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad \ldots(1)$$

Here h (n) is the impulse response of system. By the use of h (n); we can calculate output y (n) for any input x (n) using Equation (1).

Now in this section we will discuss the difference equation which are useful to implement (realize) the discrete time system. These difference equations are especially used for designing the digital filter.

We discussed that impulse response h (n) is used to calculate output of discrete time system. Now if the system is causal then impulse response h (n) is zero for negative values of n. So for causal LTI system equation of convolution becomes,

$$y(n) = \sum_{k=0}^{\infty} x(k)h(n-k) \quad \ldots(2)$$

Equation (2) can also be written as,

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) \quad \ldots(3)$$

Since output depends on impulse response h (n); the number of samples, present in h (n) plays an important role to determine output y (n). The number of samples present in h (n) is called as length of impulse response. Depending upon the length of impulse response; there are two types of discrete time systems :

1. Finite impulse response (FIR) systems.
2. Infinite impulse response (IIR) systems.

1. Finite impulse response systems (FIR) :

As the name indicates; impulse response h (n) is having a finite length. That means h (n) contains some finite number of samples. Let the range of h (n) is from $-M$ to $+M$. Thus equation of convolution for FIR system becomes,

$$y(n) = \sum_{k=-M}^{M} h(k)\,x(n-k) \quad \text{(Non-causal)} \qquad ...(4)$$

Note that in Equation (4), negative terms are present, so this equation is for the non-causal system. Now for the causal FIR system we will consider only positive samples of h (n). Thus range of h (n) becomes '0' to M instead of $-M$ to M. Equation (4) becomes,

$$y(n) = \sum_{k=0}^{M} h(k)\,x(n-k) \qquad ...(5)$$

Note that the summation is taken from 0 to M. So there will be total $M+1$ terms in h (n) or h (k). These terms are :

h (0), h (1), h (2) ....h (M) .... Number of terms in h (n).

Now expanding Equation (5) we get,

$$y(n) = h(0)\,x(n) + h(1)\,x(n-1) + h(2)\,x(n-2) + ... + h(M)\,x(n-M) \qquad ...(6)$$

In Equation (6) x (n) is present input and x ($n-1$), x ($n-2$) ... etc. are past inputs. Every input is multiplied by corresponding value of h (n). Thus Equation (6) indicates that output y (n) is weighted sum of inputs.

Note that in this case the system has to view input samples only from x (n) to x ($n-M$) and not all input samples. These input samples are most recent input samples. Thus this system has to store x ($n-1$), x ($n-2$) ... x ($n-M$), past samples in the memory. Since these are finite number of samples; FIR has limited or finite memory requirement.

2. Infinite impulse response (IIR) systems :

As the name indicates, impulse response h (n) contain '∞' samples. That means length of h (n) is infinite. Now the equation of convolution for non-causal IIR system becomes.

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k) \quad ...\text{(Non-causal)} \qquad ...(7)$$

For the causal system, limits of summation will be from 0 to ∞.

$$\therefore \quad y(n) = \sum_{k=0}^{\infty} h(k)\,x(n-k) \quad ... \text{ (Causal)} \qquad ...(8)$$

Here the range of $h(n) = h(k)$ is as follows :

$h(0), h(1), h(2) ...., h(\infty)$. So this is infinite impulse response system.

Now expanding Equation (8) we get,

$$y(n) = h(0)x(n)+h(1)x(n-1)+h(2)x(n-2)+....+h(\infty)x(n-\infty) \quad ...(9)$$

In this case IIR system has to store all past samples from $x(n-1)$ to $x(n-\infty)$. Thus IIR system needs infinite memory. This is practically not possible.

Note : We discussed that FIR system needs limited memory. So it is possible to design FIR systems using convolution equation. But IIR systems cannot be designed using convolution equation. This is because of the infinite memory requirement. So to implement such IIR systems; difference equations are used.

Now depending upon length of $h(n)$ we have classified the systems as FIR and IIR. Similarly depending upon output of system; the discrete time systems are classified as follows :

1. Non-recursive systems 2. Recursive systems.

Why this type of classification is required ?

Equation of convolution indicates that for causal systems; the output depends on present and past input as well as it depends on the impulse response, $h(n)$. But this is not the case always. There are some systems whose present output depends on present and past inputs as well as past (previous) output. For such systems we should take into account, the previous output to calculate present output. This type of classification gives the idea about the requirement of previous output to calculate present output.

1. Non-recursive systems :

Non-recursive systems do not require any past output sample to calculate the present output. Consider the equation of FIR causal system (Equation (5)). This equation is,

$$y(n) = \sum_{k=0}^{M} h(k)\,x(n-k)$$

To calculate present output we need present and past inputs only as shown by Equation (6). We will rewrite Equation (6) as follows :

$$y(n) = h(0)x(n)+h(1)x(n-1)+....+h(m)x(n-M)$$

This equation does not contain any previous output term like $y(n-1), y(n-2)$ ... etc. So this is the equation of non-recursive system. Thus we can conclude that causal FIR system is non-recursive system. In other words we can say that non-recursive implementation of FIR causal system is possible.

Now for non-causal FIR system; the limits of summation will be from $-M$ to M. Thus this system contain present, past and future inputs. But it does not contain future term. So this is also non-recursive system.

In case of non-recursive systems; past output terms are not required. Past (previous) output terms are obtained by taking feedback from output to the input. Thus non-recursive system do not require any feedback.

Is it possible to implement non-recursive IIR system ?

We know that,

Non-recursive $\Leftrightarrow$ No feedback or no past output term present

and IIR $\Leftrightarrow$ Infinite impulse response.

The equation of convolution for causal IIR system is,

$$y(n) = \sum_{k=0}^{\infty} h(k)\,x(n-k)$$

Expanding the summation we get,

$$y(n) = h(0)\,x(n)+h(1)\,x(n-1)+h(2)\,x(n-2)\ldots.+h(\infty)\,x(n-\infty)$$

We have discussed earlier that such systems require to store all past inputs. That means it is required to store infinite input terms in the memory. So infinite memory is required. This is not possible practically. Thus non-recursive implementation of causal IIR system is not possible.

The representation of non-recursive system is shown in Fig. P-36(a).

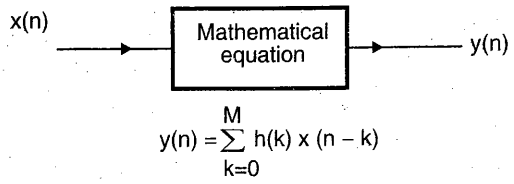


Fig. P-36(a) : Non-recursive system

2. Recursive systems : A discrete time system in which output y (n) depends on present input, past inputs as well as previous outputs; is called as recursive system.

Let us consider one example. Consider a system represented by block schematic as shown in Fig. P-36(b).

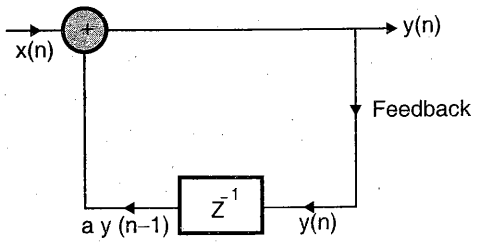


Fig. P-36(b) : Example of recursive system

First we will write the equation for system.

(i) As shown in Fig. P-36(b); a connection is drawn from output y (n) to the input side. This is a feedback.

(ii) This feedback signal, y (n) is passed through an unit delay (z^{-1}) block. So the output of delay block is y (n – 1). This is past output compared to y (n).

(iii) The signal y (n – 1) is multiplied by constant multiplier 'a' to obtain ay (n – 1).

(iv) An adder is used to add input x (n) and signal ay (n – 1).

(v) Now the addition of x (n) and ay (n – 1) produces the output.

Thus the output of system is expressed as,

$$y(n) = x(n) + ay(n-1) \quad ...(10)$$

At instant n = 0 we get,

$$y(0) = x(0) + ay(-1)$$

Here $y(0) \Rightarrow$ present output

$x(0) \Rightarrow$ present input

and $y(-1) \Rightarrow$ past output

Similarly at n = 1 we get,

$$y(1) = x(1) + ay(0)$$

Thus $y(0)$ is past output.

So at any instant present output depends on past output. Hence it is recursive system. The representation of non-recursive system is shown in Fig. P-36(c).

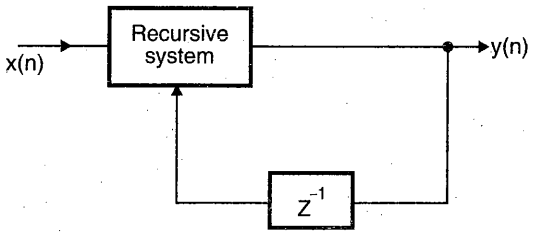


Fig. P-36(c) : Representation of recursive system

1.4.2 LTI Systems Characterized by Constant Coefficient Difference Equations :

In the previous sub section, we discussed that, the behaviour of system depends on its impulse response. In this subsection we will discuss how LTI systems are characterized by input-output relations; called as difference equation with constant coefficients.

Consider a recursive system shown in Fig. P-36(b). The equation of output is,

$$y(n) = x(n) + ay(n-1) \quad ...(11)$$

This is first order difference equation. Here 'a' is a constant coefficient. Now output $y(n)$ is calculated by putting different values of n in Equation (11).

For $n = 0 \Rightarrow y(0) = x(0) + ay(-1)$ ...(i)

For $n = 1 \Rightarrow y(1) = x(1) + ay(0)$ ...(ii)

Putting value of $y(0)$ from Equation (i) we get,

$$y(1) = x(1) + a[x(0) + ay(-1)]$$

$$\therefore \quad y(1) = x(1) + ax(0) + a^2 y(-1)$$

Rearranging the terms, $y(1) = a^2 y(-1) + ax(0) + x(1)$ ...(iii)

For $n = 2 \Rightarrow y(2) = x(2) + ay(1)$ ...(iv)

Putting value of $y(1)$ from Equation (iii) we get,

$$y(2) = x(2) + a[a^2 y(-1) + ax(0) + x(1)]$$

$$\therefore \quad y(2) = x(2) + a^3 y(-1) + a^2 x(0) + ax(1)$$

Rearranging the terms,

$$y(2) = a^3 y(-1) + a^2 x(0) + ax(1) + x(2) \quad ...(v)$$

Observe Equations (iii) and (v) carefully. Now we can directly write the equation for value 'n'.

$$\therefore \quad y(n) = a^{n+1} y(-1) + a^n x(0) + a^{n-1} x(1) + ... + ax(n-1) + x(n) \quad ...(12)$$

In terms of summation Equation (12) can be written as,

$$y(n) = a^{n+1} y(-1) + \sum_{k=0}^{n} a^k x(n-k) \quad ...(13)$$

Note that we have obtained output $y(n)$ by putting only positive values of n. That means for $n \geq 0$. So these are the equations of causal system. Every equation contains the term $y(-1)$. It is assumed that, we are starting the operation of system at $n = 0$. Then the term $y(-1)$ is called as initial condition of a system.

Important terms related to Equation (13) :

We know that Equation (13) is representing the behaviour of causal system. It consists of two parts as follows :

(i) The first term, $y(-1)$ is called initial condition of the system.

(ii) The second term, is the response of system to the input $x(n)$.

Now depending upon initial conditions, two types of responses are obtained.

(1) Zero state response or forced response or steady state response :

If initial condition, $y(-1)$ of the system is zero then the response of the system to the input signal is called as zero state response. It is denoted by $y_{zs}(n)$. In this case, whatever output we are getting, that is because of input $x(n)$ only. And since the system is forced to the input; it is also called as forced response. Since $y(-1) = 0$; the system is said to be relaxed system.

Thus putting initial condition, $y(-1) = 0$ in Equation (13) we get,

$$\boxed{y(n) = y_{zs}(n) = \sum_{k=0}^{n} a^k x(n-k)} \quad \text{... for } n \geq 0 \quad ...(14)$$

(2) Zero input response or natural response or transient response :

As the name indicates; input is not applied to the system and initial condition $y(-n) \neq 1$. Thus, this kind of output without applying input $x(n)$ is called as zero input response or natural response. Since $y(-n) \neq 1$, this is called as non-relaxed initial condition. This output is denoted by y_{zi}.

Putting $x(n) = 0$ in Equation (13) we get,

$$\boxed{y(n) = y_{zi}(n) = a^{n+1} y(-1)} \quad ...(15)$$

This equation indicates that output is not dependent on input $x(n)$ but it depends on initial condition, $y(-1)$

(3) Total response :

The total response of system is obtained by adding zero state response and zero input response.

$$\text{Total response, } y(n) = y_{zi}(n) + y_{zs}(n) \quad \ldots(16)$$

Equation (11) is basically a linear constant coefficient difference equation. The general form of this equation can be expressed as,

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \quad \ldots(17)$$

Here a_k and b_k are constant coefficients. And 'N' is the order of difference equation. That means 'N' represents the order of system.

1.4.3 Calculation of Response of LTI System using Z Transform :

To calculate the transient and steady state response, unilatered Z tranform is used.

Definition : Unilateral or one sided Z-transform of DT signal $x(n)$ is defined by,

$$X(Z) = \sum_{n=0}^{\infty} x(n) Z^{-n} \quad \ldots(1)$$

For unilateral Z-transform, some-times the notation Z^+ is also used. Basically one sided Z-transform is used for causal system.

Characteristics :

1. ROC : for unilateral Z-transform $[X^+(Z)]$; is always exterior part of the circle. Thus for unilateral Z-transform ; it is not necessary to refer ROC.
2. Z-transform is unique for causal signal.
3. It does not contain information for $n < 0$.

The shifting property of unilateral Z transform is used to calculate response of LTI system.

Shifting property of unilateral Z- transform :

The properties of unilateral Z-transform are same as two sided Z- transform properties except shifting property. Under shifting we will consider two cases as follows :

Case - I : Time delay :

Statement : If $x(n) \overset{Z}{\longleftrightarrow} X(Z)$

$$\text{then } x(n-k) \overset{Z}{\longleftrightarrow} Z^{-k}\left[X(Z) + \sum_{n=1}^{k} x(-n) Z^{n}\right] \quad \text{for } k > 0$$

Proof : According to the definition of one-sided Z-transform we have,

$$Z\{x(n)\} = X(Z) = \sum_{n=0}^{\infty} x(n)Z^{-n} \quad ...(1)$$

$$\therefore \quad Z\{x(n-k)\} = \sum_{n=0}^{\infty} x(n-k)Z^{-n} \quad ..(2)$$

Put $n-k = l$ $\therefore$ $n = k+l$ then limits will change as follows :

When $n = 0$ $\Rightarrow 0-k = l$ $\therefore$ $l = -k$

When $n = \infty$ $\Rightarrow \infty - k = l$ $\therefore$ $l = \infty$

Thus Equation (2) becomes,

$$Z\{x(n-k)\} = \sum_{l=-k}^{\infty} x(l)Z^{-(k+l)} = \sum_{l=-k}^{\infty} x(l)Z^{-k}\cdot Z^{-l}$$

$$\therefore \quad Z\{x(n-k)\} = Z^{-k}\sum_{l=-k}^{\infty} x(l)Z^{-l} \quad ...(3)$$

The summation in Equation (3) can be splitted into two parts as follows :

$$Z\{x(n-k)\} = Z^{-k}\left[\sum_{l=0}^{\infty} x(l)Z^{-l} + \sum_{l=-k}^{-1} x(l)Z^{-l}\right] \quad ...(4)$$

The first summation term at R.H.S. indicates $X(Z)$.

$$\therefore \quad Z\{x(n-k)\} = Z^{-k}X(Z) + Z^{-k}\sum_{l=-k}^{-1} x(l)Z^{-l} \quad ...(5)$$

To make the limits of summation positive put $l = -m$ $\therefore$ $m = -l$.

$$\therefore \quad Z\{x(n-k)\} = Z^{-k}X(Z) + Z^{-k}\sum_{m=k}^{1} x(-m)Z^{m}$$

$$\therefore \quad Z\{x(n-k)\} = Z^{-k}\left[X(Z) + \sum_{m=k}^{1} x(-m)Z^{m}\right] \quad ...(6)$$

In Equation (6), m is a variable. Instead of m we can put another variable n.

$$\therefore Z\{x(n-k)\} = Z^{-k}\left[X(Z) + \sum_{n=k}^{1} x(-n)Z^{n}\right] \quad ...(7)$$

Case - II : Time advance :

Statement : If $x(n) \overset{Z}{\longleftrightarrow} X(Z)$

$$\text{then} \quad x(n+k) \overset{Z}{\longleftrightarrow} Z^k \left[X(Z) - \sum_{n=0}^{k-1} x(n) Z^{-n} \right], \; k > 0 \qquad \ldots(8)$$

Proof : According to the definition of one-sided Z- transform,

$$Z\{x(n)\} = X(Z) = \sum_{n=0}^{\infty} x(n) Z^{-n}$$

$$\therefore \quad Z\{x(n+k)\} = \sum_{n=0}^{\infty} x(n+k) Z^{-n} \qquad \ldots(9)$$

Put $n + k = l$ $\therefore$ $n = l - k$. The limits of summation will change as follows :

When $n = 0 \Rightarrow 0 + k = l$ $\quad \therefore \quad l = k$

When $n = \infty \Rightarrow \infty + k = l$ $\quad \therefore \quad l = \infty$

Thus Equation (9) becomes,

$$Z\{x(n+k)\} = \sum_{l=k}^{\infty} x(l) Z^{-(l-k)} = \sum_{l=k}^{\infty} x(l) Z^{-l} \cdot Z^k$$

$$\therefore \quad Z\{x(n+k)\} = Z^k \left[\sum_{l=k}^{\infty} x(l) Z^{-l} \right] \qquad \ldots(10)$$

In Equation (10), the limits of summation are from $l = k$ to $l = \infty$.

These limits can be written as follows,

$$\sum_{l=k}^{\infty} = \sum_{l=0}^{\infty} - \sum_{l=0}^{k-1}$$

Thus Equation (10) becomes,

$$Z\{x(n+k)\} = Z^k \left[\sum_{l=0}^{\infty} x(l) Z^{-l} - \sum_{l=0}^{k-1} x(l) Z^{-l} \right]$$

$$\therefore \quad Z\{x(n+k)\} = Z^k \sum_{l=0}^{\infty} x(l) Z^{-l} - Z^k \sum_{l=0}^{k-1} x(l) Z^{-l} \qquad \ldots(11)$$

The first summation in Equation (11) indicates $X(Z)$

$$\therefore \quad Z\{x(n+k)\} = Z^k X(Z) - Z^k \sum_{l=0}^{k-1} x(l) Z^{-l}$$

$$\therefore \quad Z\{x(n+k)\} = Z^k \left[X(Z) - \sum_{l=0}^{k-1} x(l) Z^{-l} \right] \qquad ...(12)$$

Replacing the variable 'l' by 'n' we get,

$$Z\{x(n+k)\} = Z^k \left[X(Z) - \sum_{l=0}^{k-1} x(n) Z^{-n} \right] \qquad ...(13)$$

Solved Problems :

Prob. 1 : A LTI is described by equation,

$2y(n) + 3y(n-1) + y(n-2) = u(n) + u(n-1) - u(n-2)$.

Find response of system when initial conditions are given by $y(-1) = 2$ and $y(-2) = -1$ and when unit step is applied at the input.

Soln. : Given difference equation is,

$$2y(n) + 3y(n-1) + y(n-2) = u(n) + u(n-1) - u(n-2) \qquad ...(1)$$

Now according to shifting property of unilateral Z-transform;

$$x(n-K) = Z^{-K} \left[X(Z) + \sum_{n=1}^{K} x(-n) Z^n \right] \qquad ...(2)$$

Applying this property to Equation (1) we get,

$$2Y(Z) + 3Z^{-1}[Y(Z) + y(-1) \cdot Z] + Z^{-2} \left[Y(Z) + \sum_{n=1}^{2} y(-n) Z^n \right]$$

$$= U(Z) + Z^{-1}[U(Z) + u(-1)Z] - Z^{-2} \left[U(Z) + \sum_{n=1}^{2} u(-n) Z^n \right]$$

$$\therefore \quad 2Y(Z) + 3Z^{-1}[Y(Z) + y(-1) \cdot Z] + Z^{-2} \left[Y(Z) + y(-1)Z^1 + y(-2)Z^2 \right]$$

$$= U(Z) + Z^{-1}[U(Z) + u(-1)Z] - Z^{-2} \left[U(Z) + u(-1)Z + u(-2)Z^2 \right] \qquad ...(3)$$

Given initial conditions are,

$$y(-1) = 2, \quad y(-2) = -1$$

Since unit step is applied at the input,

$$u(-1) = u(-2) = 0.$$

Putting these values in Equation (3) we get,

$$2Y(Z)+3Z^{-1}[Y(Z)+2Z]+Z^{-2}\left[Y(Z)+2Z-Z^2\right] = U(Z)+Z^{-1}U(Z)-Z^{-2}U(Z)$$

$$\therefore \quad 2Y(Z)+3Z^{-1}Y(Z)+6+Z^{-2}Y(Z)+2Z^{-1}-1 = U(Z)\left[1+Z^{-1}-Z^{-2}\right]$$

$$\therefore \quad Y(Z)\left[2+3Z^{-1}+Z^{-2}\right]+5+2Z^{-1} = U(Z)[1+Z^{-1}-Z^{-2}]$$

$$\therefore \quad Y(Z)[2+3Z^{-1}+Z^{-2}] = -5-2Z^{-1}+U(Z)[1+Z^{-1}-Z^{-2}]$$

$$\therefore \quad Y(Z) = \frac{-5-2Z^{-1}}{2+3Z^{-1}+Z^{-2}} + U(Z)\frac{1+Z^{-1}-Z^{-2}}{2+3Z^{-1}+Z^{-2}} \qquad ...(4)$$

Now we have, $U(Z) = Z\{u(n)\} = \frac{Z}{Z-1}$.

Putting this value in Equation (4) and converting all terms into positive powers of Z we get,

$$Y(Z) = \frac{-5Z^2-2Z}{2Z^2+3Z+1} + \frac{Z}{Z-1}\,\frac{(Z^2+Z-1)}{(2Z^2+3Z+1)}$$

$$\therefore \quad Y(Z) = \frac{(Z-1)(-5Z^2-2Z)+Z(Z^2+Z-1)}{(Z-1)(2Z^2+3Z+1)}$$

$$\therefore \quad Y(Z) = \frac{-5Z^3-2Z^2+5Z^2+2Z+Z^3+Z^2-Z}{(Z-1)(2Z^2+3Z+1)}$$

$$\therefore \quad Y(Z) = \frac{-4Z^3+4Z^2+Z}{(Z-1)(2Z^2+3Z+1)}$$

$$\therefore \quad \frac{Y(Z)}{Z} = \frac{-4Z^2+4Z+1}{(Z-1)(2Z^2+3Z+1)}$$

To adjust second term in the denominator divide numerator and denominator by 2;

$$\therefore \quad \frac{Y(Z)}{Z} = \frac{-2Z^2+2Z+\frac{1}{2}}{(Z-1)\left(Z^2+\frac{3}{2}Z+\frac{1}{2}\right)} \qquad ...(5)$$

Consider the term $Z^2+\frac{3}{2}Z+\frac{1}{2}$. We will obtain the roots as follows :

$$\text{roots} = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-\frac{3}{2}\pm\sqrt{\frac{9}{4}-4\left(\frac{1}{2}\right)}}{2}$$

$$= \frac{-\frac{3}{2}\pm\sqrt{\frac{9}{4}-2}}{2} = \frac{-\frac{3}{2}\pm\frac{1}{2}}{2} = -\frac{1}{2}, -1$$

$$\therefore \quad \frac{Y(Z)}{Z} = \frac{-2Z^2+2Z+0.5}{(Z-1)(Z+0.5)(Z+1)} \qquad ...(6)$$

In P.F.E. form we can write,

$$\frac{Y(Z)}{Z} = \frac{A_1}{Z-1} + \frac{A_2}{Z+0.5} + \frac{A_3}{Z+1} \qquad ...(7)$$

Here $$A_1 = (Z-1)\left[\frac{-2Z^2+2Z+0.5}{(Z-1)(Z+0.5)(Z+1)}\right]\Bigg|_{Z=1}$$

$$\therefore A_1 = \frac{-2+2+0.5}{(1+0.5)(1+1)} = \frac{0.5}{(1.5)(2)} = 0.17$$

$$A_2 = (Z+0.5)\left[\frac{-2Z^2+2Z+0.5}{(Z-1)(Z+0.5)(Z+1)}\right]\Bigg|_{Z=-0.5}$$

$$\therefore A_2 = \frac{-2(-0.5)^2+2(-0.5)+0.5}{(-0.5-1)(-0.5+1)} = \frac{-0.5-1+0.5}{(-1.5)(0.5)} = 1.33$$

and $$A_3 = (Z+1)\left[\frac{-2Z^2+2Z+0.5}{(Z-1)(Z+0.5)(Z+1)}\right]\Bigg|_{Z=-1}$$

$$\therefore A_3 = \frac{-2(-1)^2+2(-1)+0.5}{(-1-1)(-1+0.5)} = \frac{-2-2+0.5}{(-2)(-0.5)} = -3.5$$

Putting these values in Equation (7) we get,

$$\frac{Y(Z)}{Z} = \frac{0.17}{Z-1} + \frac{1.33}{Z+0.5} - \frac{3.5}{Z+1}$$

$$\therefore Y(Z) = \frac{0.17Z}{Z-1} + \frac{1.33Z}{Z+0.5} - \frac{3.5Z}{Z+1}$$

Taking IZT we get,

$$y(n) = 0.17(1)^n u(n) + 1.33(-0.5)^n u(n) - 3.5(-1)^n u(n)$$

This is the response of system.

Prob. 2 : A DT system is represented as,

$y(n) = 0.5\,y(n-1) - 0.06\,y(n-2) + x(n)$ with $y(-1) = y(-2) = 1$ as initial condition.

Find the response of system in a closed form for an input

$x(n) = (0.2)^n u(n)$.

Soln. : Considering initial conditions, using shifting property of unilateral Z-transform we have,

$$x(n-K) \overset{Z}{\longleftrightarrow} Z^{-K}\left[X(Z) + \sum_{n=1}^{K} x(-n)Z^n\right] \qquad ...(1)$$

Given difference is,

$$y(n) = 0.5\,y(n-1) - 0.06\,y(n-2) + x(n)$$

$$Y(Z) = 0.5\,Z^{-1}\left[Y(Z) + y(-1)Z^{1}\right] - 0.06\,Z^{-2}\left[Y(Z) + \sum_{n=1}^{2} y(-n)Z^{n}\right] + X(Z)$$

$$\therefore Y(Z) = 0.5\,Z^{-1}\left[Y(Z) + y(-1)Z^{1}\right] - 0.06\,Z^{-2}\left[Y(Z) + y(-1)Z + y(-2)Z^{2}\right] + X(Z) \quad ...(2)$$

Given initial conditions are,

$$y(-1) = y(-2) = 1.$$

Putting these values in Equation (2) we get,

$$Y(Z) = 0.5\,Z^{-1}\left[Y(Z) + Z^{+1}\right] - 0.06\,Z^{-2}\left[Y(Z) + Z + Z^{2}\right] + X(Z)$$

$$\therefore Y(Z) = 0.5\,Z^{-1}Y(Z) + 0.5 - 0.06\,Z^{-2}Y(Z) - 0.06\,Z^{-1} - 0.06 + X(Z)$$

$$\therefore Y(Z) - 0.5\,Z^{-1}Y(Z) + 0.06\,Z^{-2}Y(Z) = 0.44 - 0.06\,Z^{-1} + X(Z) \quad ...(3)$$

Given input is, $x(n) = (0.2)^{n}u(n)$

$$\therefore X(Z) = \frac{Z}{Z-0.2}$$

Putting this value in Equation (3) we get,

$$Y(Z) - 0.5\,Z^{-1}Y(Z) + 0.06\,Z^{-2}Y(Z) = 0.44 - 0.06\,Z^{-1} + \frac{Z}{Z-0.2}$$

$$\therefore Y(Z)\left[1 - 0.5\,Z^{-1} + 0.06\,Z^{-2}\right] = 0.44 - 0.06\,Z^{-1} + \frac{Z}{Z-0.2}$$

$$\therefore Y(Z) = \frac{0.44}{1 - 0.5\,Z^{-1} + 0.06\,Z^{-2}} - \frac{0.06\,Z^{-1}}{1 - 0.5\,Z^{-1} + 0.06\,Z^{-2}} + \left(\frac{Z}{Z-0.2}\right)\left(\frac{1}{1 - 0.5\,Z^{-1} + 0.06\,Z^{-2}}\right)$$

$$\therefore Y(Z) = \frac{0.44\,Z^{2}}{Z^{2} - 0.5Z + 0.06} - \frac{0.06Z}{Z^{2} - 0.5Z + 0.06} + \left(\frac{Z}{Z-0.2}\right)\left(\frac{Z^{2}}{Z^{2} - 0.5Z + 0.06}\right) \quad ...(4)$$

Let $Y(Z) = Y_1(Z) - Y_2(Z) + Y_3(Z)$ ...(5)

We will obtain IZT of each term separately.

Here $Y_1(Z) = \dfrac{0.44\,Z^{2}}{Z^{2} - 0.5Z + 0.06}$

The roots of denominator term are as follows :

$$\text{roots} = \frac{-b \pm \sqrt{b^{2} - 4ac}}{2a} = \frac{0.5 \pm \sqrt{(0.5)^{2} - 4(0.06)}}{2} = \frac{0.5 \pm 0.1}{2}$$

$$\therefore P_1, P_2 = 0.3,\ 0.2.$$

$$\therefore \frac{Y_1(Z)}{Z} = \frac{0.44Z}{(Z-0.3)(Z-0.2)} \quad ...(6)$$

In PFE form Equation (6) can be written as,

$$\frac{Y_1(Z)}{Z} = \frac{A_1}{Z-0.3} + \frac{A_2}{Z-0.2} \quad \ldots(7)$$

Here $A_1 = (Z-0.3)\frac{Y_1(Z)}{Z}\Big|_{Z=0.3}$

$$\therefore A_1 = (Z-0.3)\cdot\frac{0.44Z}{(Z-0.3)(Z-0.2)}\Big|_{Z=0.3}$$

$$\therefore A_1 = \frac{0.44(0.3)}{(0.3-0.2)} = 1.32$$

and $A_2 = (Z-0.2)\frac{Y_1(Z)}{Z}\Big|_{Z=0.2}$

$$\therefore A_2 = (Z-0.2)\cdot\frac{0.44Z}{(Z-0.3)(Z-0.2)}\Big|_{Z=0.2}$$

$$\therefore A_2 = \frac{0.44(0.2)}{(0.2-0.3)} = -0.88$$

$$\therefore \frac{Y_1(Z)}{Z} = \frac{1.32}{Z-0.3} - \frac{0.88}{Z-0.2}$$

$$\therefore Y_1(Z) = \frac{1.32Z}{Z-0.3} - \frac{0.88Z}{Z-0.2} \quad \ldots(8)$$

Taking IZT of Equation (8) we get,

$$y_1(n) = 1.32(0.3)^n u(n) - 0.88(0.2)^n u(n) \quad \ldots(9)$$

Consider the term $Y_2(Z)$.

$$Y_2(Z) = \frac{0.06Z}{Z^2 - 0.5Z + 0.06}$$

$$\therefore \frac{Y_2(Z)}{Z} = \frac{0.06}{(Z-0.3)(Z-0.2)} \quad \ldots(10)$$

In PFE form we can write,

$$\frac{Y_2(Z)}{Z} = \frac{A_1}{Z-0.3} + \frac{A_2}{Z-0.2}$$

Here $A_1 = (Z-0.3)\frac{Y_2(Z)}{Z}\Big|_{Z=0.3}$

$$\therefore A_1 = \frac{(Z-0.3)(0.06)}{(Z-0.3)(Z-0.2)}\Big|_{Z=0.3}$$

$$\therefore A_1 = \frac{0.06}{0.3-0.2} = 0.6$$

$$\text{and} \quad A_2 = (Z-0.2)\frac{Y_2(Z)}{Z}\Big|_{Z=0.2}$$

$$\therefore \quad A_2 = (Z-0.2)\frac{(0.06)}{(Z-0.3)(Z-0.2)}\Big|_{Z=0.2}$$

$$\therefore \quad A_2 = \frac{0.06}{0.2-0.3} = -0.6$$

$$\therefore \quad \frac{Y_2(Z)}{Z} = \frac{0.6}{Z-0.3} - \frac{0.6}{Z-0.2}$$

$$\therefore \quad Y_2(Z) = \frac{0.6Z}{Z-0.3} - \frac{0.6Z}{Z-0.2}$$

Taking IZT we get,

$$y_2(n) = 0.6(0.3)^n u(n) - 0.6(0.2)^n u(n) \qquad \ldots(11)$$

Now consider the term $Y_3(Z)$.

$$Y_3(Z) = \left(\frac{Z}{Z-0.2}\right)\left(\frac{Z^2}{Z^2-0.5Z+0.06}\right)$$

$$\therefore \quad \frac{Y_3(Z)}{Z} = \frac{Z^2}{(Z-0.2)(Z-0.3)(Z-0.2)}$$

$$\therefore \quad \frac{Y_3(Z)}{Z} = \frac{Z^2}{(Z-0.2)^2(Z-0.3)} \qquad \ldots(12)$$

This is the case of repeated poles. In P.F.E. we can write,

$$\frac{Y_3(Z)}{Z} = \frac{A_1}{Z-0.3} + \frac{A_2}{Z-0.2} + \frac{A_3}{(Z-0.2)^2} \qquad \ldots(13)$$

$$\text{Here} \quad A_1 = (Z-0.3)\frac{Y_3(Z)}{Z}\Big|_{Z=0.3}$$

$$\therefore \quad A_1 = (Z-0.3)\cdot\frac{Z^2}{(Z-0.2)^2(Z-0.3)}\Big|_{Z=0.3}$$

$$\therefore \quad A_1 = \frac{(0.3)^2}{(0.3-0.2)^2} = 9$$

$$A_2 = \frac{d}{dZ}\left[(Z-0.2)^2\frac{Y_3(Z)}{Z}\right]\Big|_{Z=0.2}$$

$$\therefore \quad A_2 = \frac{d}{dZ}\left[(Z-0.2)^2 \cdot \frac{Z^2}{(Z-0.2)^2 (Z-0.3)} \right] \Bigg|_{Z = 0.2}$$

$$\therefore \quad A_2 = \frac{d}{dZ}\left[Z^2 \cdot (Z-0.3)^{-1} \right] \Bigg|_{Z = 0.2}$$

$$A_2 = \left[Z^2 \cdot (-1)(Z-0.3)^{-2} + (Z-0.3)^{-1} \cdot 2Z \right] \Bigg|_{Z = 0.2}$$

$$\therefore \quad A_2 = \left[\frac{-Z^2}{(Z-0.3)^2} + \frac{2Z}{Z-0.3} \right] \Bigg|_{Z = 0.2}$$

$$\therefore \quad A_2 = \frac{-(0.2)^2}{(0.2-0.3)^2} + \frac{2(0.2)}{(0.2-0.3)} = -4-4 = -8$$

$$\text{and} \quad A_3 = (Z-0.2)^2 \frac{Y_3(Z)}{Z} \Bigg|_{Z = 0.2}$$

$$\therefore \quad A_3 = (Z-0.2)^2 \cdot \frac{Z^2}{(Z-0.2)^2 (Z-0.3)} \Bigg|_{Z = 0.2}$$

$$\therefore \quad A_3 = \frac{(0.2)^2}{(0.2-0.3)} = -0.4$$

Putting these values in Equation (13),

$$\frac{Y_3(Z)}{Z} = \frac{9}{Z-0.3} - \frac{8}{Z-0.2} - \frac{0.4}{(Z-0.2)^2}$$

$$\therefore \quad Y_3(Z) = \frac{9Z}{Z-0.3} - \frac{8Z}{Z-0.2} - \frac{0.4Z}{(Z-0.2)^2}$$

For the last term we have to use the Z-transform pair

$$n a^n u(n) \overset{Z}{\longleftrightarrow} \frac{aZ}{(Z-a)^2}$$

$$\therefore \quad Y_3(Z) = \frac{9Z}{Z-0.3} - \frac{8Z}{Z-0.2} - \frac{0.4}{0.2} \cdot \frac{(0.2Z)}{(Z-0.2)^2}$$

Taking IZT, $y_3(n) = 9(0.3)^n u(n) - 8(0.2)^n u(n) - 2n(0.2)^n u(n)$ ...(14)

Now final output is,

$$y(n) = y_1(n) + y_2(n) + y_3(n)$$

Prob. 3 : A system is described by difference equation

$$y(n) = \frac{1}{2}y(n-1)+x(n)$$

Input given to system is $\left(\frac{1}{3}\right)^n u(n)$ and initial condition is $y(-1) = 1$.

Determine : (i) Zero input response (ii) Zero state response (ii) Total response.

Soln. :

(i) Zero input response is the response of system when input is zero. That means this is the response due to initial conditions only. When initial conditions are present, we have to apply the shifting property of unilateral Z-transform. It is,

$$x(n-K) \overset{Z}{\longleftrightarrow} Z^{-K}\left[X(Z)+\sum_{n=1}^{K} x(-n)Z^n\right] \quad ...(1)$$

Given difference equation is,

$$y(n) = \frac{1}{2}y(n-1)+x(n) \quad ...(2)$$

Applying Equation (1) to Equation (2) we get,

$$Y(Z) = \frac{1}{2}Z^{-1}\left[Y(Z)+y(-1)Z^1\right]+X(Z)$$

But initial condition is $y(-1) = 1$

$$\therefore \quad Y(Z) = \frac{1}{2}Z^{-1}[Y(Z)+Z]+X(Z) \quad ...(3)$$

For zero input response, $x(n)$ should be zero. That means $X(Z)$ is zero. Thus Equation (3) becomes,

$$Y(Z) = \frac{1}{2}Z^{-1}[Y(Z)+Z] \qquad \therefore \quad Y(Z) = \frac{1}{2}Z^{-1}Y(Z)+\frac{1}{2}$$

$$\therefore \quad Y(Z)-\frac{1}{2}Z^{-1}Y(Z) = \frac{1}{2} \qquad \therefore \quad Y(Z)\left[1-\frac{1}{2}Z^{-1}\right] = \frac{1}{2}$$

$$\therefore \quad Y(Z) = \frac{1}{2}\cdot\frac{1}{1-\frac{1}{2}Z^{-1}} = \frac{1}{2}\cdot\frac{Z}{Z-\frac{1}{2}} \quad ...(4)$$

IZT of Equation (4) can be written as,

$$\boxed{y(n) = \frac{1}{2}\cdot\left(\frac{1}{2}\right)^n u(n)}$$

This is the zero input response of system.

(ii) Zero state response is the response of system when, initial conditions are zero and input is $x(n) = \left(\frac{1}{3}\right)^n u(n)$.

The given difference equation is,

$$y(n) = \frac{1}{2}y(n-1)+x(n)$$

Taking Z-transform without considering initial conditions,

$$Y(Z) = \frac{1}{2}Z^{-1}Y(Z)+X(Z) \qquad ...(5)$$

$$\text{But} \quad X(Z) = Z\left\{\left(\frac{1}{3}\right)^{n} u(n)\right\} = \frac{Z}{Z-\frac{1}{3}}$$

$$\therefore \quad Y(Z) = \frac{1}{2}Z^{-1}Y(Z)+\frac{Z}{Z-\frac{1}{3}}$$

$$\therefore \quad Y(Z)-\frac{1}{2}Z^{-1}Y(Z) = \frac{Z}{Z-\frac{1}{3}}$$

$$\therefore \quad Y(Z)\left[1-\frac{1}{2}Z^{-1}\right] = \frac{Z}{Z-\frac{1}{3}} \qquad \therefore \quad Y(Z) = \frac{1}{1-\frac{1}{2}Z^{-1}}\cdot\frac{Z}{Z-\frac{1}{3}}$$

$$\therefore \quad Y(Z) = \frac{Z}{Z-\frac{1}{2}}\cdot\frac{Z}{Z-\frac{1}{3}} = \frac{Z^2}{\left(Z-\frac{1}{2}\right)\left(Z-\frac{1}{3}\right)}$$

$$\therefore \quad \frac{Y(Z)}{Z} = \frac{Z}{\left(Z-\frac{1}{2}\right)\left(Z-\frac{1}{3}\right)} \qquad ...(6)$$

In PFE form Equation (6) can be written as,

$$\frac{Y(Z)}{Z} = \frac{A_1}{Z-\frac{1}{2}}+\frac{A_2}{Z-\frac{1}{3}} \qquad ...(7)$$

The values of A_1 and A_2 are calculated as follows :

$$A_1 = \left(Z-\frac{1}{2}\right)\cdot\frac{Y(Z)}{Z}\Bigg|_{Z=\frac{1}{2}}$$

$$\therefore \quad A_1 = \left(Z-\frac{1}{2}\right)\cdot\frac{Z}{\left(Z-\frac{1}{2}\right)\left(Z-\frac{1}{3}\right)}\Bigg|_{Z=\frac{1}{2}}$$

$$\therefore \quad A_1 = \frac{\frac{1}{2}}{\frac{1}{2}-\frac{1}{3}} = \frac{\frac{1}{2}}{\frac{1}{6}} = 3 \qquad \text{and} \qquad A_2 = \left(Z-\frac{1}{3}\right)\frac{Y(Z)}{Z}\Bigg|_{Z=\frac{1}{3}}$$

$$\therefore \quad A_2 = \left(Z-\frac{1}{3}\right)\cdot\frac{Z}{\left(Z-\frac{1}{2}\right)\left(Z-\frac{1}{3}\right)}\Bigg|_{Z=\frac{1}{3}}$$

$$\therefore \quad A_2 = \frac{1/3}{\frac{1}{3}-\frac{1}{2}} = -2$$

Putting these values in Equation (7) we get,

$$\frac{Y(Z)}{Z} = \frac{3}{Z-\frac{1}{2}} - \frac{2}{Z-\frac{1}{3}}$$

$$\therefore \quad Y(Z) = \frac{3Z}{Z-\frac{1}{2}} - 2\frac{Z}{Z-\frac{1}{3}} \qquad ...(8)$$

IZT of Equation (8) can be written as,

$$y(n) = 3\left(\frac{1}{2}\right)^n u(n) - 2\left(\frac{1}{3}\right)^n u(n)$$

This is zero state response of system.

(iii) Total response is addition of zero input response (ZIR) and zero state response (ZSR).

$$\therefore \quad y(n) = ZIR + ZSR$$

$$\therefore \quad y(n) = \frac{1}{2}\left(\frac{1}{2}\right)^n u(n) + 3\left(\frac{1}{2}\right)^n u(n) - 2\left(\frac{1}{3}\right)^n u(n)$$

This is the total response of system.

Prob. 4 : Obtain a closed form expression of n^{th} term of Fibonacci sequence where a system is described by means of difference equation as :

$$y(n) = y(n-1) + y(n-2)$$

Soln. : Fibonacci series is the sequence in which the value at any instant is obtained by adding previous two values.

$$\therefore \quad y(n) = y(n-1) + y(n-2) \qquad ...(1)$$

Using Uni-lateral Z-transform we have,

$$\therefore \quad Y(Z) = \left[Z^{-1}Y(Z) + Y(-1)\right] + \left[Z^{-2}Y(Z) + Y(-2) + y(-1)Z^{-1}\right]$$

Let $y(-1) = 0$ and $y(-2) = 1$

$$\therefore \quad Y(Z) = \left[Z^{-1}Y(Z)\right] + \left[Z^{-2}Y(Z) + 1\right]$$

$$\therefore \quad Y(Z) - Z^{-1}Y(Z) - Z^{-2}Y(Z) = 1 \qquad \therefore \quad Y(Z)\left[1 - Z^{-1} - Z^{-2}\right] = 1$$

$$\therefore \quad Y(Z) = \frac{1}{1 - Z^{-1} - Z^{-2}} \qquad ...(2)$$

Multiplying numerator and denominator by Z^2 we get,

$$Y(Z) = \frac{Z^2}{Z^2 - Z - 1} \qquad ...(3)$$

$$\frac{Y(Z)}{Z} = \frac{Z}{Z^2 - Z - 1} \qquad ...(4)$$

We will obtain the roots (poles) of denominator.

$$P_1, P_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1-4}}{2}$$

$$\therefore \quad P_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad P_2 = \frac{1-\sqrt{5}}{2}$$

$$\therefore \quad \frac{Y(Z)}{Z} = \frac{Z}{\left[Z - \left(\frac{1+\sqrt{5}}{2}\right)\right]\left[Z - \left(\frac{1-\sqrt{5}}{2}\right)\right]} \qquad ...(5)$$

In the partial fraction expansion form we can write,

$$\frac{Y(Z)}{Z} = \frac{A_1}{Z - \left(\frac{1+\sqrt{5}}{2}\right)} + \frac{A_2}{Z - \left(\frac{1-\sqrt{5}}{2}\right)} \qquad ...(6)$$

$$\text{Here} \quad A_1 = (Z - P_1)\frac{Y(Z)}{Z}\Bigg|_{Z = P_1}$$

$$= Z - \left(\frac{1+\sqrt{5}}{2}\right) \frac{Z}{\left[Z - \left(\frac{1+\sqrt{5}}{2}\right)\right]\left[Z - \left(\frac{1-\sqrt{5}}{2}\right)\right]}\Bigg|_{Z = P_1}$$

$$\therefore \quad A_1 = \frac{Z}{Z - \left(\frac{1-\sqrt{5}}{2}\right)}\Bigg|_{Z = P_1} = \frac{\frac{1+\sqrt{5}}{2}}{\frac{1+\sqrt{5} - 1 + \sqrt{5}}{2}} = \frac{1+\sqrt{5}}{2\sqrt{5}}$$

$$\text{And} \quad A_2 = \frac{-1+\sqrt{5}}{2\sqrt{5}}$$

Putting values of A_1 and A_2 in Equation (6) we get,

$$\frac{Y(Z)}{Z} = \frac{\frac{1+\sqrt{5}}{2\sqrt{5}}}{Z - \left(\frac{1+\sqrt{5}}{2}\right)} + \frac{\frac{-1+\sqrt{5}}{2\sqrt{5}}}{Z - \left(\frac{1-\sqrt{5}}{2}\right)}$$

$$\therefore \quad Y(Z) = \frac{\left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)Z}{Z - \left(\frac{1+\sqrt{5}}{2}\right)} + \frac{\left(\frac{-1+\sqrt{5}}{2\sqrt{5}}\right)Z}{Z - \left(\frac{1-\sqrt{5}}{2}\right)}$$

Taking IZT we get,

$$y(n) = \left[\frac{1+\sqrt{5}}{2\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n} - \frac{1-\sqrt{5}}{2\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{n} \right] u(n)$$

1.5 Introduction to Analog Filter Design :

The basic function of filter is to eliminate the noise and to extract the signal of interest from other signals. A digital filter is a basic device used in digital signal processing.

There are several techniques available to design the digital filters. But generally while designing a digital filter; first an analog filter is designed and then it is converted into the corresponding digital filter.

1.5.1 Review of Analog Filters :

Before studying digital filters, we will study some fundamental things related to analog filters. This is necessary because, generally digital filters are designed using analog filters.

Some parameters related to analog filters :

1. Pass band : It passes certain range of frequencies. In the pass band, attenuation is zero.
2. Stop band : It suppresses certain range of frequencies. In the stop band, attenuation is infinity.
3. Cut-off frequency : This is the frequency which seperates pass band and stop band.

Types of analog filters :

The different types of analog filters are as follows :

1. Low Pass Filter (L.P.F.) :

It passes the frequency from zero upto some designated frequency, called as cut-off frequency. After cut-off frequency; it will not allow any signal to pass through it. An ideal low pass filter characteristic is shown in Fig. P-37(a).

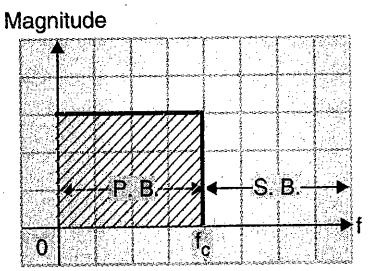


Fig. P-37(a) : Ideal L.P.F. characteristic

2. High Pass Filter (H.P.F.) :

It passes the frequency above some designated frequency called as cut-off frequency. If input signal frequency is less than the cut-off frequency; then this signal is not allowed to pass through it. An ideal H.P.F. characteristic is shown in Fig. P-37(b).

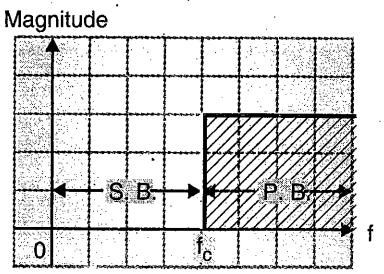


Fig. P-37(b) : Ideal H.P.F. characteristic

3. Band Pass Filter (B.P.F.) :

It allows the frequencies between two designated cut-off frequencies (say f_{c1} and f_{c2}). An ideal B.P.F. characteristic is shown in Fig. P-37(c).

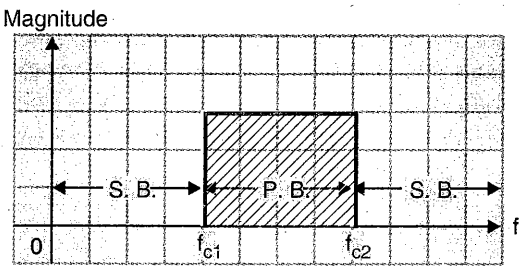


Fig. P-37(c) : Ideal B.P.F. characteristic

4. Band Stop Filter (B.S.F.) :

It attenuates all frequencies between two designated cut-off frequencies; while it passes all other frequencies. An ideal B.S.F. characteristic is shown in Fig. P-37(d).

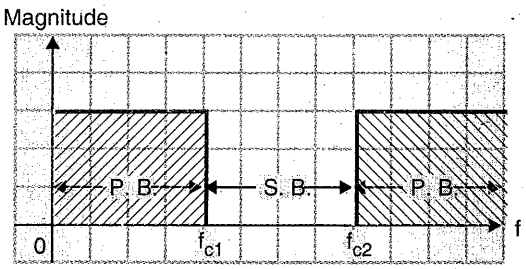


Fig. P-37(d) : Ideal B.S.F. characteristic

5. All pass filter :

It passes all the frequencies. But by using this filter the phase of input signal can be modified. All pass filter characteristic is shown in Fig. P-37(e).

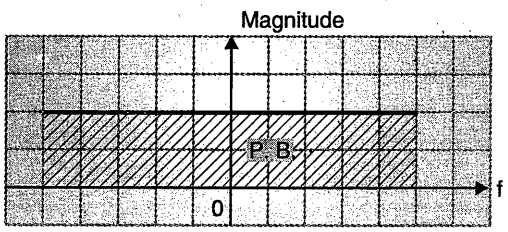


Fig. P-37(e) : All pass filter characteristic

1.5.2 Ideal Filters and Approximations :

First will study why ideal frequency selective filters are practically not realizable ?

Two important characteristics of ideal filters are :

(1) Ideal filters have constant gain in the passband and zero gain in the stop band.

(2) Ideal filter has linear phase response.

In order to realize (design) digital filter, an important condition is that; the response of filter should be causal. We have studied a causal LTI system. Impulse response of causal LTI system is denoted by h (n). Now system is causal when h (n) has some value for positive values of 'n' and h (n) is zero for negative values of 'n'. Thus the condition of causality can be summarized as,

$$h(n) = 0 \quad \text{for } n < 0 \qquad \ldots(1)$$

As an example, let us consider the magnitude response of ideal low pass filter (L.P.F.). It is shown in Fig. P-38(a).

Here, ω_c = Cut-off frequency

and $|H(\omega)|$ = Magnitude of filter

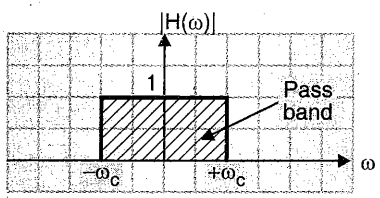


Fig. P-38(a) : Magnitude response of ideal LPF

As shown in Fig. P-38(a), the magnitude is '1' in the frequency range $-\omega_c$ to $+\omega_c$. That means,

$$H(\omega) = \begin{cases} 1 & \text{for } -\omega_c \le \omega \le \omega_c \\ 0 & \text{otherwise} \end{cases} \qquad \ldots(2)$$

Now the value of h (n) is obtained by taking inverse fourier of H (ω). According to the definition of inverse fourier we have,

$$h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\omega)\, e^{j\omega n}\, d\omega \qquad \ldots(3)$$

Using Equation (2) we get,

$$h(n) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} 1 \cdot e^{j\omega n}\, d\omega \qquad \ldots(4)$$

Now we will consider two conditions as follows :

Condition (i) : When n = 0 :

Putting $n = 0$ in Equation (4) we get,

$$h(n) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} 1 \cdot e^0 \, d\omega$$

$$\therefore \quad h(n) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} d\omega \qquad \ldots(\because e^0 = 1)$$

$$\therefore \quad h(n) = \frac{1}{2\pi} [\omega]_{-\omega_c}^{\omega_c}$$

$$\therefore \quad h(n) = \frac{\omega_c}{\pi} \qquad \ldots\text{when } n = 0 \qquad \ldots(5)$$

Condition (ii) : When n ≠ 0 :

Taking integration of Equation (4) we get,

$$h(n) = \frac{1}{2\pi} \left[\frac{e^{j\omega n}}{jn} \right]_{-\omega_c}^{\omega_c} = \frac{1}{2\pi jn} \left[e^{j\omega_c n} - e^{-j\omega_c n} \right]$$

$$\therefore \quad h(n) = \frac{1}{\pi n} \left[\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j} \right] \qquad \ldots(6)$$

But according to Euler's identity,

$$\frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin\theta$$

Thus Equation (6) becomes,

$$h(n) = \frac{\sin \omega_c n}{\pi n} \qquad \ldots\text{when } n \neq 0 \qquad \ldots(7)$$

Thus combining two conditions we can write,

$$h(n) = \begin{cases} \dfrac{\sin \omega_c n}{\pi n} & \text{For } n \neq 0 \\ \dfrac{\omega_c}{\pi} & \text{For } n = 0 \end{cases} \qquad \ldots(8)$$

Now we will sketch $h(n)$ for different values of n by taking $\omega_c = \frac{\pi}{4}$. It is shown in Fig. P-38(b).

From Fig. P-38(b), we can easily conclude that $h(n)$ is present for negative values of 'n'. That means it is noncausal. We had discussed that if the frequency response of filter is causal then only it can be realized. Thus ideal filters are practically not realizable.

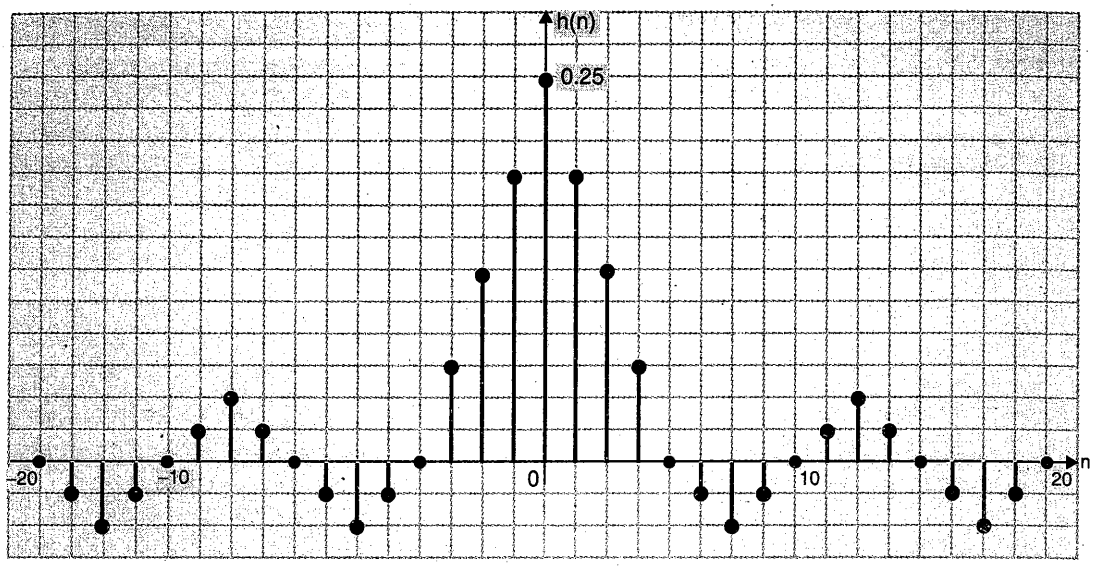


Fig. P-38(b) : Response h (n) of ideal LPF

We have studied that, in practical cases it is not possible to realize idle filter. An important reason for this is, idle filters are non-causal.

Specifications :

The frequency response characteristic of a filter is denoted by H (ω). An ideal filter cannot be realized because the requirements of ideal filter cannot be fulfilled. But if these requirements are relaxed then it is possible to realize causal filters having characteristics which are approximately similar to ideal filters. The magnitude characteristics of physically realizable filter is shown in Fig. P-39.

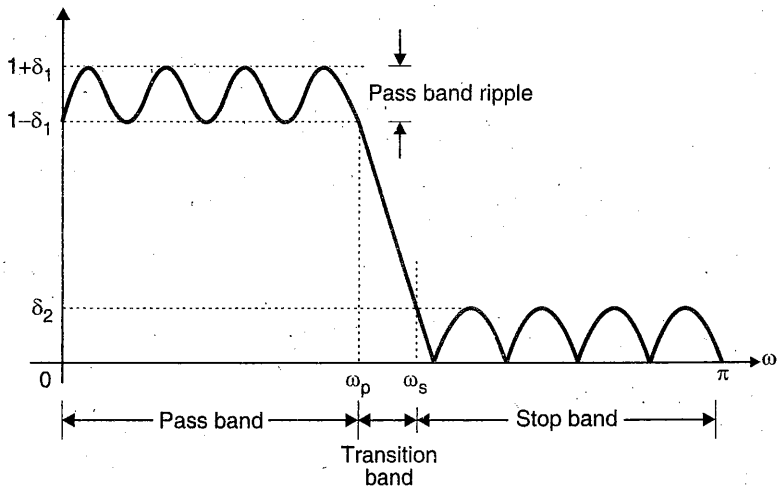


Fig. P-39

The different notations used in Fig. P-39 are as follows :

δ_1 = Pass band ripple $\quad$ δ_2 = Stop band ripple

ω_p = Pass band edge frequency $\quad$ ω_s = Stop band edge frequency.

The different conditions that can be relaxed to design such filters are as follows :

1. It is not necessary to insist that the magnitude $|H(\omega)|$ is constant in the entire range of passband.
2. A small amount of ripple in the passband is allowed.
3. It is not necessary that the filter response in the stop band should be perfectly zero. A small amount of ripple in the stopband is tolerable.

As shown in Fig. P-39, the frequency response from zero to ω_p is called as passband. The frequency ω_p denotes the edge of passband; while the frequency ω_s denotes the beginning of stopband. The transition between pass band to stop band is called as transition band. Thus the frequency range of transition band is $\omega_p - \omega_s$. The width of passband is called as bandwidth of a filter. Thus as shown in Fig. P-39, the bandwidth of filter is ω_p.

A ripple in the passband is denoted by δ_1 and ripple in stop and is denoted by δ_2. The magnitude in the passband varies between the limits $1 \pm \delta_1$.

Specifications :

In any filter design, following specifications are required :

1. The maximum tolerable pass band ripple (δ_1)
2. The maximum tolerable stop band ripple (δ_2).
3. Passband edge frequency (ω_p).
4. Stopband edge frequence (ω_s).

1.5.3 Butterworth and Chebyshev Approximations :

Many times it is necessary to approximate the characteristics of analog filter. This approximation is required because the practical characteristic of a filter is not identical to the ideal characteristics. There are three different types of approximation techniques as follows :

(1) Butterworth filter approximation.

(2) Chebyshev filter approximation.

1 Butterworth filter approximation :

A typical characteristic of a butterworth low pass filter is as shown in Fig. P-40.

This type of response is called as butterworth response because its main characteristic is that the passband is maximally flat. That means there are no variations (ripples) in the passband.

Now the magnitude squared response of low pass butterworth filter is given by,

$$|H(\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}} \qquad \ldots(1)$$

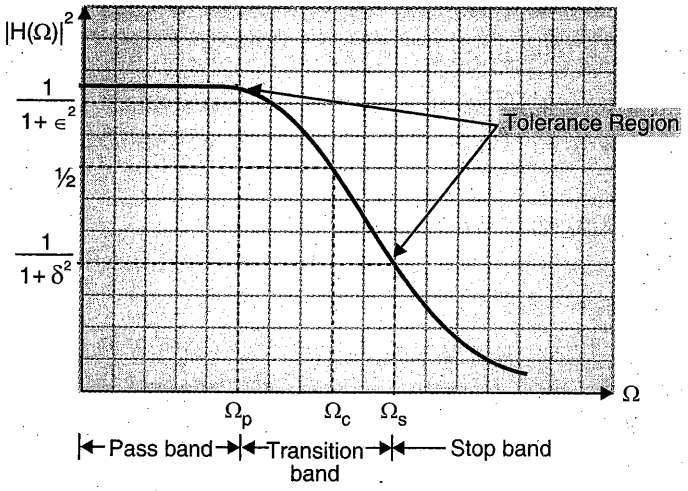


Fig. P-40 : Typical characteristics of analog L.P.F.

This equation is also expressed as,

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}} \quad ..(2)$$

Here $|H(\Omega)|$ = Magnitude of analog low pass filter.

Ω_c = Cut-off frequency (– 3dB frequency).

Ω_p = Pass band edge frequency.

$1+\epsilon^2$ = Pass band edge value.

$1+\delta^2$ = Stop band edge value.

ϵ = Parameter related to ripples in pass band.

δ = Parameter related to ripples in stop band.

N = Order of the filter.

We know that, in case of low pass filter the frequencies will pass upto the value of cut-off frequency (Ω_c). This is called as pass band. After that the frequencies are attenuated. This is called as stop band. Ideal characteristic is shown by dotted line in Fig. P-40. Ideally, at the value of cut-off frequency (Ω_c) the frequencies should be stopped. But in practical cases this is not happening.

Now the order of filter is denoted by 'N'. Roughly we can say order of filter means, the number of stages used in the design of analog filter. As the order of filter 'N' increase's, the response of filter is more close to the ideal response as shown in Fig. P-41.

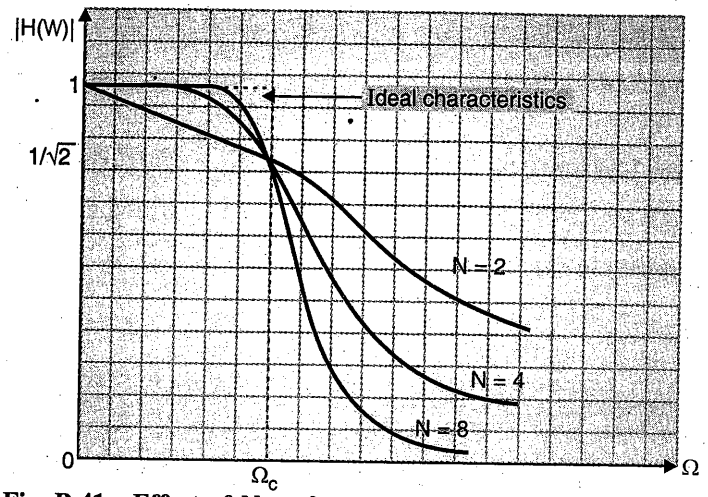


Fig. P-41 : Effect of N on frequency response characteristics

Salient features of low pass butterworth filter :

1. The magnitude response is nearly constant (equal to 1) at lower frequencies. That means pass band is maximally flat.
2. There are no ripples in the pass band and stop band.
3. The maximum gain occurs at $\Omega = 0$ and it is $|H(0)| = 1$.
4. The magnitude response is monotonically decreasing.

2. Chebyshev filter :

Using chebyshev filter design, there are two subgroups,

1. Type-1 chebyshev filter
2. Type-2 chebyshev filter

1. Type-1 Chebyshev Filter :

These filters are all pole filters. In the passband, these filters show equiripple behaviour and they have monotonic characteristics in the stopband.

The filter characteristics for odd and even values of N are shown in Fig. P-42.

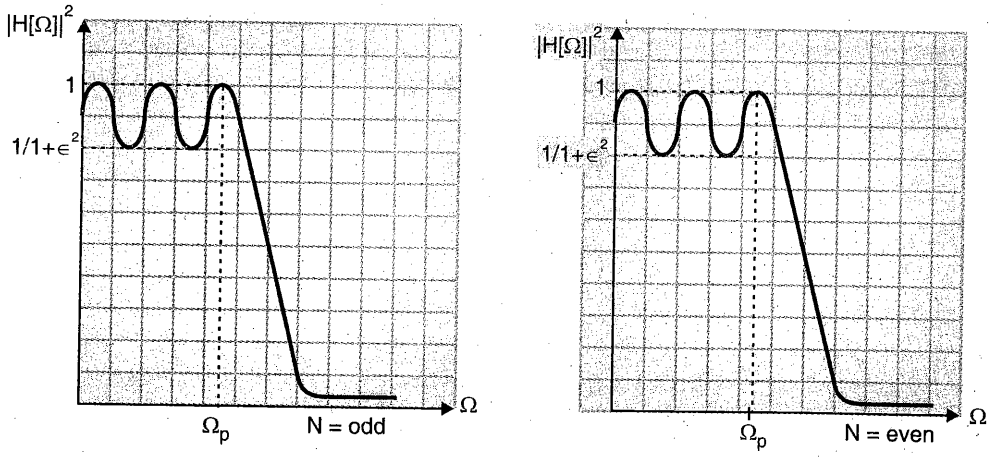


Fig. P-42 : Type-1 chebyshev filter characteristics

The magnitude squared frequency response is given by,

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2 C_N^2(\Omega/\Omega_p)} \quad ...(1)$$

Here ϵ = Ripple parameter in the passband; Ω_p = Passband frequency

$C_N(x)$ = Chebyshev polynomial of order N.

The chebyshev polynomials are determined by using the equation,

$$C_{N+1}(x) = 2x\,C_N(x) - C_{N-1}(x) \quad ...(2)$$

with $C_0(x) = 1$ and $C_1(x) = x$

The different polynomials are given in Table P-3.

Table P-3

N	$C_N(x)$
0	1
1	x
2	$2x^2 - 1$
3	$4x^3 - 3x$
4	$8x^4 - 8x^2 + 1$
5	$16x^5 - 20x^3 + 5x$

The major difference between butterworth and chebyshev filter is that the poles of butterworth filter lie on the circle, while the poles of chebyshev filter lie on ellipse.

2. Type-II Chebyshev Filter :

This filter contains zeros as well as poles. The characteristic of such filters for even and odd values of N is shown in Fig. P-43

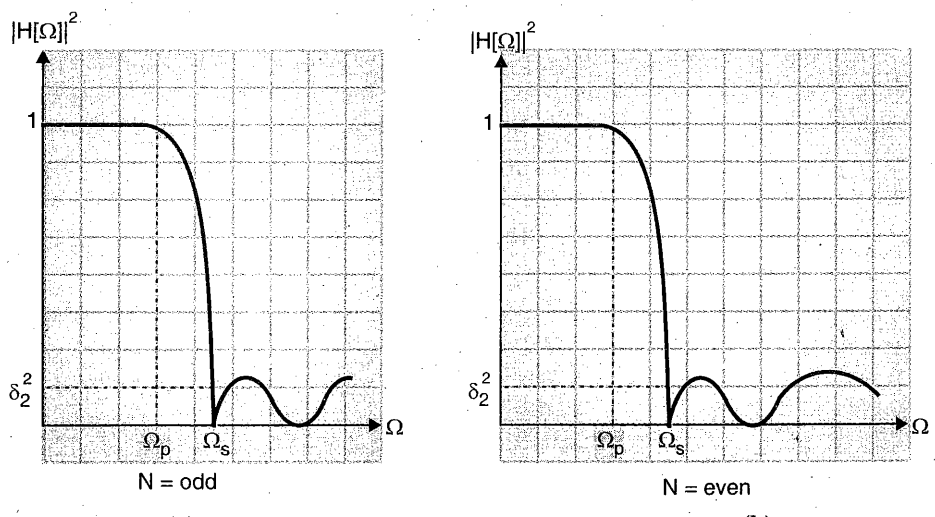


Fig. P-43 : Type-II chebyshev filter characteristic

The magnitude squared response is given by,

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2\left[C_N^2\left(\frac{\Omega_s}{\Omega_p}\right)\Big/C_N^2\left(\frac{\Omega_s}{\Omega}\right)\right]}$$

Here $C_N(x)$ = N^{th} order polynomial; Ω_s = Stopband frequency

Ω_p = Passband frequency

Theory Questions

Q. 1 Write difference equation of system which generates a ramp discrete time signal as an output for $n \geq 0$.

Q. 2 Explain the system classification tests for stability and causality, in time domain and z-domain.

Q. 3 Explain significance of pole-zero plot.

Unsolved Numericals

Q. 1 Determine all possible signals $x(n)$ associated with Z transform.

(i) $X(Z) = \dfrac{1}{1-\frac{1}{2}Z^{-1}+\frac{1}{4}Z^{-2}}$ (ii) $X(Z) = \dfrac{5Z^{-1}}{(1-2Z^{-1})(1-3Z^{-1})}$

Ans. :

(i) $x(n) = \left(\frac{1}{2}-\frac{j}{2\sqrt{3}}\right)\cdot\left(\frac{1}{4}+\frac{j\sqrt{3}}{4}\right)^n u(n)+\left(\frac{1}{2}+\frac{i}{2\sqrt{3}}\right)\left(\frac{1}{4}-\frac{j\sqrt{3}}{4}\right)^n u(n)$

(ii) $x(n) = -5(2)^n u(n)+5(3)^n u(n)$ ... for casual sequence.

$x(n) = 5(2)^n u(n)-5(3)^n u(n)$ ... for anticasual sequence.

Q. 2 Find all possible sequences (DT sequences) for

$$X(Z) = \frac{5Z}{6Z^2-Z-1}$$

Ans. :

(i) $|Z| > \frac{1}{2}$ $x(n) = 5\left(\frac{1}{2}\right)^n u(n)-5\left(\frac{1}{3}\right)^n u(n)$

(ii) $|Z| < \frac{1}{3}$ $x(n) = -5\left(\frac{1}{2}\right)^n u(-n-1)+5\left(\frac{1}{3}\right)^n u(-n-1)$

(iii) $|Z| > \frac{1}{3}$ $x(n) = -5\left(\frac{1}{2}\right)^n u(-n-1)-5\left(\frac{1}{3}\right)^n u(n)$

Q. 3 Determine the unit step response of a DT - LTI system whose I/O relation is described by $y(n) = x(n)-3y(n-1)$ with $y(-1) = 1$

Ans. : $y(n) = \frac{1}{4}(1)^n u(n)-\frac{9}{4}(-3/4)^n u(n)$

Q. 4 A DT–LTI system is described by a difference equation

$y(n) + b^2 y(n-2) = 0$ for $n \geq 0$ with

initial conditions – $y(-1) = 0$ and $y(-2) = -1$

Prove that : $y(n) = b^{n+2} \cos(n\pi/2)$

Ans. : $y(n) = b^{n+2} \cos\left(\frac{n\pi}{2}\right)$

Q. 5 Determine the response of the system

$$y(n) = \frac{5}{6} y(n-1) - \frac{1}{6} y(n-2) + x(n)$$

to the input signal

$$x(n) = \delta(n) - \frac{1}{3}\delta(n-1)$$

Ans. : $y(n) = 0.087\,(0.4)^n u(n) + 0.9127\,(-0.4)^n u(n)$

Q. 6 Determine the inverse Z-transform of

$$X(Z) = \frac{1}{1 - 1.5 z^{-1} + 0.5 z^{-2}} \text{ if}$$

(i) ROC : $|Z| > 1$

(ii) ROC : $|Z| < 0.5$

(iii) ROC : $0.5 < |z| < 1$

Ans. :

$$x(n) = 2\left[-(1)^n u(-n-1)\right] - (0.5)^n u(n)$$

Q. 7 Find $y(Z)$ if difference equation is known to be

$$y(n) = x(n) + \frac{3}{4} y(n-1) - \frac{1}{8} y(n-2)$$

with initial conditions $y(-1) = 2$ and $y(-2) = 4$ and $x(n) = u(n)$

Ans. :

$$Y(Z) = \frac{8Z^2 - 5Z + 1}{(Z-1)} \times \frac{2Z^2}{8Z^2 - 6Z + 1}$$

Q. 8 Consider a DT system given by a difference equation

$y(n) = x(n) - 7.2\,x(n-1) - 0.3y(n-1) + 0.1\,y(n-2)$

Find transfer function and Impulse response of the system.

Ans. :

$$H(z) = \frac{-10Z}{Z-0.2} + \frac{11Z}{Z+0.5}$$

$$h(n) = -10\,(0.2)^n u(n) + 11\,(-0.5)^n u(n)$$

❑❑❑

Frequency Analysis of Continuous Time Signal

Section No.	Name of the Topic
1.1	Introduction
1.2	Frequency Analysis of Continuous Time Signals

• Code : DSP20

1.1 Introduction :

The frequency analysis of signal involves the resolution of signals into its frequency components. These frequency components are basically sinusoidal components. If any signal is decomposed into sinusoidal components; the sum of all these sinusoidal components results in original waveform. The basic aim for performing the frequency analysis is to provide the mathematical representation of frequency components contained in a given signal.

One of the important technique to do the frequency analysis is the fourier representation of signals. These are four distinct fourier representation.

(1) **Fourier Series (FS) :**

It applies to continuous time periodic signals.

(2) **Fourier Transform (FT) :**

It applies to continuous time non periodic signals.

(3) **Discrete Time Fourier Series (DTFS) :**

It applies to discrete time periodic signals.

(4) **Discrete Time Fourier transform (DTFT) :**

It applies to discrete time non periodic signals.

1.2 Frequency Analysis of Continuous Time Signals :

In this section we will study the fourier series and fourier transform of continuous time signals.

1.2.1 Continuous Time Fourier Series (CTFS) :

Sinewaves and cosinewaves are the basic building functions for any periodic signal. That means any periodic signal basically consists of sinewaves having different amplitudes, of different frequencies and having different relative phase shifts.

Fourier series represents a periodic waveform in the form of sum of infinite number of sine and cosine terms. It is a representation of the signal in a time domain series form.

Fourier series is a "tool" used to analyze any periodic signal. After the "analysis" we obtain the following information about the signal :

(i) What all frequency components are present in the signal ?

(ii) Their amplitudes and

(iii) The relative phase difference between these frequency components.

All the "frequency components" are nothing else but sinewaves at those frequencies.

Types of fourier series :

There are three types of fourier series used for the analysis of periodic signals. They are :

- Trignometric or quadrature fourier series
- Polar fourier series

- Exponential fourier series.

Let us understand them one-by-one.

1. Trignometric or quadrature Fourier series :

A periodic signal $x(t)$ with a period of " T_o" is represented by the trignometric fourier series as,

$$x(t) = a_o + \sum_{n=1}^{\infty} a_n \cos(n \omega_o t) + \sum_{n=1}^{\infty} b_n \sin(n \omega_o t) \quad ...(1)$$

where a_o, a_n and b_n are known as the fourier coefficients and $\omega_o = 2 \pi f_o = \frac{2\pi}{T_o}$

The expression for the fourier coefficients are as follows :

(i) The value of $a_o = \frac{1}{T_o} \int_{t}^{t+T_o} x(t)\, dt$ ...(2)

This shows that a_o is the average value of $x(t)$. It is also called as the dc component of $x(t)$.

(ii) The value of $a_n = \frac{2}{T_o} \int_{t}^{t+T_o} x(t) \cdot \cos(n \omega_o t)\, dt$ ...(3)

(iii) The value of $b_n = \frac{2}{T_o} \int_{t}^{t+T_o} x(t) \cdot \sin(n \omega_o t)\, dt$ ...(4)

2. Polar Fourier series :

The polar fourier series representation of $x(t)$ is as follows :

$$x(t) = C_o + \sum_{n=1}^{\infty} \cos(n \omega_o t + \phi_n) \quad ...(5)$$

This is also known as the polar fourier series.

where, $C_n = \left[a_n^2 + b_n^2 \right]^{1/2}$ ...(6)

and $\phi_n = -\tan^{-1} \left[\frac{b_n}{a_n} \right]$ ...(7)

and C_0 = Average value of $x(t)$.

The Equation (5) can be expanded by opening the summation sign as follows :

$$x(t) = C_0 + C_1 \cos(\omega_o t + \phi_1) + C_2 \cos(2\omega_o t + \phi_2) + C_3 \cos(3\omega_o t + \phi_3) + \ldots\ldots \quad \ldots(8)$$

Fundamental Second harmonic Third harmonic

Conclusions :

The important conclusions from the Equation (8) are :

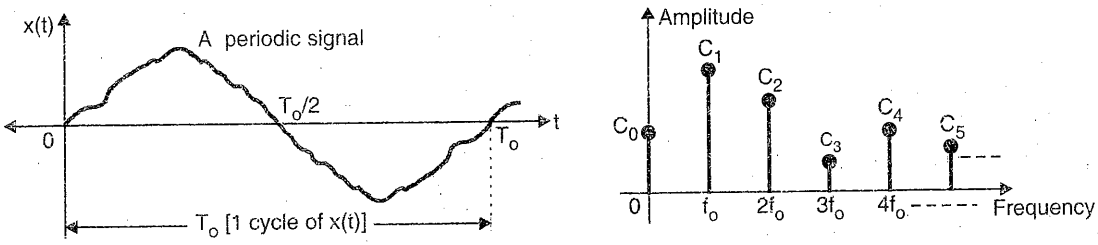
(i) The periodic signal $x(t)$ is represented as the sum of cosinewaves having different frequencies f_o, $2f_o$, $3f_o$ ..... The cosine wave of frequency f_o is called as the "fundamental component" and all its multiples i.e. $2f_o$, $3f_o$... are called as the "harmonic components".

(ii) "C_0" is the average value of $x(t)$. Because it is a dc component and its frequency is zero.

(iii) C_1, C_2, C_3 ... are the peak amplitudes of the fundamental and the harmonic components respectively.

Spectrum or frequency domain representation :

The periodic signal $x(t)$ can be plotted in the time domain as shown in Fig. T-1(a). This is amplitude versus time graph. After the analysis of signal $x(t)$, it is possible to plot its frequency spectrum as shown in Fig. T-1(b).



(a) $x(t)$ represented in time domain **(b) Frequency spectrum of $x(t)$**

Fig. T-1

As seen from Fig. T-1(b) the frequency spectrum of a continuous signal is discrete in nature. The frequency components f_o, $2f_o$, $3f_o$....etc. are called as the "spectral components". The adjacent spectral components are spaced by "f_o" from each other. As the spectrum is consisting of vertical lines, (C_1, C_2, ...) this spectrum is called as the line spectrum.

Dirichlet conditions for the existence of Fourier series :

The fourier series stated in the Equations (1) and (5) will exist if and only if the periodic signal $x(t)$ satisfies the following conditions. These are known as the "Dirichlet" conditions. They are as follows :

(i) The periodic signal $x(t)$ and its integrals are finite and single valued in the interval (t to $t+T_o$), i.e. over a period of one cycle T_o.

(ii) $x(t)$ must have only finite number of discontinuities in the given interval of time.

(iii) $x(t)$ should have only finite number of maxima and minima in the given interval of time.

(iv) The function $x(t)$ is absolutely integrable, that is

$$\int_{-T_o/2}^{T_o/2} |x(t)|\,dt < \infty$$

3. Exponential Fourier series [OR complex exponential Fourier series] :

The sine and cosine terms can be expressed in terms of the exponential terms using Euler's equations as follows :

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad ...(9)$$

and

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2} \qquad ...(10)$$

Substituting the sine and cosine functions in terms of exponential function in the expression for the quadrature fourier series, Equation (1), we can obtain another type of fourier series called the exponential fourier series.

A periodic signal $x(t)$ is expressed in the exponential fourier series form as follows :

Exponential fourier series :

$$x(t) = \sum_{n=-\infty}^{\infty} C_n\, e^{j2\pi n t/T_o} \qquad ...(11)$$

where,

$$C_n = \frac{1}{T_o}\int_{t}^{t+T_o} x(t)\cdot e^{-j2\pi n t/T_o}\,dt \qquad ...(12)$$

As seen from the Equation (11), this fourier series consists of exponential terms. Therefore it is called as the exponential fourier series.

Concept of negative frequency :

In Equation (11), observe that "n" is extending from $-\infty$ to $+\infty$ instead of from 0 to ∞. Due to this the frequencies in the frequency spectrum will extend from $-\infty$ to $+\infty$. Thus if we express the signal $x(t)$ using exponential fourier series then we obtain a double sided frequency spectrum.

However the negative frequency signals do not exist physically. They are used as an important mathematical concept and for mathematical convenience.

Amplitude and Phase Spectrums :

The amplitude spectrum of the signal $x(t)$ is denoted by,

$$|C_n| = \left[(\text{Real part of } C_n)^2 + (\text{Imaginary part of } C_n)^2\right]^{1/2} \qquad ...(13)$$

The amplitude spectrum is a graph of $|C_n|$ on the y-axis versus frequency $f = n f_o$ on the x-axis.

The phase spectrum of $x(t)$ is denoted by,

$$\phi_n = \arg(C_n) = \tan^{-1}\left[\frac{\text{Imaginary part of } C_n}{\text{Real part of } C_n}\right] \quad ...(14)$$

The phase spectrum is a graph of ϕ_n on the y-axis versus frequency $f = n f_o$ on the x-axis.

The amplitude spectrum is a symmetric or even function. That means $|C_n| = |C_{-n}|$. But the phase spectrum is an asymmetric or odd function. That means $\arg(C_n) = -\arg(C_{-n})$.

Examples on Fourier series :

Prob. 1 : Obtain the quadrature fourier series for the rectangular pulse train shown in Fig. T-2.

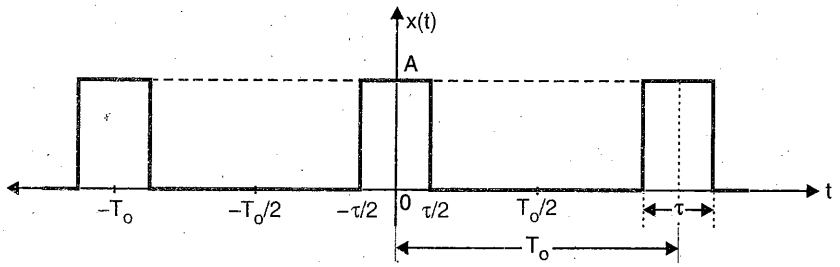


Fig. T-2

Soln. : Let us obtain the quadrature fourier series for the given rectangular pulse. The quadrature fourier series is given by the following expression :

$$x(t) = a_o + \sum_{n=1}^{\infty} a_n \cos(n\omega_o t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_o t) \quad ...(1)$$

Substituting $\omega_o = 2\pi f_o = \frac{2\pi}{T_o}$ into Equation (1) we get,

$$x(t) = a_o + \sum_{n=1}^{\infty} a_n \cos\left[\frac{2\pi nt}{T_o}\right] + \sum_{n=1}^{\infty} b_n \sin\left[\frac{2\pi nt}{T_o}\right] \quad ...(2)$$

To find the fourier coefficients, we must consider one complete cycle of $x(t)$ for integration. Here we will consider one cycle from $t = -\frac{T_o}{2}$ to $t = \frac{T_o}{2}$. Let us obtain the fourier coefficients now.

(i) To obtain the value of a_o :

$$a_o = \frac{1}{T_o}\int_{-T_o/2}^{T_o/2} x(t)\,dt$$

$$= \frac{1}{T_o}\int_{-\tau/2}^{\tau/2} x(t)\,dt \qquad \text{.....As } x(t) \text{ exists from } -\tau/2 \text{ to } \tau/2 \text{ only}$$

$$= \frac{1}{T_o}\int_{-\tau/2}^{\tau/2} A\,dt \qquad \text{.....As } x(t) = A \text{ from } -\tau/2 \text{ to } \tau/2$$

$$\therefore \quad a_o = \frac{A\tau}{T_o} \qquad \text{...(3)}$$

(ii) To obtain the value of a_n :

a_n is given as,

$$a_n = \frac{2}{T_o}\int_{-T_o/2}^{T_o/2} x(t)\cdot\cos\left[\frac{2\pi nt}{T_o}\right]dt = \frac{2}{T_o}\int_{-\tau/2}^{\tau/2} A\cos\left[\frac{2\pi nt}{T_o}\right]dt$$

$$= \frac{2A}{T_o}\cdot\frac{1}{\frac{2\pi n}{T_o}}\left[\sin\left(\frac{2\pi nt}{T_o}\right)\right]_{-\tau/2}^{\tau/2} = \frac{A}{\pi n}\left[\sin\left(\frac{2\pi n\tau}{2T_o}\right) - \sin\left(\frac{-2\pi n\tau}{2T_o}\right)\right]$$

$$= \frac{A}{\pi n}\left[\sin\left(\frac{\pi n\tau}{T_o}\right) + \sin\left(\frac{\pi n\tau}{T_o}\right)\right]$$

$$\therefore \quad a_n = \frac{2A}{\pi n}\sin\left[\frac{\pi n\tau}{T_o}\right] \qquad \text{...(4)}$$

(iii) To obtain the value of b_n :

b_n is given as,

$$b_n = \frac{2}{T_o}\int_{-T_o/2}^{T_o/2} x(t)\sin\left[\frac{2\pi nt}{T_o}\right]dt = \frac{2}{T_o}\int_{-\tau/2}^{\tau/2} A\sin\left[\frac{2\pi nt}{T_o}\right]dt$$

$$= \frac{-2A}{T_o}\cdot\frac{1}{\frac{2\pi n}{T_o}}\left[\cos\left(\frac{2\pi nt}{T_o}\right)\right]_{-\tau/2}^{\tau/2}$$

$$= \frac{-A}{\pi n}\left[\cos\left(\frac{2\pi n\tau}{2T_o}\right) - \cos\left(\frac{-2\pi n\tau}{2T_o}\right)\right]$$

$$\therefore \quad b_n = \frac{-A}{\pi n}\left[\cos\left(\frac{\pi n\tau}{T_o}\right) - \cos\left(\frac{\pi n\tau}{T_o}\right)\right]$$

$$\therefore \quad b_n = 0 \qquad \ldots(5)$$

(iv) Substituting the values of a_o, a_n and b_n in the Equation (2) we get the quadrature fourier series for $x(t)$ as,

$$x(t) = \frac{A\tau}{T_o} + \sum_{n=1}^{\infty} \frac{2A}{n\pi} \sin\left[\frac{n\pi\tau}{T_o}\right] \cos\left[\frac{2\pi nt}{T_o}\right] \qquad \ldots\text{Ans.}$$

Prob. 2 : Obtain the exponential fourier series for the rectangular pulse train shown in Fig. T-3 and sketch the spectrum.

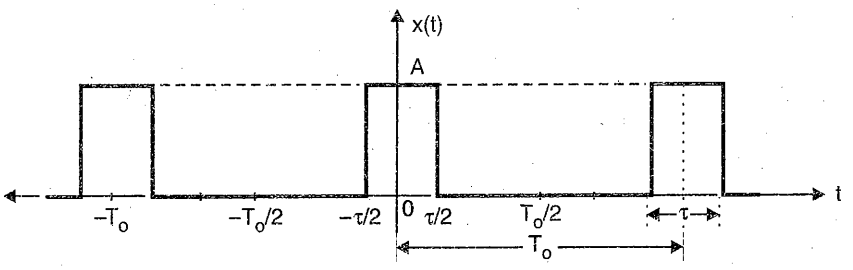


Fig. T-3 : A rectangular pulse train

Soln. : The exponential fourier series is given as,

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi nt/T_o} \qquad \ldots(1)$$

where, $$C_n = \frac{1}{T_o}\int_{t}^{t+T_o} x(t)\cdot e^{-j2\pi nt/T_o}\, dt \qquad \ldots(2)$$

Let us consider one cycle of $x(t)$ extending from $t = \frac{-T_o}{2}$ to $t = \frac{T_o}{2}$.

(i) To obtain the value of C_n :

$$C_n = \frac{1}{T_o}\int_{-T_o/2}^{T_o/2} x(t)\cdot e^{-j2\pi nt/T_o}\, dt$$

But $x(t) = A$, from $t = -\tau/2$ to $t = \tau/2$

$$\therefore \quad C_n = \frac{1}{T_o}\int_{-\tau/2}^{\tau/2} A\, e^{-j2\pi nt/T_o}\, dt = \frac{-A}{j\frac{2\pi n}{T_o}\times T_o}\left[e^{-j2\pi nt/T_o}\right]_{-\tau/2}^{\tau/2}$$

$$= \frac{-A}{j2\pi n}\left[e^{-j2\pi n\tau/2T_o} - e^{j2\pi n\tau/2T_o}\right] = \frac{A}{\pi n}\left[\frac{e^{j\pi n\tau/T_o} - e^{-j\pi n\tau/T_o}}{2j}\right]$$

$$\therefore \quad C_n = \frac{A}{\pi n}\sin\left(\frac{\pi n\tau}{T_o}\right) \qquad ...(3)$$

Multiply and divide the RHS of Equation (3) by (τ/T_o) we get,

$$C_n = \frac{A(\tau/T_o)}{(\pi n\tau/T_o)}\sin(\pi n\tau/T_o)$$

$$\therefore \quad C_n = A\left(\frac{\tau}{T_o}\right)\text{sinc}\left[\frac{n\tau}{T_o}\right] \qquad ...(4)$$

(ii) To obtain the fourier series :

Substitute the value of C_n into Equation (1) to obtain the exponential fourier series as,

$$x(t) = \sum_{n=-\infty}^{\infty}\frac{\tau A}{T_o}\text{sinc}\left[\frac{n\tau}{T_o}\right]e^{j2\pi nt/T_o} \qquad ...(5)$$

(iii) Spectrum of the signal $x(t)$:

From the value of C_n in Equation (4), it is clear that C_n does not have any imaginary part. Therefore the amplitude spectrum of $x(t)$ is given as,

$$|C_n| = C_n = \frac{A\tau}{T_o}\text{sinc}\left[\frac{n\tau}{T_o}\right] \qquad ...(6)$$

Thus the amplitude spectrum of a rectangular pulse of duration τ is a sinc function. The spectrum is as shown in Fig. T-4.

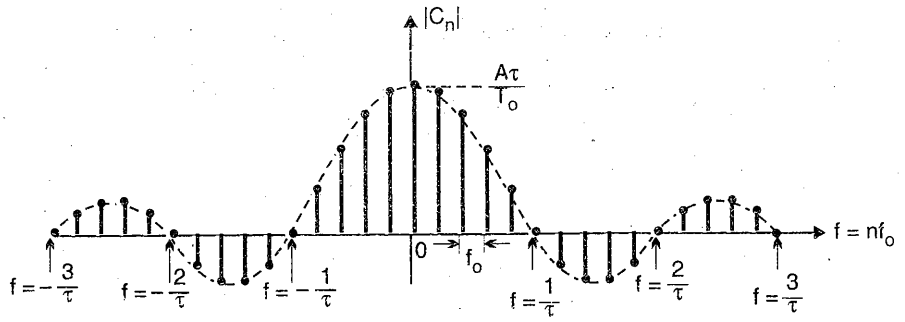


Fig. T-4 : Amplitude spectrum of a rectangular pulse

The imaginary part of C_n is zero therefore the phase spectrum is zero for all the values of f. The phase spectrum is shown in Fig. T-5.

Important points related to the amplitude and phase spectrums of Fig. T-5 are :

(1) The spacing between the adjacent spectral components is f_o.

(2) The shape of the envelope of amplitude spectrum $|C_n|$ is determined by A, τ and f_o. The envelope is "sinc" shaped.

(3) Zero crossings occur in the envelope of the amplitude spectrum at frequencies of $f = \pm 1/\tau, \pm 2/\tau, \pm 3/\tau$ ...etc.

(4) The phase spectrum takes on values of 0° corresponding to the positive values of $|C_n|$. However it takes on values of $\pm 180^\circ$ corresponding to the negative values of $|C_n|$ e.g. between $f = \pm 1/\tau$ to $\pm 2/\tau$. The negative values in the amplitude spectrum are made positive by assigning phase shifts of $\pm 180^\circ$.

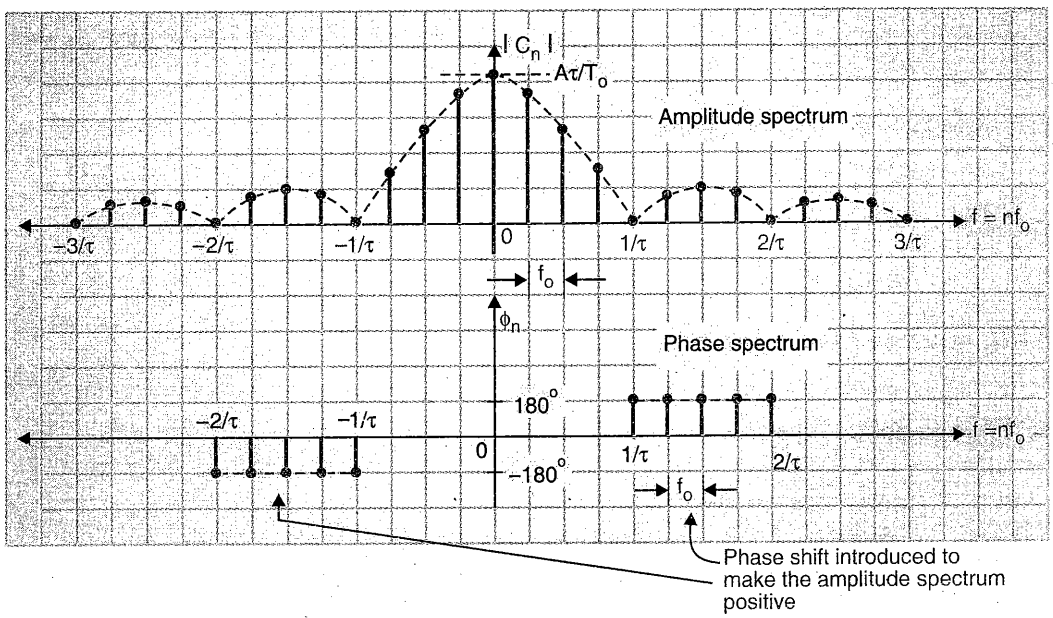


Fig. T-5 : Amplitude and phase spectrum of a periodic pulse train

(5) Note that the choice of phase shift $+180^\circ$ or -180° is arbitrary. However we have used both of them to preserve the "antisymmetry" of the phase spectrum.

Prob. 3 : A train of rectangular pulses making excursions from 0 to 10 volts has a duration of 50 msec and are separated by intervals of 500 msec. Assuming that the center of a pulse is located at t = 0, obtain the fourier series of the above signal and sketch the spectrum.

Soln. :

It is given that : $T_o = 500$ msec.

τ = 50 msec. A = 10 volts, $\frac{\tau}{T_o} = 0.1$

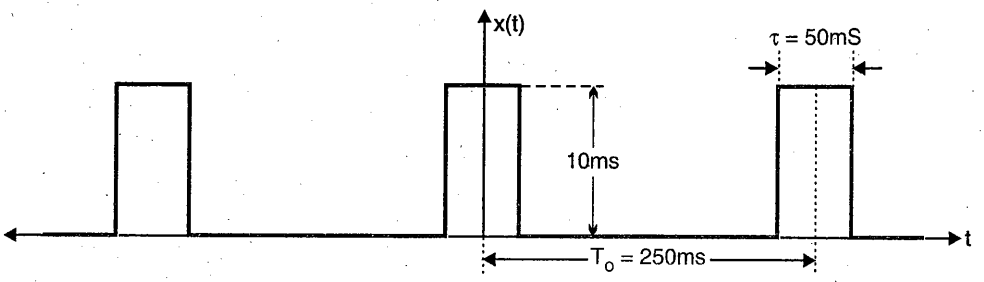


Fig. T-6 : Train of rectangular pulses

(i) To obtain the value of C_n :

We have already obtained the expression of C_n in the Prob. 2. It is as follows :

$$C_n = \frac{A}{n\pi} \sin\left[\frac{n\pi\tau}{T_o}\right] \quad ...(1)$$

$$\text{OR} \quad C_n = \frac{\tau A}{T_o} \text{sinc}\left[\frac{n\tau}{T_o}\right] \quad ...(2)$$

and the amplitude spectrum is given as,

$$|C_n| = C_n \quad ...(3)$$

(ii) How to plot the amplitude spectrum ?

Substitute n = 0 in Equation (2) to get,

$$C_o = \frac{\tau A}{T_o} \text{ sinc } (0).$$

But we know that $\frac{\tau}{T_o} = 0.1$ and A = 10 and sinc (0) = 1

$$\therefore \quad C_o = 1 \quad ...(4)$$

The other values of $|C_n|$ are found using Equation (1), as shown in the Table T-1.

$$|C_n| = \frac{A}{n\pi} \sin\left[\frac{n\pi\tau}{T_o}\right] = \frac{10}{n\pi} \sin\left[\frac{n\pi}{10}\right] \quad ...(5)$$

Using the Equation (5), we can calculate the amplitude response as shown in Table T-1.

Table : T-1 : Amplitude and phase spectrums for positive value of "n"

Sr. No.	n	$f = n f_o = \frac{n}{T_o}$	Amplitude spectrum $\lvert C_n \rvert = \frac{10}{n\pi} \sin\left[\frac{n\pi}{10}\right]$	Phase spectrum $\phi_n = \arg C_n$
1.	1	$f_o = 1/T_o$	$\lvert C_1 \rvert = 0.9836$	0
2.	2	$2 f_o = 2/T_o$	$\lvert C_2 \rvert = 0.9354$	0
3.	3	$3 f_o = 3/T_o$	$\lvert C_3 \rvert = 0.8584$	0
4.	4	$4 f_o = 4/T_o$	$\lvert C_4 \rvert = 0.7568$	0
5.	5	$5 f_o = 5/T_o$	$\lvert C_5 \rvert = 0.6366$	0
6.	6	$6 f_o = 6/T_o$	$\lvert C_6 \rvert = 0.5045$	0
7.	7	$7 f_o = 7/T_o$	$\lvert C_7 \rvert = 0.3678$	0
8.	8	$8 f_o = 8/T_o$	$\lvert C_8 \rvert = 0.2338$	0
9.	9	$9 f_o = 9/T_o$	$\lvert C_9 \rvert = 0.1092$	0
10.	10	$10 f_o = 10/T_o$	$\lvert C_{10} \rvert = 0$	0

To obtain the values of $\lvert C_n \rvert$ for negative values of "n", substitute negative "n" in Equation (5).

$$\therefore \quad \lvert C_{-n} \rvert = \frac{10}{-n\pi} \sin\left[\frac{-n\pi}{10}\right]$$

But, $\sin(-\theta) = -\sin\theta.$

$$\therefore \quad \lvert C_{-n} \rvert = \frac{10}{n\pi} \sin\left[\frac{n\pi}{10}\right] = \lvert C_n \rvert \qquad \ldots(6)$$

Thus the values of amplitude spectrum for various negative values of "n" will be same as that for the corresponding positive values. The amplitude and phase spectrums are as shown in Fig. T-7.

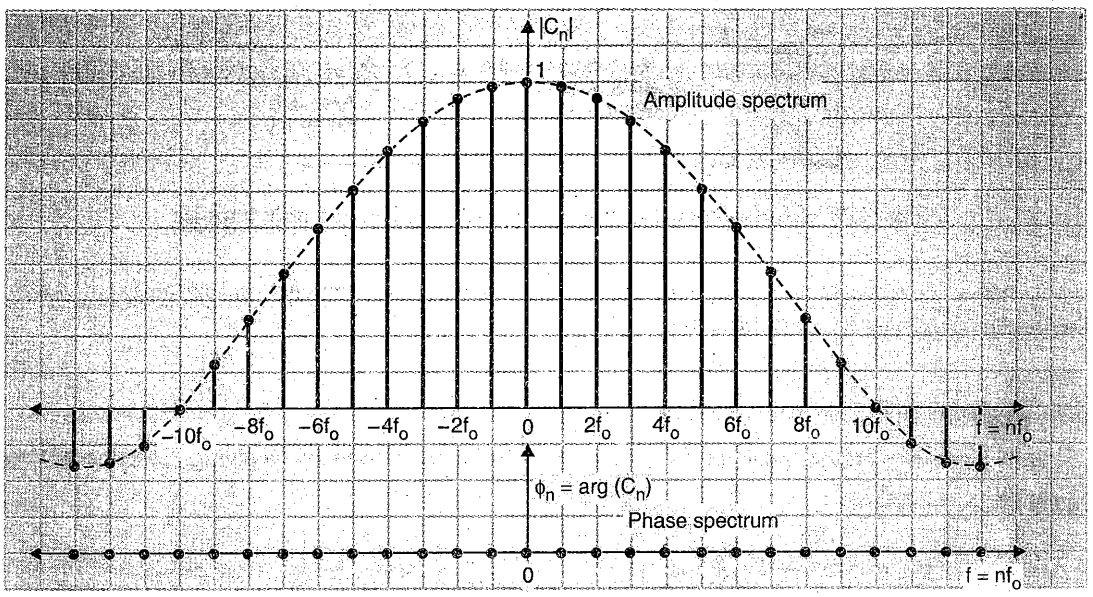


Fig. T-7 : Amplitude and phase spectrums

Prob. 4 : Obtain the fourier series of the unit impulse train shown in Fig. T-8. Also plot the amplitude and phase spectrums for the same.

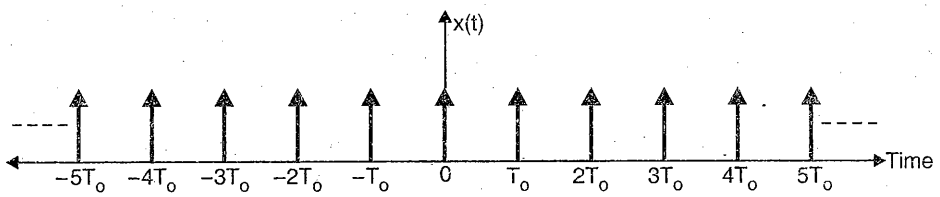


Fig. T-8 : Unit impulse train

Soln. :

(i) Value of C_n for a train of rectangular pulses :

In the Prob. 2, we have already obtained the exponential fourier series for a train of rectangular pulses of duration τ and period T_o. Its value is,

$$C_n = \frac{A\tau}{T_o} \text{sinc}\left[\frac{n\tau}{T_o}\right] \quad ...(1)$$

We will use it to find the value of C_n for an impulse train.

(ii) Equivalence between a rectangular pulse and an impulse :

The rectangular pulse of amplitude A and width τ is as shown in Fig. T-9.

$\therefore$ Area under a rectangular pulse = $A\tau$ ...(2)

As we know, area under an impulse = 1 ...(3)

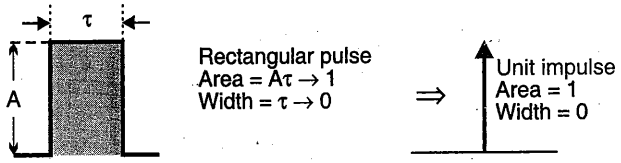


Fig. T-9 : Equivalence between rectangular pulse and unit impulse

Hence a rectangular pulse will be equivalent to a unit impulse as its area i.e. "Aτ" approaches unity and as its width τ approaches zero.

$$\therefore \quad A\tau \to 1 \text{ and } \tau \to 0 \qquad ...(4)$$

(iii) To obtain C_n for the unit impulse train :

We can obtain the value of "C_n" for the unit impulse train by applying Equation (4) to Equation (1) as follows :

$$\therefore \quad C_n = \lim_{\substack{A\tau \to 1 \\ \tau \to 0}} \frac{A\tau}{T_o} \text{sinc}\left[\frac{n\tau}{T_o}\right] = \lim_{\tau \to 0} \frac{1}{T_o} \text{sinc}\left[\frac{n\tau}{T_o}\right] \quad \text{.....as } A\tau = 1$$

$$\text{But} \quad \lim_{\tau \to 0} \text{sinc}\left[\frac{n\tau}{T_o}\right] = \text{sinc}\,(0) = 1.$$

$$\therefore \quad C_n = \frac{1}{T_o} \qquad ...(5)$$

This is the value of C_n for the unit impulse train.

(iv) To obtain the exponential fourier series :

Substitute the value of "C_n" from Equation (5) into the standard expression of exponential fourier series.

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi nt/T_o}$$

$$\therefore \quad x(t) = \frac{1}{T_o} \sum_{n=-\infty}^{\infty} e^{j2\pi nt/T_o} \qquad ...(6)$$

This is the required fourier series for the unit impulse train.

Amplitude spectrum :

$$\text{Amplitude spectrum : } |C_n| = C_n = \frac{1}{T_o} \qquad ...(7)$$

This means that for every value of "n" the value of C_n is going to be the same, equal to $(1/T_o)$. The amplitude spectrum also is train of impulses each having amplitude of $(1/T_o)$, as shown in Fig. T-10.

Phase spectrum :

The phase spectrum $\phi_n = \arg(C_n) = 0$ as C_n is constant. The phase spectrum is as shown in Fig. T-10.

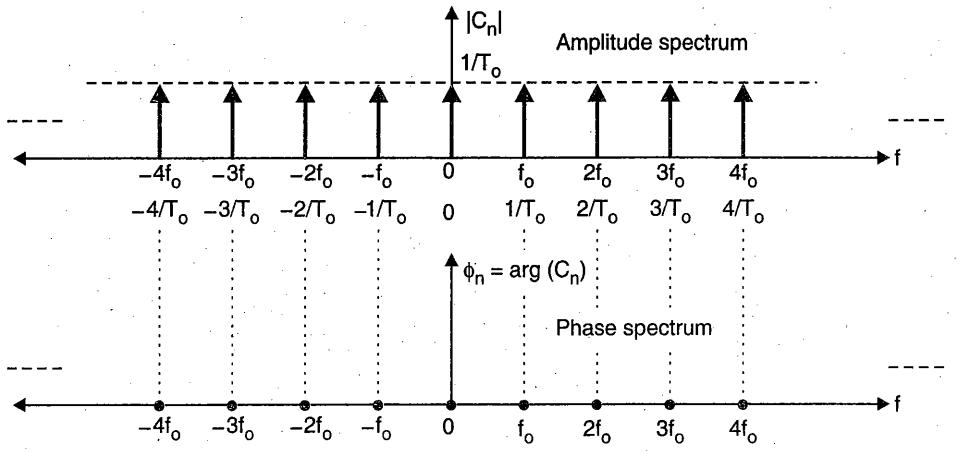


Fig. T-10 : Amplitude and phase spectrums of the unit impulse train

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Prob. 5 : Obtain the fourier series of the sawtooth waveform shown in Fig. T-11 and plot its spectrum.

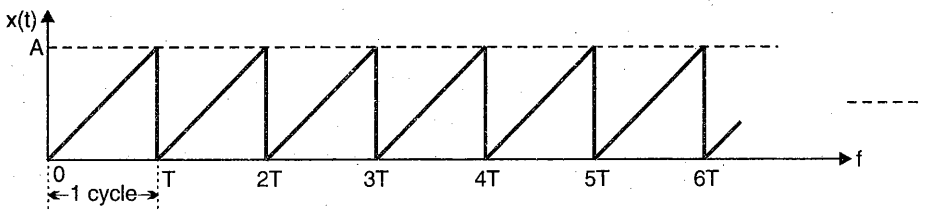


Fig. T-11 : A sawtooth waveform

Soln. :

(i) Representation of the signal :

The sawtooth signal x (t) shown in Fig. T-11 can be represented over one cycle as,

$$x(t) = \frac{A}{T} t \qquad \text{for } 0 < t < T \qquad ...(1)$$

and one cycle period $T_o = T$.

(ii) To find C_n :

In order to represent the signal in the form of an exponential fourier series, let us find "C_n" using the following equation :

$$C_n = \frac{1}{T_o} \int_{T}^{t+T_o} x(t) \cdot e^{-j2\pi nt/T_o} dt \qquad ...(2)$$

Substitute $T_o = T$ and $x(t) = \frac{A}{T} t$ and integrate from t = 0 to t = T.

$$\therefore \quad C_n = \frac{1}{T} \int_{0}^{T} \frac{At}{T} e^{-j2\pi nt/T} dt \qquad ...(3)$$

At this point, let us use a standard identity which states that,

$$\int x e^{ax} dx = e^{ax} \left[\frac{x}{a} - \frac{1}{a^2} \right] \qquad ...(4)$$

Let : x = t and $a = -j2\pi n/T$. Then "C_n" can be expressed as,

$$C_n = \frac{A}{T^2} \int_{0}^{T} t e^{-j2\pi nt/T} dt$$

Using Equation (4) we get,

$$= \frac{A}{T^2} \cdot \left[e^{at} \left(\frac{t}{a} - \frac{1}{a^2} \right) \right]_0^T$$

$$\therefore \quad C_n = \frac{A}{T^2} \left[e^{aT} \left(\frac{T}{a} - \frac{1}{a^2} \right) - e^{0} \left(0 - \frac{1}{a^2} \right) \right] = \frac{A}{T^2} \left[e^{aT} \left(\frac{T}{a} - \frac{1}{a^2} \right) + \frac{1}{a^2} \right]$$

$$\therefore \quad C_n = \frac{A}{T^2} \left[\frac{e^{aT}(aT-1)+1}{a^2} \right] \qquad ...(5)$$

$$= \frac{A}{a^2 T^2} [e^{aT}(aT-1)+1]$$

Now substitute : $a = -j2\pi n/T$

$$\therefore \quad a^2 = j^2 4\pi^2 n^2/T^2 = -4\pi^2 n^2/T^2$$

and $aT = -j2\pi n$ in Equation (5) to get,

$$C_n = \frac{A}{-4\pi^2 n^2} \left[e^{-j2\pi n}(-j2\pi n - 1) + 1 \right]$$

But $e^{-j2\pi n} = \cos(2\pi n) - j\sin(2\pi n) = 1$

Substituting we get, $C_n = \frac{A}{-4\pi^2 n^2} [-j2\pi n - 1 + 1] = \frac{j2\pi nA}{4\pi^2 n^2}$

$$\therefore \quad C_n = \frac{jA}{2\pi n} \qquad [\text{for } n \neq 0] \qquad ...(6)$$

At n = 0, let us find the value of C_o.

$$C_o = \frac{1}{T}\int_0^T x(t)\, e^0\, dt = \frac{1}{T}\int_0^T \frac{A}{T}\, t\, dt = \frac{A}{T^2} \times \left[\frac{t^2}{2}\right]_0^T$$

$$= \frac{A}{2T^2} \times T^2 \qquad \therefore \quad C_o = \frac{A}{2} \qquad \ldots(6a)$$

(iii) To obtain the exponential fourier series :

The exponential fourier series is given by,

$$x(t) = \sum_{n=-\infty}^{\infty} C_n\, e^{j 2\pi nt/T}$$

$$\therefore \quad x(t) = \sum_{n=-\infty}^{\infty} \frac{jA}{2\pi n} \cdot e^{j 2\pi nt/T} \qquad \ldots(7)$$

(iv) Amplitude spectrum :

$$\text{Amplitude spectrum :} \quad |C_n| = \left[0 + \frac{A^2}{4\pi n^2}\right]^{1/2} = \frac{A}{2\sqrt{\pi}\, n} \qquad \ldots(8)$$

The amplitude spectrum is as shown in Fig. T-12.

(v) Phase spectrum :

$$\text{Phase spectrum :} \quad \phi_n = \arg(C_n) = \tan^{-1}\left[\frac{A/2\sqrt{\pi}\, n}{0}\right] = \pm 90^\circ$$

$$\therefore \quad \phi_n = +90^\circ \text{ for } n > 0 \qquad \text{and} \qquad \phi_n = -90^\circ \text{ for } n < 0$$

The phase spectrum is as shown in Fig. T-12.

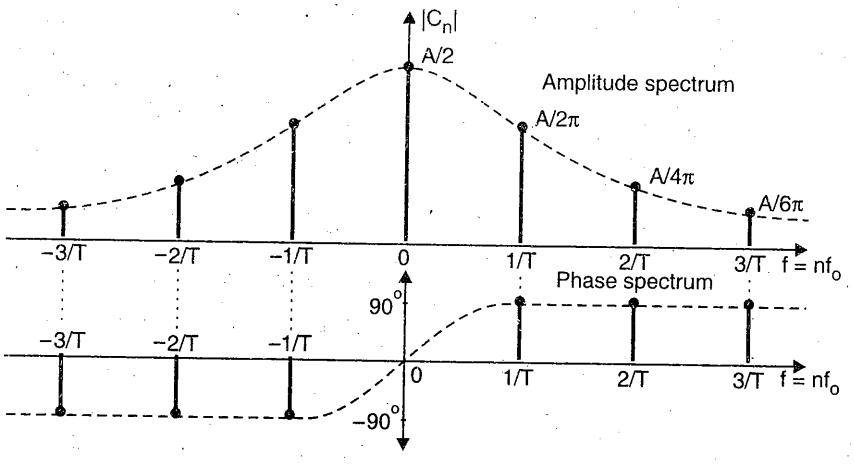


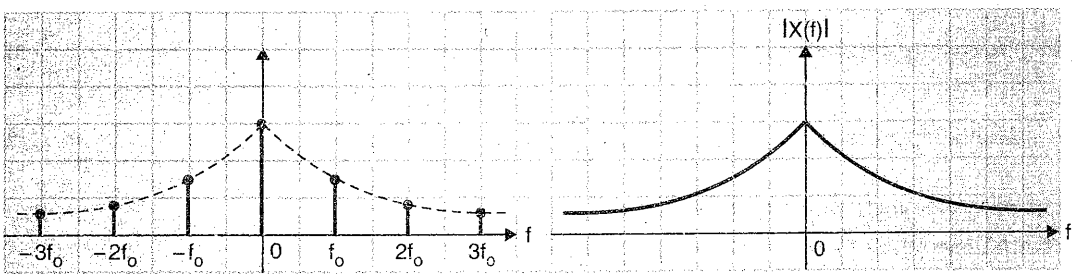
Fig. T-12 : Amplitude and phase spectrums

1.2.2 Fourier Transform for Continuous Time Aperiodic Signals :

Till now we have seen how to represent the periodic signals extended over the interval $(-\infty, \infty)$, using the fourier series. Non-periodic time limited signal can also be represented by the fourier series.

However the non-periodic signals which extend from $-\infty$ to ∞ can be represented more conveniently using the "Fourier Transform" in the frequency domain.

It is possible to find the fourier transform of periodic signal for $T_0 \to \infty$. Hence the frequency $f_0 = \frac{1}{T_0} \to 0$. Therefore the difference between the spectral components which is f_0 (as seen in the line spectrum) becomes extremely small and they come very close to each other. Due to this the frequency spectrum appears to be continuous as shown in Fig. T-13(a) and (b).



(a) Line spectrum showing vertical spectral lines at f_0 , $2f_0$ ..... **(b) Continuous spectrum as $f_0 \to 0$**

Fig. T-13

1.2.2.1 What does a Fourier Transform do ?

Any signal is build up by addition of elementary signals which are at different frequencies, have different amplitudes and relative phases. Using the fourier transform we can plot the amplitude and phase spectrums which provide us all the information about amplitudes and relative phases of such elementary signals. Thus fourier transform can be used for the "analysis" of a signal. It is used for transformation from the time domain to frequency domain.

1.2.2.2 Definition of Fourier Transform :

The fourier transform of a signal $x(t)$ is defined as follows :

$$\text{Fourier transform :} \quad X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\text{OR} \quad X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt \qquad \ldots(15)$$

These equations are known as "analysis" equations.

1.2.2.3 Definition of Inverse Fourier Transform :

The signal $x(t)$ can be obtained back from fourier transform $X(f)$ by using the inverse fourier transform. The inverse fourier transform (IFT) is defined as follows :

$$\text{Inverse fourier transform : } x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$\text{OR} \quad x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\, df \qquad \ldots(16)$$

Representation :

The signal $x(t)$ and its fourier transform $X(f)$ form a fourier transform pair which can be represented as,

$$x(t) \xleftrightarrow{F} X(f) \qquad \ldots(17)$$

The other way to represent is as follows :

$$X(f) = F[x(t)] \qquad \ldots(18)$$

or it can be represented as,

$$x(t) = F^{-1}[X(f)] \qquad \ldots(19)$$

The fourier transform is a complex function of frequency f. Therefore it is possible to express it in the complex exponential form as follows :

$$X(f) = |X(f)| \cdot e^{j\theta(f)} \qquad \ldots(20)$$

In this expression :

$|X(f)|$ = The amplitude spectrum of $x(t)$ and

$\theta(f)$ = The phase spectrum.

The amplitude spectrum is a graph of amplitude versus frequency. Whereas the phase spectrum is a graph of phase angle versus frequency.

1.2.2.4 Conditions for the Existence of Fourier Transform :

We have studied the **Dirichletconditions** in section 1.2.1. Similarly these conditions should be satisfied by a signal $x(t)$, then only it is possible to obtain the fourier transform of $x(t)$. For the periodic signals the integration is obtained over one period however for the non-periodic signals, it will be obtained over a range $-\infty$ to ∞.

The signal $x(t)$ will have to satisfy the following conditions so that it's fourier transform can be obtained :

(i) The function $x(t)$ should be single valued in any finite time interval T.

(ii) It should have a finite number of discontinuities in any finite interval T.

(iii) The function x (t) should have a finite number of maxima and minima in any finite interval of time T.

(iv) The function x (t) should be an absolutely integrable function.

$$\text{That means } \int_{-\infty}^{\infty} |x(t)|\, dt < \infty \qquad \ldots(21)$$

The conditions stated above are sufficient conditions, but they are not the necessary conditions.

Amplitude and phase spectrums :

1. The amplitude and phase spectrums are continuous rather than being discrete in nature. Out of them, the amplitude spectrum of a real valued function x (t) exhibits an even symmetry.

$$\therefore \quad X(f) = X(-f) \qquad \ldots(22)$$

2. And the phase spectrum has an odd symmetry. That means,

$$\theta(f) = -\theta(-f) \qquad \ldots(23)$$

3. The two Equations (22) and (23) are due to the fact that for a real valued function x (t) its fourier transform is given by,

$$X(f) = X^*(-f) \qquad \ldots(24)$$

Proof :

Step 1 : Fourier transform of a real valued function x (t) is given by,

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft}\, dt$$

Step 2 : Take the complex conjugate of both sides,

$$X^*(f) = \left[\int_{-\infty}^{\infty} x(t) e^{-j2\pi ft}\, dt \right]^*$$

$$\therefore \quad X^*(f) = \int_{-\infty}^{\infty} x^*(t) e^{+j2\pi ft}\, dt \qquad \ldots(25)$$

Step 3 : Substitute $-f$ in place of f in Equation (25) to get,

$$X^*(-f) = \int_{-\infty}^{\infty} x^*(t) e^{-j2\pi ft}\, dt$$

But $x^*(t) = x(t)$ as the function x (t) is real valued.

$$\therefore \quad X^*(-f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j2\pi ft} dt$$

$$\therefore \quad X^*(-f) = X(f) \qquad \text{...Proved}$$

Looking at the Equations (22) and (23) we can make the following statements about the spectrums of a real valued signal.

1. The amplitude spectrum $|X(f)|$ of a real valued signal $x(t)$, is an even function of frequency. That means the amplitude spectrum is symmetric about the vertical axis.
2. The phase spectrum $\theta(f)$ is an odd function of frequency. That means the phase spectrum is antisymmetric about the Y-axis.

Properties of Fourier transform :

Following table shows summary of properties of fourier transform.

Table T-2 : Fourier transform properties

Sr. No.	Property	Mathematical expression
1	Linearity or superposition	$[a_1 x_1(t) + a_2 x_2(t)] \leftrightarrow [a_1 X_1(f) + a_2 X_2(f)]$ a_1 and a_2 are constants.
2	Time scaling	$x(\alpha t) \xleftrightarrow{F} \frac{1}{\lvert\alpha\rvert} X(f/\alpha)$. α is constant.
3	Duality or symmetry	If $x(t) \xleftrightarrow{F} X(-f)$ then $X(t) \xleftrightarrow{F} x(-f)$
4	Time shifting	$x(t-t_d) \xleftrightarrow{F} e^{-j2\pi f t_d} X(f)$
5	Area under $x(t)$	$\int_{-\infty}^{\infty} x(t)\,dt = X(0)$
6	Area under $X(f)$	$\int_{-\infty}^{\infty} X(f)\,df = x(0)$
7	Frequency shifting	$e^{j2\pi f_c t} x(t) \xleftrightarrow{F} X(f-f_c)$
8	Differentiation in time domain	$\frac{d}{dt} x(t) \xleftrightarrow{F} j2\pi f X(f)$
9	Integration in time domain	$\int_{-\infty}^{t} x(\lambda)\,d\lambda \xleftrightarrow{F} \frac{1}{j2\pi f} X(f)$

Sr. No.	Property	Mathematical expression
10	Conjugate functions	If $x(t) \overset{F}{\longleftrightarrow} X(f)$ then $x^*(t) \overset{F}{\longleftrightarrow} X(-f)$
11	Multiplication in time domain	$x_1(t)\,x_2(t) \overset{F}{\longleftrightarrow} \int_{-\infty}^{\infty} X_1(\lambda)\,X_2(f-\lambda)\,d\lambda$
12	Convolution in time domain	$x_1(t) * x_2(t) \overset{F}{\longleftrightarrow} X_1(f)\,X_2(f)$

1.2.2.5 Examples on Fourier Transform :

Prob. 6 : Obtain the fourier transform of the double exponential pulse shown in the Fig. T-14(a).

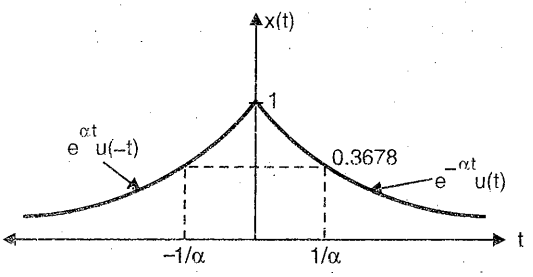


Fig. T-14(a) : Double exponential pulse

Soln. : The signal $x(t)$ is the sum of two exponential pulses

i.e. $$x(t) = x_1(t) + x_2(t) = e^{\alpha t}u(-t) + e^{-\alpha t}u(t)$$

Taking the fourier transform of both sides,

$$F[x(t)] = F[e^{\alpha t}u(-t)] + F[e^{-\alpha t}u(t)]$$

Substitute fourier transforms of the two terms on the RHS we get,

$$X(f) = \frac{1}{\alpha - j2\pi f} + \frac{1}{\alpha + j2\pi f}$$

$$= \frac{\alpha + j2\pi f + \alpha - j2\pi f}{\alpha^2 + (2\pi f)^2}$$

$$\therefore \quad X(f) = \frac{2\alpha}{\alpha^2 + (2\pi f)^2}$$

(i) The amplitude spectrum $$|X(f)| = \frac{2\alpha}{\alpha^2 + (2\pi f)^2}$$

(ii) The phase spectrum $$\theta(f) = 0.$$

The amplitude and phase spectrums are plotted as shown in Fig. T-14(b) and T-14(c).

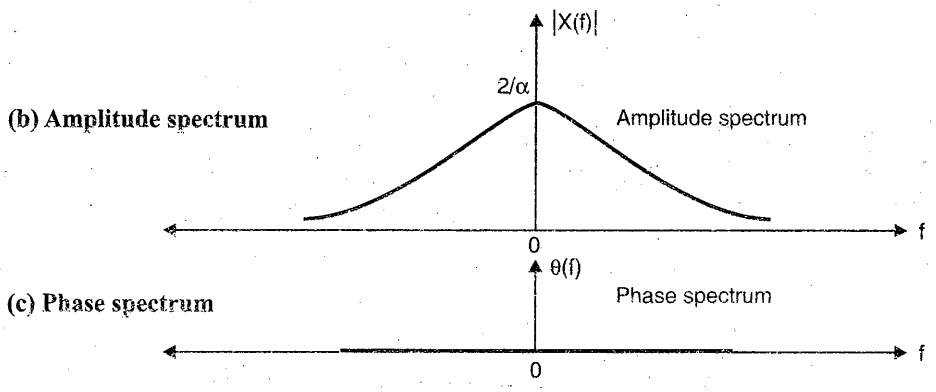


Fig. T-14

Prob. 7 : Obtain the fourier transform of the delta function shown in Fig. T-15(a).

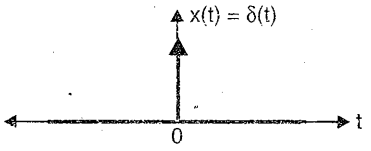


Fig. T-15(a) : Delta function

Soln. :

(i) By the definition of fourier transform,

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt = \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi ft}\, dt \qquad ...(1)$$

We cannot substitute the value of $\delta(t)$ directly in the Equation (1) because it is infinitely large at $t = 0$. Therefore let us use the shifting property of the delta function.

(ii) Shifting property of delta function :

The shifting property states that

$$\int_{-\infty}^{\infty} f(t)\, \delta(t - t_d)\, dt = f(t_d) \qquad ...(2)$$

Let us use this property in Equation (1) as follows :

(iii) In Equation (2) assume that $t_d = 0$ and $f(t) = e^{-j2\pi ft}$

$$\therefore \quad X(f) = \int_{-\infty}^{\infty} e^{-j2\pi ft} \cdot \delta(t - 0) \quad \therefore \text{ by using Equation (2).}$$

$$X(f) = e^{-j2\pi f t_d}, \qquad \text{but } t_d = 0$$

$$\therefore \quad X(f) = e^{-j2\pi t \cdot 0} = 1 \qquad ...(3)$$

$$\text{Thus } \delta(t) \xleftrightarrow{F} 1$$

The amplitude spectrum of the delta function is as shown in the Fig. T-15(b) This shows that the delta function contains all the frequencies from $-\infty$ to ∞ with equal amplitudes.

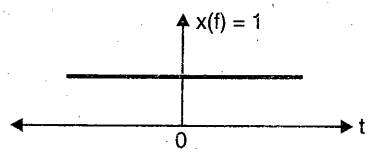


Fig. T-15(b) : Amplitude spectrum of delta function

Prob. 8 : Obtain the F.T. of the antisymmetrical pulse shown in the Fig. T-16.

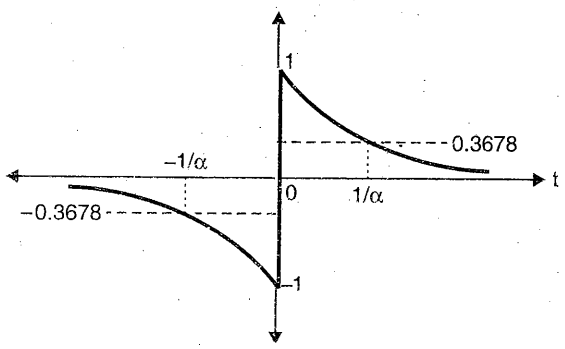


Fig. T-16

Soln. : The antisymmetric pulse can be represented as,

$$\begin{aligned} x(t) &= e^{-\alpha t} && t > 0 \\ &= |1| && t = 0 \\ &= -e^{\alpha t} && t < 0 \end{aligned}$$

Therefore the fourier transform is given by,

$$F[x(t)] = \int_{-\infty}^{0^-} -e^{\alpha t} e^{-j2\pi ft} dt + \int_{0^-}^{0^+} |1| e^{-j2\pi ft} dt + \int_{0^+}^{\infty} e^{-\alpha t} e^{-j2\pi ft} dt$$

$$= \int_{-\infty}^{0^-} -e^{(\alpha - j2\pi f)t} dt + \int_{0^+}^{\infty} e^{-(\alpha + j2\pi f)t} dt$$

$$= \frac{-1}{\alpha - j2\pi f}\left[e^{(\alpha - j2\pi f)t}\right]_{-\infty}^{0^-} + \frac{1}{-(\alpha + j2\pi f)}\left[e^{-(\alpha + j2\pi f)t}\right]_{0^+}^{\infty}$$

$$= \frac{-1}{(\alpha - j2\pi f)}\left[e^{0} - e^{-\infty}\right] - \frac{1}{(\alpha + j2\pi f)}\left[e^{-\infty} - e^{0}\right]$$

$$= \frac{-1}{(\alpha - j2\pi f)} + \frac{1}{(\alpha + j2\pi f)} = \frac{-\alpha - j2\pi f + \alpha - j2\pi f}{\alpha^2 + (2\pi f)^2}$$

$$\therefore \quad X(f) = \frac{-j4\pi f}{\alpha^2 + (2\pi f)^2}$$

Prob. 9 : Find the fourier transform of the signum function shown in Fig. T-17(a).

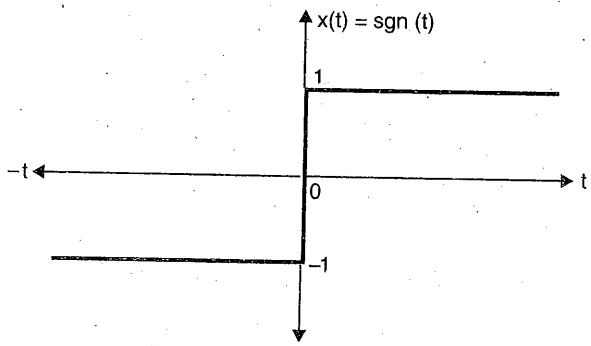


Fig. T-17(a) : Signum function

Soln. : In the Prob. 8, we have found FT of an antisymmetrical exponential pulse. The antisymmetrical pulse can be represented as,

$$x(t) = e^{-|\alpha| t} \operatorname{sgn}(t) \qquad ...(1)$$

Therefore as proved in Prob. 8, the FT of the antisymmetric exponential pulse is given by,

$$F\left[e^{-|\alpha| t} \operatorname{sgn}(t)\right] = \frac{-j4\pi f}{\alpha^2 + (2\pi f)^2} \qquad ...(2)$$

In the Equation (2), substitute $\alpha = 0$ to get,

$$F[e^{0} \operatorname{sgn}(t)] = \frac{-j4\pi f}{4\pi^2 f^2}$$

$$\therefore \quad F[\operatorname{sgn}(t)] = \frac{-j}{\pi f}$$

$$\therefore \text{ Signum function } \operatorname{sgn}(t) \overset{F}{\longleftrightarrow} \frac{-j}{\pi f}$$

...Ans.

Amplitude and phase spectrums :

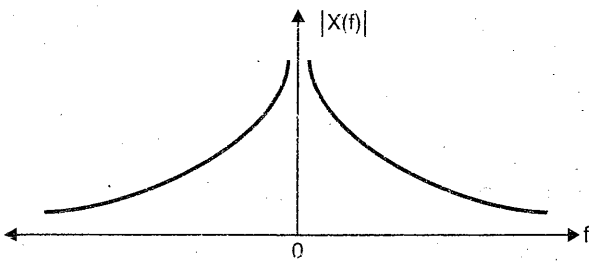
The amplitude spectrum is given by,

$$|X(f)| = \frac{1}{\pi f}$$

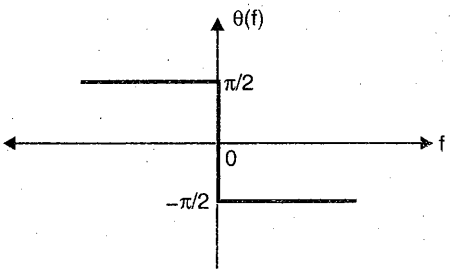
and the phase spectrum is given by,

$$\theta(f) = \tan^{-1}\left[\frac{-1/\pi f}{0}\right] = \begin{matrix} -\pi/2 & \text{for positive f} \\ +\pi/2 & \text{for negative f} \end{matrix}$$

The amplitude and phase spectrums of the signum function are shown in Figs. T-17(b) and (c) respectively.



(b) Amplitude spectrum of signum function



(c) Phase spectrum of signum function

Fig. T-17

Prob. 10 : Obtain the fourier transform of a unit step function.

Soln. : A unit step function is mathematically defined as,

$$u(t) = 1 \quad \text{for } t \geq 0$$
$$= 0 \quad \text{elsewhere}$$

Using the definition of the fourier transform we get,

$$F[u(t)] = \int_{-\infty}^{\infty} u(t) \cdot e^{-j2\pi ft}\, dt$$

But unit step function is present only for $t \geq 0$

$$\therefore \quad F[u(t)] = \int_{0}^{\infty} 1 \cdot e^{-j2\pi ft}\, dt = \frac{1}{-j2\pi f}\left[e^{-j2\pi ft}\right]_0^{\infty}$$

$$= \frac{1}{-j2\pi f}\left[e^{-\infty} - e^{0}\right] = \frac{1}{-j2\pi f}[0-1] = \frac{1}{j2\pi f}$$

$$\therefore \quad u(t) \overset{F}{\longleftrightarrow} \frac{1}{j2\pi f}$$ ...Ans.

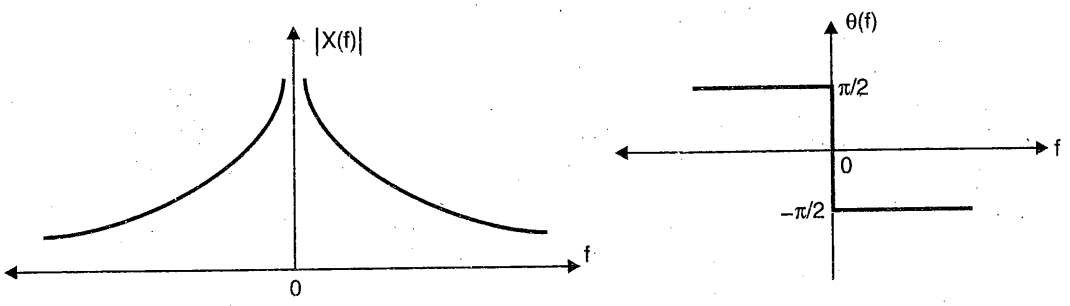
The amplitude spectrum of the unit step function is given by,

$$|X(f)| = \frac{1}{2\pi f}$$

and the phase spectrum is given by,

$$\theta(f) = \tan^{-1}\left[\frac{-1/2\pi f}{0}\right]$$

The amplitude spectrum of a unit step function is shown in Fig. T-18(a).



(a) Amplitude spectrum **(b) Phase spectrum**

Fig. T-18

Prob. 11 : Obtain the fourier transform of a rectangular pulse of duration T and amplitude A as shown in Fig. T-19(a).

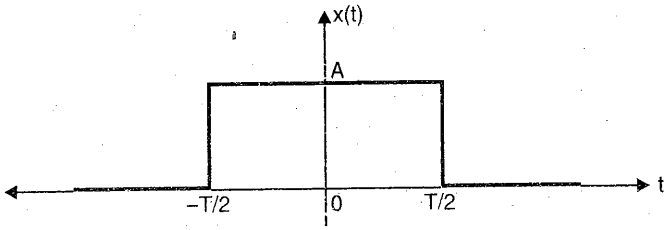


Fig. T-19(a) : Rectangular pulse

Soln. : The rectangular pulse shown in Fig. T-19(a) can be expressed mathematically as,

$$\text{rect}\,(t/T) = A \quad \text{for } -T/2 \le t \le T/2$$
$$= 0 \quad \text{elsewhere}$$

This is also known as the gate function.

Therefore the fourier transform will be,

$$F[x(t)] = X(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j2\pi ft}\, dt \qquad \text{...by definition of F.T.}$$

$$= \int_{-T/2}^{T/2} A\, e^{-j2\pi ft}\, dt = \frac{A}{-j2\pi f}\left[e^{-j2\pi ft} \right]_{-T/2}^{T/2}$$

$$= \frac{A}{-j2\pi f}\left[e^{-j\pi fT} - e^{j\pi fT} \right] = \frac{A}{j2\pi f}\left[e^{j\pi fT} - e^{-j\pi fT} \right]$$

$$= \frac{A}{\pi f}\left[\frac{e^{j\pi fT} - e^{-j\pi fT}}{2j} \right] \qquad \ldots(1)$$

As per the Euler's theorem,

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Applying this to Equation (1)

$$\text{we get,} \quad F[x(t)] = \frac{A}{\pi f}[\sin(\pi fT)] \qquad ...(2)$$

Multiply and divide the RHS of Equation (2) by T to get,

$$F[x(t)] = AT\frac{\sin(\pi fT)}{\pi fT} \qquad ...(3)$$

In the above equation,

$$\frac{\sin(\pi fT)}{\pi fT} = \text{sinc}(fT) \qquad \therefore \quad F[x(t)] = AT\,\text{sinc}(fT)$$

$$\therefore \quad A\,\text{rect}(t/T) \xleftrightarrow{F} AT\,\text{sinc}(fT)$$

Thus the rectangular pulse transforms into a sinc function.

Amplitude spectrum :

The amplitude spectrum of the rectangular function is as shown in Fig. T-19(b).

As we already know, sinc (0) = 1 $\qquad \therefore$ AT sinc (0) = AT

The sinc function will have zero value for the following values of "fT":

$$\text{sinc}(fT) = 0 \qquad \text{for } fT = \pm 1, \pm 2, \pm 3, ...$$

$$\text{i.e. for} \quad f = \pm\frac{1}{T}, \pm\frac{2}{T}, \pm\frac{3}{T}.....$$

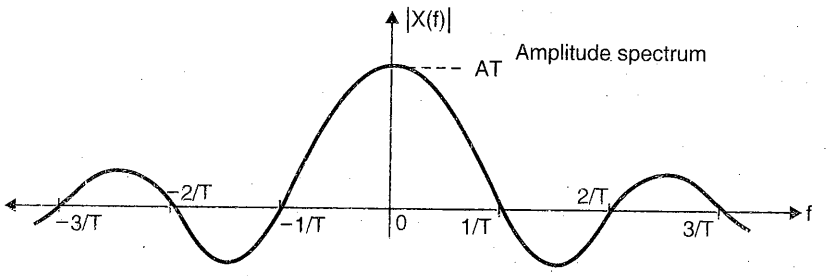


Fig. T-19(b) : Amplitude spectrum of a rectangular pulse

The phase spectrum has not been shown as it has zero value for all the values of f.

To absorb negative values of |X(f)| in the phase shift :

The negative amplitude of the amplitude spectrum |X(f)| can be made positive by introducing a phase shift of $\pm 180^{\circ}$ in the phase spectrum. This is as shown in Fig. T-19(c). A negative phase shift for positive frequency and positive phase shift for the negative frequency is introduced in order to maintain symmetry of the phase spectrum.

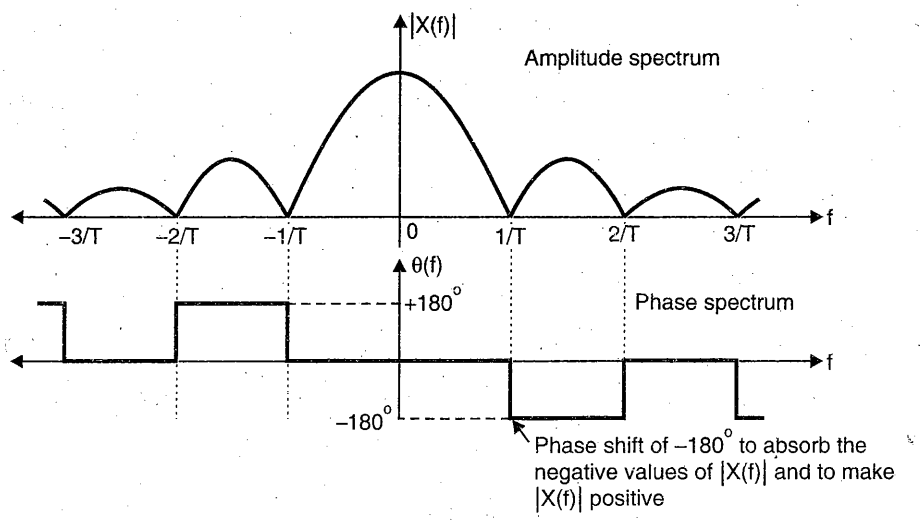


Fig. T-19(c) : Amplitude and phase spectrums for a rectangular pulse. Negative values of $|X(f)|$ have been absorbed in the additional phase shift of $\pm 180^\circ$ in the phase spectrum

Note : The choice of the phase shift of $\pm 180^\circ$ is completely arbitrary.

Prob. 12 : Obtain the fourier transform of a dc signal shown in Fig. T-20(a).

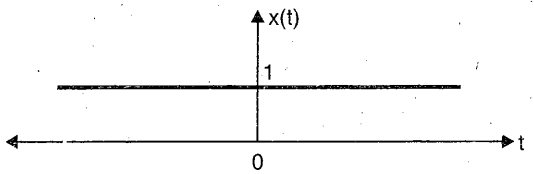


Fig. T-20(a) : A dc signal

Soln. : We have already obtained the fourier transform of the unit impulse or delta function $\delta(t)$.

$$\delta(t) \overset{F}{\longleftrightarrow} 1 \qquad ...(1)$$

To solve this example we are going to use the duality property.

It states that :

Duality property :

$$\text{If } x(t) \overset{F}{\longleftrightarrow} X(-f) \quad \text{then} \quad X(t) \overset{F}{\longleftrightarrow} x(-f).$$

From the Equation (1) it is clear that $X(f) = 1$

$$\text{i.e. } \delta(t) \overset{F}{\longleftrightarrow} X(f) \qquad ...(2)$$

Because $X(f) = 1$ $\quad X(-f)$ also is equal to 1

$$\therefore \quad \delta(t) \overset{F}{\longleftrightarrow} X(-f) \qquad ...(3)$$

Hence we can use the duality property as,

$$X(t) \overset{F}{\longleftrightarrow} x(-f) \qquad ...(4)$$

In Equation (4), $X(t) = 1$ and $x(-f) = \delta(-f)$

$$\therefore \quad 1 \overset{F}{\longleftrightarrow} \delta(-f) \qquad ...(5)$$

But as the delta function is a symmetrical (even) function

$$\delta(-f) = \delta(f)$$

$$\text{Therefore } 1 \overset{F}{\longleftrightarrow} \delta(f) \qquad ...\text{Ans.}$$

Meaning :

The above equation states that a dc signal of unity amplitude is transformed into a delta function in the frequency domain.

Important deduction :

The dc signal shown in Fig. T-20(a) is represented mathematically as,

$$x(t) = 1 \text{ for } -\infty < t < \infty \qquad ...(6)$$

Therefore its fourier transform is given as,

$$X(f) = \int_{-\infty}^{\infty} 1 \cdot e^{-j2\pi ft}\, dt \qquad ...(7)$$

But we have already obtained the fourier transform of a dc signal

$$\therefore \quad X(f) = \delta(f) \qquad ...(8)$$

Substituting this in Equation (7) we get,

$$\int_{-\infty}^{\infty} 1 \cdot e^{-j2\pi ft}\, dt = \delta(f) \qquad ...(9)$$

The spectrum of the dc signal is as shown in Fig. T-20(b).

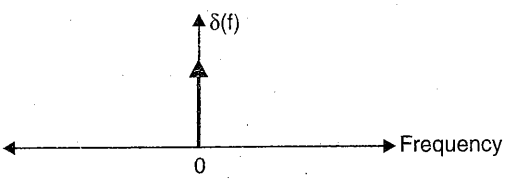


Fig. T-20(b) : Spectrum of a dc signal

Prob. 13 : Find the fourier transform of a cosine wave shown in Fig. T-21(a). Assume the frequency to be f_o and peak amplitude to be unity.

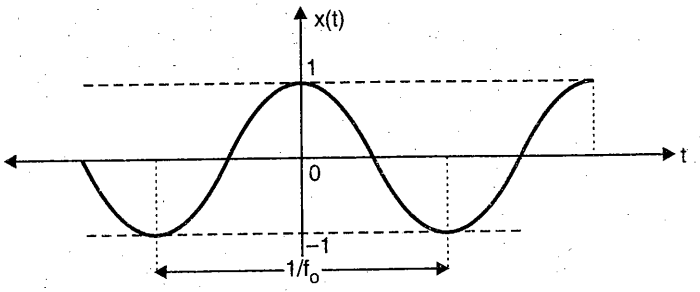


Fig. T-21(a)

Soln. : A cosine wave can be mathematically represented as,

$$x(t) = A\cos(2\pi f_o t) \qquad \text{But } A = 1.$$

$$\therefore \quad x(t) = \cos(2\pi f_o t) \qquad \ldots(1)$$

By Euler's identity we can write,

$$\therefore \quad x(t) = \cos(2\pi f_o t) = \frac{e^{j2\pi f_o t} + e^{-j2\pi f_o t}}{2} \qquad \ldots(2)$$

The fourier transform of $x(t)$ is given by,

$$X(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j2\pi ft}\, dt$$

Substituting the value of $x(t)$ we get,

$$X(f) = \int_{-\infty}^{\infty} \left[\frac{e^{j2\pi f_o t} + e^{-j2\pi f_o t}}{2}\right] e^{-j2\pi ft}\, dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \left[e^{-j2\pi (f-f_o)t} + e^{-j2\pi (f+f_o)t}\right] dt$$

$$\therefore \quad X(f) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-j2\pi (f-f_o)t}\, dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-j2\pi (f+f_o)t}\, dt \qquad \ldots(3)$$

Refer to Prob. 12 in which we have found the fourier transform of a dc signal. In that example we have proved that,

$$\int_{-\infty}^{\infty} e^{-j2\pi ft}\, dt = \delta(f) \qquad \ldots(4)$$

Using this property for the RHS of Equation (3) we get,

$$X(f) = \frac{1}{2}\delta(f-f_o)+\frac{1}{2}\delta(f+f_o) \quad \text{...Ans.}$$

The frequency spectrum is as shown in the Fig. T-21(b) which shows that two impulses are present one at f_o and the other at $-f_o$.

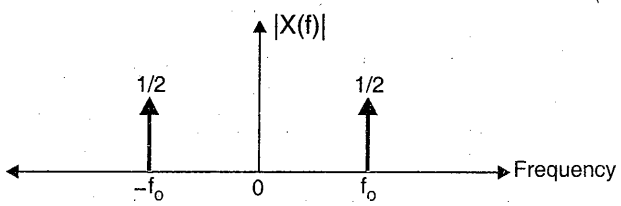


Fig. T-21(b) : Spectrum of a cosine wave

1.2.3 Power and Energy Theorems : :

In this section, we are going to learn two important theorems regarding the energy and power of signals. The two theorems are :

(i) Parseval's power theorem and

(ii) Rayleigh's energy theorem.

1.2.3.1 Parseval's Power Theorem :

This theorem relates the average power P of a periodic signal to its "fourier series" coefficients. The Parseval's power theorem states that the total average power of a periodic signal x (t) is equal to the sum of the average powers of the individual fourier coefficients i.e. C_n.

$\therefore$ Average power of x (t) = (Power of C_1) + (Power of C_2) + ...(26)

$$\text{or total average power : } P = \sum_{n=-\infty}^{\infty} |C_n|^2 \quad \text{...(27)}$$

Proof of the theorem :

(i) The total average normalized power of signal x (t) is given as follows :

$$P = \frac{1}{T_o}\int_{t}^{t+T_o} |x(t)|^2\,dt \quad \text{...(28)}$$

(ii) Substitute $|x(t)|^2 = x(t)\,x^*(t)$ in the Equation (28). Where $x^*(t)$ is the complex conjugate of x (t).

$$\therefore \quad P = \frac{1}{T_o}\int_{-T_o/2}^{T_o/2} x(t)\,x^*(t)\cdot dt \quad \text{...(29)}$$

(iii) We have defined x (t) using the exponential fourier series as follows :

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j 2\pi n f_o t}$$

Therefore the complex conjugate of x (t) is given as,

$$x^*(t) = \left[\sum_{n=-\infty}^{\infty} C_n e^{j 2\pi n f_o t} \right]^*$$

$$= \left[\sum_{n=-\infty}^{\infty} C_n^* e^{-j 2\pi n f_o t} \right] \quad ...(30)$$

(iv) Substitute the value of $x^*(t)$ into the Equation (29) to get,

$$P = \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} x(t) \cdot \left[\sum_{n=-\infty}^{\infty} C_n^* e^{-j 2\pi n f_o t} \right] dt$$

Rearranging the equation we get,

$$= \frac{1}{T_o} \sum_{n=-\infty}^{\infty} C_n^* \left[\int_{-T_o/2}^{T_o/2} x(t) \cdot e^{-j 2\pi n f_o t} dt \right]$$

$$P = \sum_{n=-\infty}^{\infty} \left[\frac{1}{T_o} \int_{-T_o/2}^{T_o/2} x(t) \cdot e^{-j 2\pi n f_o t} dt \right] C_n^* \quad ...(31)$$

(v) The term inside the square bracket in Equation (31) is nothing but "C_n".

$$\therefore \quad P = \sum_{n=-\infty}^{\infty} C_n \cdot C_n^*$$

But $C_n \cdot C_n^* = |C_n|^2$

$$\therefore P = \sum_{n=-\infty}^{\infty} |C_n|^2 \quad \text{... Proved} \quad ...(32)$$

The interpretation of the Parseval's theorem is that the total average power of the signal x (t) can be found by squaring and adding the heights $|C_n|$ of the amplitude lines in the spectrum of the periodic signal x (t). Thus the Parseval's theorem implies "superposition" of the average powers.

1.2.3.2 Rayleigh's Energy Theorem :

The Rayleigh's energy theorem is analogous to the Parseval's power theorem. It states that the total energy of the signal x (t) is equal to the sum of energies of the individual spectral components in the frequency domain. The total normalized energy of a signal x (t) is given by,

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

According to the Rayleigh's energy theorem,

$$\text{Total normalized energy : } E = \int_{-\infty}^{\infty} |X(f)|^2 df \qquad ...(33)$$

Thus the total normalized energy is equal to the area under the signal corresponding to the square of the amplitude spectrum | X (f) | of the signal.

Proof of the theorem :

(i) We know that the normalized energy of the signal x (t) is given by,

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t) \cdot x^*(t) dt \qquad ...(34)$$

(ii) Using the definition of inverse fourier transform (IFT),

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j 2\pi ft} df$$

Therefore the complex conjugate of x (t) can be obtained as follows :

$$x^*(t) = \left[\int_{-\infty}^{\infty} X(f) e^{j 2\pi ft} df \right]^* = \int_{-\infty}^{\infty} X^*(f) e^{-j 2\pi ft} df$$

(iii) Substitute the value of $x^*(t)$ into the Equation (34) to get,

$$E = \int_{-\infty}^{\infty} x(t) \cdot \left[\int_{-\infty}^{\infty} X^*(f) e^{-j 2\pi ft} df \right] dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(t) \cdot e^{-j 2\pi ft} dt \right] X^*(f) df \qquad ...(35)$$

In the Equation (35), the term inside the square bracket is the fourier transform X (f) of the signal x (t).

$$\therefore \quad \text{Total energy : } E = \int_{-\infty}^{\infty} X(f) \cdot X^*(f)\, df$$

$$\therefore \quad E = \int_{-\infty}^{\infty} |X(f)|^2\, df \quad \text{... Proved}$$

Thus it is possible to obtain the energy of a signal x (t) by calculating the area under the square of its amplitude spectrum | X (f) |

1.2.3.3 Examples on Parseval's and Rayleigh Theorems :

Prob. 14 : Calculate the energy of a causal exponential pulse as shown in Fig. T-22.

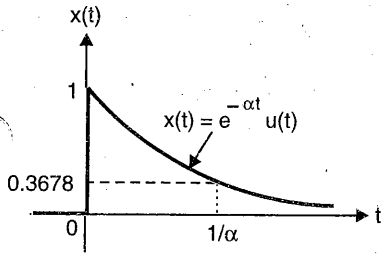


Fig. T-22

Soln. :

The normalized energy, $E = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{0}^{\infty} (e^{-\alpha t})^2\, dt$

$$= \int_{0}^{\infty} e^{-2\alpha t}\, dt = \frac{1}{-2\alpha} [e^{-2\alpha t}]_0^{\infty} = -\frac{1}{2\alpha} [e^{-\infty} - e^{0}]$$

$$\therefore \quad E = \frac{1}{2\alpha} \quad \text{...Ans.}$$

Prob. 15 : For the same signal shown in Fig. 1.22, calculate the energy in the frequency band $-W \le t \le W$ where $W = \alpha / 2\pi$.

Soln. :

The fourier transform of $x(t) = e^{-\alpha t} u(t)$ is given by,

$$X(f) = \frac{1}{\alpha + j\, 2\pi f} \quad \text{...(1)}$$

To obtain the amplitude spectrum :

Therefore the amplitude spectrum | X (f) | is obtained as follows :

$$X(f) = \frac{1}{(\alpha + j2\pi f)} \times \frac{(\alpha - j2\pi f)}{(\alpha - j2\pi f)}$$

$$= \frac{\alpha}{\alpha^2 + (2\pi f)^2} \times \frac{j2\pi f}{\alpha^2 + (2\pi f)^2} \quad ...(2)$$

$\therefore$ Amplitude spectrum $|X(f)| = \left\{ [\text{Re}\, X(f)]^2 + [\text{Im}\, X(f)]^2 \right\}^{1/2}$

$$= \left[\frac{\alpha^2}{(\alpha^2 + 4\pi^2 f^2)^2} + \frac{4\pi^2 f^2}{(\alpha^2 + 4\pi^2 f^2)^2} \right]^{1/2}$$

$$= \left[\frac{\alpha^2 + 4\pi^2 f^2}{(\alpha^2 + 4\pi^2 f^2)^2} \right]^{1/2}$$

$$|X(f)| = \frac{1}{[\alpha^2 + 4\pi^2 f^2)]^{1/2}} \quad ...(3)$$

$\therefore$ Square of the amplitude spectrum is given as,

$$|X(f)|^2 = \frac{1}{\alpha^2 + 4\pi^2 f^2} \quad ...(4)$$

To obtain the energy of the signal :

The energy in the frequency range of – W to W is given by,

$$\text{Energy} \quad E = \int_{-W}^{W} \frac{1}{\alpha^2 + 4\pi^2 f^2} df \quad ...(5)$$

Let us use the standard identity as,

$$\int \frac{dx}{\alpha^2 + b^2 x^2} = \frac{1}{\alpha b} \tan^{-1}\left(\frac{bx}{\alpha}\right)$$

Hence applying this to Equation (5), x = f, b = 2 π

$$E = \frac{1}{2\pi\alpha}\left[\tan^{-1}\left(\frac{2\pi f}{\alpha}\right)\right]_{-W}^{W} = \frac{1}{2\pi\alpha}\left[\tan^{-1}\left(\frac{2\pi W}{\alpha}\right) - \tan^{-1}\left(\frac{-2\pi W}{\alpha}\right)\right]$$

$$= \frac{1}{2\pi\alpha} \times 2\tan^{-1}\left(\frac{2\pi W}{\alpha}\right)$$

Substitute, $W = \frac{\alpha}{2\pi}$

$$\therefore \quad E = \frac{1}{\pi\alpha}\tan^{-1}(1) = \frac{1}{\pi\alpha} \times \frac{\pi}{4}$$

$$\therefore \quad E = \frac{1}{4\alpha} \quad ...\text{Ans.}$$

Prob. 16 : Calculate the energy of the sinc pulse shown in Fig. T-23.

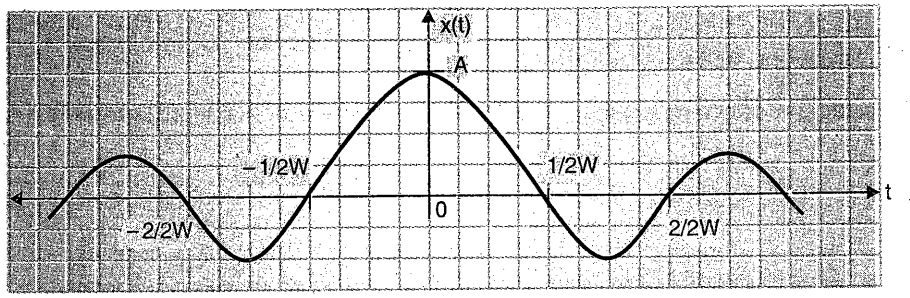


Fig. T-23 : Sinc pulse

Soln. : The fourier transform of the sinc pulse shown in Fig. T-23 is given by,

$$X(f) = F[A \operatorname{sinc}(2Wt)] = \frac{A}{2W} \operatorname{rect}(f/2W) \qquad ...(1)$$

The energy of the signal $x(t)$ is given by,

$$E = \int_{-\infty}^{\infty} |X(f)|^2 \, df \qquad ...(2)$$

Substituting the expression for $X(f)$ we get,

$$E = \int_{-\infty}^{\infty} [(A/2W) \operatorname{rect}(f/2W)]^2 \, df \qquad ...(3)$$

But $\frac{A}{2W} \operatorname{rect}(f/2W)$ is a rectangular pulse of duration 2W, extending from $f = -W$ to $f = +W$ with a constant amplitude of $(A/2W)$.

Therefore Equation (3) gets modified to,

$$E = \int_{-W}^{W} [A/2W]^2 \, df = \frac{A^2}{4W^2} (W + W)$$

$$\therefore \quad E = \frac{A^2}{2W} \qquad ...Ans.$$

1.2.4 Spectral Density Functions :

In this section, we will categorize signals and systems by concentrating on the "energy" or "power" of a signal. In order to do so let us get introduced to a new term called "spectral density". The spectral density of a signal, defines the distribution of energy or power per unit bandwidth as a function of frequency.

The spectral density of energy signals is called as "Energy Spectral Density (ESD)" while

that of the power signals is called as "Power Spectral Density (PSD)". Let us discuss about them now.

1.2.4.1 Energy Spectral Density (ESD) :

According to Rayleigh's energy theorem, the total energy of a signal is given by,

$$E = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

This means that the energy of a signal $x(t)$ is a result of energies contributed by all the spectral components of the signal $x(t)$. Thus it is possible to obtain the energy of the signal $x(t)$ by calculating the area under the square of its amplitude spectrum $|X(f)|$.

The Rayleigh energy theorem not just provides a useful method of evaluating the total energy but it also tells us that $|X(f)|^2$ can provide us the "distribution of energy" of signal $x(t)$ in the frequency domain.

Definition of ESD :

Therefore the squared amplitude spectrum $|X(f)|^2$ is called as the "Energy Spectral Density (ESD)" or "Energy Density Spectrum".

$$\therefore \text{ESD} = \psi(f) = |X(f)|^2 \qquad \ldots(36)$$

1.2.4.2 Properties of Energy Spectral Density :

Some of the important properties of ESD are as follows :

Property 1 :

The energy spectral density of an energy signal $x(t)$ is a non-negative real valued function of frequency. That means,

$$\psi(f) \geq 0, \quad \text{for all values of } f \qquad \ldots(37)$$

This is because $\psi(f) = |X(f)|^2$ i.e. square of the amplitude spectrum which cannot be negative for any value of f.

Property 2 :

The ESD of a real valued energy signal $x(t)$ is an even (symmetric) function of frequency. That means,

$$\psi(f) = \psi(-f) \qquad \ldots(38)$$

This is true because the amplitude spectrum $|X(f)|$ of a real valued signal is a symmetric (even) function of f. That means $|X(f)| = |X(-f)|$, hence $|X(f)|^2 = |X(-f)|^2$ i.e. $\psi(f) = \psi(-f)$.

Property 3 :

The total area under the curve of ESD of an energy signal $x(t)$ is equal to the signal energy. That means,

$$\text{Total energy } E = \text{Area under } \psi(f) = \int_{-\infty}^{\infty} \psi(f)\, df \qquad ...(39)$$

We have defined $\psi(f) = |X(f)|^2$

$$\therefore \int_{-\infty}^{\infty} \psi(f)\, df = \int_{-\infty}^{\infty} |X(f)|^2\, df$$

$$= E \qquad \text{... According to Rayleigh's theorem.}$$

Therefore area under $\psi(f)$ is equal to the signal energy E.

Property 4 :

When an energy signal $x(t)$ is passed through a linear time-invariant (LTI) system, the energy spectral density of the output signal is equal to the product of energy spectral density of the input and squared amplitude response of the system. (Refer Fig. T-24). That means,

$$\psi_o(f) = |H(f)|^2 \psi_i \qquad ...(40)$$

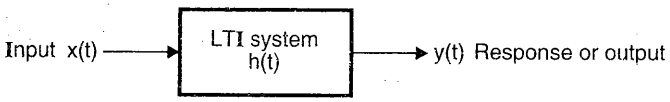


Fig. T-24 : Input and output of an LTI system

Proof :

(i) Let $x(t)$ be the input, $y(t)$ be the output and $h(t)$ be the impulse response of an LTI system. These are related to each other as,

$$y(t) = x(t) * h(t) \qquad ...(41)$$

where $*$ represents the convolution of $x(t)$ with $h(t)$.

(ii) Taking fourier transform of both the sides of Equation (41) we get,

$$Y(f) = X(f) \cdot H(f) \qquad ...(42)$$

This is because convolution in time domain gets transformed into multiplication of transforms in the frequency domain.

(iii) Taking the squared amplitude of both the sides of Equation (42) we get,

$$|Y(f)|^2 = |H(f)|^2 |X(f)|^2 \quad ...(43)$$

That means,

$$\psi_o(f) = |H(f)|^2 \cdot \psi_i(f) \quad ...(44)$$

where $\psi_o(f) = |Y(f)|^2$ = Energy spectral density of output $y(t)$

$\psi_i(f) = |X(f)|^2$ = Energy spectral density of input $x(t)$.

Thus Equation (44) is nothing but the mathematical statement of property 4 stated earlier.

Property 5 :

For an energy signal $x(t)$, the auto-correlation function and energy spectral density form a fourier transform pair. That means,

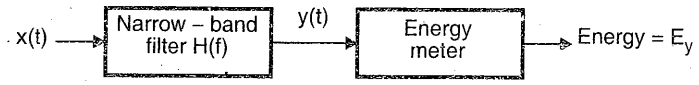
$$R(\tau) \overset{F}{\longleftrightarrow} \psi(f) \quad ...(45)$$

1.2.4.3 Physical Interpretation of the Energy Spectral Density :

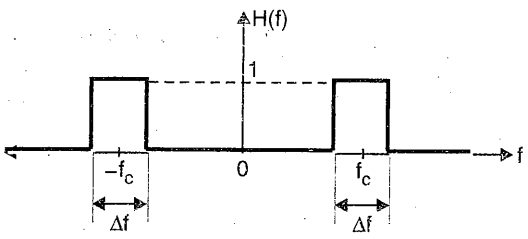
For understanding the physical interpretation of energy spectral density, we are going to use Equation (44) as the basic equation which states that for a LTI system.

$$\psi_o(f) = |H(f)|^2 \cdot \psi_i(f)$$

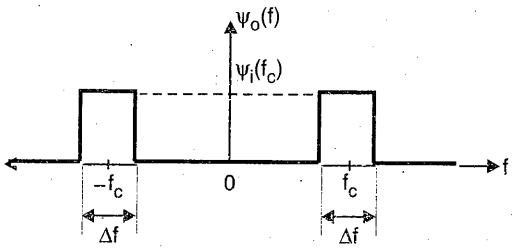
Now refer to the arrangement of an energy signal $x(t)$ is passed through a narrow band, bandpass filter followed by an energy meter.



(a) Circuit arrangement to measure energy spectral density



(b) Idealized amplitude response of the bandpass filter



(c) Energy spectral density of filter output

Fig. T-25

(i) Fig. T-25(b) shows the idealized amplitude response of a bandpass filter. The amplitude response of this filter can be expressed mathematically as,

$$|H(f)| = \begin{cases} 1, & \left(f_c - \frac{\Delta f}{2}\right) \le |f| \le \left(f_c + \frac{\Delta f}{2}\right) \\ 0, & \text{otherwise} \end{cases} \quad ...(46)$$

(ii) In this expression f_c is the center frequency and Δf is its bandwidth. We assume that Δf is very small, therefore the amplitude response of the input signal $x(t)$ is constant over the range Δf.

$$\text{i.e.} \quad |X(f)| = \text{constant} \quad \text{... over the range } \Delta f \quad ...(47)$$

(iii) The amplitude spectrum of the filter output is given by,

$$|Y(f)| = |H(f)| \cdot |X(f)|$$

$$\text{But } |H(f)| = \begin{cases} 1, & \text{... for } \left(f_c - \frac{\Delta f}{2}\right) \le |f| \le \left[f_c + \frac{\Delta f}{2}\right] \\ 0, & \text{... otherwise} \end{cases}$$

$$\therefore \quad |Y(f)| = \begin{cases} |X(f)| = |X(f_c)|, & \text{for } \left[f_c - \frac{\Delta f}{2}\right] \le |f| \le \left[f_c + \frac{\Delta f}{2}\right] \\ 0, & \text{otherwise,} \end{cases} \quad ...(48)$$

That means the amplitude spectrum of the filter output is identical to the amplitude spectrum of filter input, in the range Δf.

(iv) Naturally the energy spectral density of the filter output is related to the energy spectral density of the filter input as follows :

$$\psi_o(f) = \begin{cases} \psi_i(f_c), & \text{... for } \left[f_c - \frac{\Delta f}{2}\right] \le |f| \le \left[f_c + \frac{\Delta f}{2}\right] \\ 0, & \text{... otherwise} \end{cases} \quad ...(49)$$

This relation is plotted graphically as shown in Fig. T.25(c)

(v) From the Rayleigh's energy theorem, the energy of the filter output y (t) is given by,

$$Ey = \int_{-\infty}^{\infty} \psi_o(f)\,df$$

$$= 2\int_{-0}^{\infty} \psi_o(f)\,df = 2\int_{(f_c-\Delta f/2)}^{(f_c+\Delta f/2)} \psi_i(f_c)\,df$$

$$\therefore \quad Ey = 2\,\psi_i(f_c)\,\Delta f \qquad ...(50)$$

The multiplying factor of 2 in Equation (50) is to account for the contributions of negative as well as positive frequency components.

(vi) Equation (50) can be rearranged as follows :

$$\psi_i(f_c) = \frac{Ey}{2\,\Delta f} \qquad ...(51)$$

Equation (51) tells us that the energy spectral density of the filter input [$\psi_i(f_c)$] at some frequency f_c is equal to the energy of the filter output Ey divided by $2\,\Delta f$, where Δf is the bandwidth.

Therefore energy spectral density for an energy signal for any frequency f can be interpreted as the "energy per unit bandwidth" which is contributed by the frequency components of the signal around the frequency f.

1.2.4.4 Power Spectral Density (PSD) :

(i) Let x (t) be a real valued power signal. Then the average normalized power of this signal is defined as,

$$\mathbf{P} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\,\mathbf{dt} \qquad ...(52)$$

(ii) If the signal x (t) is periodic with a period T_o then according to the Parseval's power theorem, the average signal power is given by,

$$P = \sum_{n=-\infty}^{\infty} |C_n|^2 \qquad ...(53)$$

(iii) For a periodic signal, C_n is expressed as follows :

$$C_n = \frac{1}{T_o} X(nf_o) \qquad ...(54)$$

Substituting this value of C_n into Equation (53) we get,

$$P = \sum_{n=-\infty}^{\infty} \frac{1}{T_o^2} |X(nf_o)|^2$$

$$\therefore \quad P = \frac{1}{T_o^2} \sum_{n=-\infty}^{\infty} |X(nf_o)|^2 \qquad ...(55)$$

(iv) But the fourier transform of a periodic signal is not continuous, it is discrete and present only at frequencies $\pm f_o, \pm 2f_o, \pm 3f_o$ ....

(v) We now define the power spectral density of a periodic signal x (t) as follows :

$$\text{Power spectral density : } s(f) = \frac{1}{T_o^2} \sum_{n=-\infty}^{\infty} |X(nf_o)|^2 \delta(f - nf_o) \qquad ...(56)$$

(vi) Equation (56) shows that power spectral density is a discrete function of frequency. This is indicated by the term $\delta(f - nf_o)$ in Equation (56). The power spectral density function is present only at the harmonic frequencies of f_o.

(vii) The most important property of PSD is that the area under PSD is equal to the average power of the signal x (t).

$$\therefore \quad P = \int_{-\infty}^{\infty} S(f)\, df \qquad ...(57)$$

Let us prove this,

$$\int_{-\infty}^{\infty} S(f)\, df = \int_{-\infty}^{\infty} \frac{1}{T_o^2} \sum_{n=-\infty}^{\infty} |X(nf_o)|^2 \delta(f - nf_o)\, df \qquad ...(58)$$

Using the shifting property of delta function we get,

$$\int_{-\infty}^{\infty} S(f)\,df = \frac{1}{T_o^2} \sum_{n=-\infty}^{\infty} |X(nf_o)|^2$$

$$\therefore \quad \int_{-\infty}^{\infty} S(f)\,df = P \qquad \text{... from Equation (55)}$$

Thus the average power of a periodic signal $x(t)$ is given by the area under the power spectral density function.

1.2.4.5 Properties of Power Spectral Density :

As the role of PSD for power signals is similar to that of ESD for the energy signal, the properties of PSD will be similar to those of ESD discussed in section 1.2.4.2.

Property 1 :

The PSD of a power signal $x(t)$ is a non-negative real valued function of frequency; that means,

$$S(f) \geq 0 \text{ for all values of } f \qquad ...(59)$$

From Equation (56) it is clear that $S(f)$ is the summation of the values of $|X(nf_o)|^2$ therefore it will be always positive.

Property 2 :

The PSD of a real valued power signal $x(t)$ is an even function of frequency. That means,

$$S(f) = S(-f) \qquad ...(60)$$

From the definition of PSD (Equation (56)) it is clear that we are using the values of amplitude spectrum $|X(f)|$ at $f = nf_o$ and $X(f)$ is an even function therefore $s(f)$ also will be an even function.

Property 3 :

The total area under the curve of power spectral density signal $x(t)$ is equal to the average signal power. That means,

$$P = \int_{-\infty}^{\infty} S(f)\,df \qquad \ldots(61)$$

We have already proved this property in section 1.2.1.4.

Property 4 :

When a power signal x (t) is applied to an LTI (linear time invariant) system, the PSD of its output is equal to the PSD of its input multiplied by the squared amplitude response of the system.

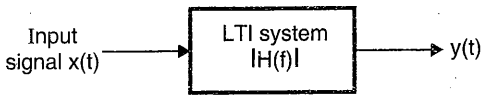


Fig. T-26 : LTI system

The statement of property 4 can be written in the mathematical form as,

$$S_o(f) = |H(f)|^2 \; S_i(f) \qquad \ldots(62)$$

where $S_o(f)$ = PSD of output of an LTI system

$S_i(f)$ = PSD of input of an LTI system

$|H(f)|$ = Amplitude response of the system.

Proof :

(i) We know that the output of an LTI system is related to its input as,

$$y(t) = x(t) * h(t) \qquad \ldots(63)$$

Taking fourier transform of both the sides we get,

$$Y(f) = H(f) \times X(f) \qquad \ldots(64)$$

Squaring both the sides we get,

$$|Y(f)|^2 = |H(f)|^2 |X(f)|^2 \qquad \ldots(65)$$

(ii) We know that for a periodic signal x (t), the fourier transform X (f) is given by,

$$X(f) = \frac{1}{T_o} \sum_{n=-\infty}^{\infty} X(nf_o)\,\delta(f - nf_o) \qquad \ldots(66)$$

and fourier transform of $y(t)$ is given by,

$$Y(f) = \frac{1}{T_o} \sum_{n=-\infty}^{\infty} Y(nf_o)\,\delta(f - nf_o) \qquad ...(67)$$

(iii) Substitute Equations (66) and (67) into Equation (65) to get,

$$\frac{1}{T_o^2} \sum_{n=-\infty}^{\infty} |Y(nf_o)|^2 \delta(f - nf_o) = |H(f)|^2 \cdot \frac{1}{T_o^2} \sum_{n=-\infty}^{\infty} |X(nf_o)|^2 \delta(f - nf_o)$$

$$\therefore \quad S_o(f) = |H(f)|^2 \cdot S_i(f) \qquad \text{... Proved}$$

Property 5 :

The auto-correlation function $R(\tau)$ and the power spectral density function form a fourier transform pair. That means,

$$R(\tau) \overset{F}{\longleftrightarrow} S(f) \qquad ...(68)$$

Prob. 17 : Determine the energy spectral density of a sinc pulse given by,

$$x(t) = A \text{ sinc}(2Wt)$$

Soln. :

(i) We know that the energy spectral density of $x(t)$ is given by,

$$\psi(f) = |X(f)|^2$$

Hence we have to obtain the expression for $X(f)$.

(ii) We know that,

$$A \text{ sinc}(2Wt) \longleftrightarrow \frac{A}{2W} \text{rect}(f/2W)$$

$$\text{i.e.} \quad X(f) = \frac{A}{2W} \text{rect}\left(\frac{f}{2W}\right)$$

(iii) $$\therefore \quad \text{ESD} = |X(f)|^2 = \frac{A^2}{4W^2} \text{rect}\left[\frac{f}{2W}\right] \qquad ...\text{Ans.}$$

Prob. 18 : Calculate the total energy for the signal given in Prob. 17.

Soln. :

$$x(t) = A \text{ sinc}(2Wt)$$

(i) According to property 3 of energy spectral density, the total energy is given by the area under $\psi(f)$.

$$\text{i.e.} \quad E = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

(ii) Substitute $|X(f)|^2 = \frac{A^2}{4W^2} \text{rect}[f/2W]$ to get,

$$E = \int_{-\infty}^{\infty} \frac{A^2}{4W^2} \text{rect}[f/2W] \, df$$

But rect $(f/2W)$ is a rectangular pulse in the frequency domain which extends from $-W$ to $+W$ with an amplitude of 1. Therefore expression for E is modified to,

$$E = \int_{-W}^{W} \frac{A^2}{4W^2} \, df = \frac{A^2}{4W^2} [f]_{-W}^{W}$$

$$\therefore \quad E = \frac{A^2}{2W} \qquad \text{...Ans.}$$

Theory Questions

Q. 1 Explain the concept of negative frequency

Q. 2 Define Fourier transform and inverse Fourier transform. Write down the conditions for existence of Fourier transform.

Q. 3 Using shifting property, define the Fourier transform of delta function.

Q. 4 State and prove Parseval's power theorem

Q. 5 State and prove Rayleigh's energy theorem

Unsolved Numericals

Q. 1 Determine Fourier transform of an impulse function. What is the effect on its Fourier transform if impulse is delayed in time by 3 sec. ?

Ans. : $\delta(t) \xleftrightarrow{\text{F.T.}} 1, \quad \delta(t-3) \xleftrightarrow{\text{F.T.}} e^{-j6\pi F}$

Q. 2 Find the Fourier transform of following signal

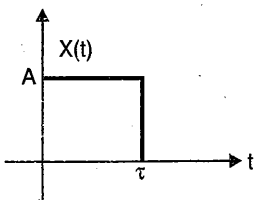


Fig. T-27

Ans. : $X(f) = e^{-j2\pi f\tau/2} A\tau \,\text{sinc}(f\tau)$

Q. 3 Using properties of Fourier transform, find Fourier transform of $x(t) = U(2t) + U(t-1)$

Ans. : $\frac{1}{j2\pi f} + \frac{e^{-j2\pi f}}{j2\pi F}$

Q. 4 Obtain exponential Fourier series expansion of gate function

Ans. : Tsinc (fT)

❑❑❑

Frequency Analysis of Discrete Time Signals

Section No.	Name of the Topic
1.1	Introduction
1.2	Fourier Series for Discrete Time Periodic Signals (DTFS)
1.3	Power Density
1.4	Fourier Transform of Discrete Time Aperiodic Signals (DTFT)
1.5	Symmetry Properties of F.T.
1.6	F.T. Theorems (Properties of DTFT)
1.7	Relationship between Fourier Transform and Z Transform

1.1 Introduction :

The DSP processors are used to perform frequency analysis of signals. To do the frequency analysis, we should convert the time domain signal into the frequency domain. For this, the different frequency transformation techniques are used.

What does a Fourier Transform do ?

Any signal is build up by addition of elementary signals which are at different frequencies, have different amplitudes and relative phases. Using the fourier transform we can plot the amplitude and phase spectrums which provide us all the information about amplitudes and relative phases of such elementary signals. Thus fourier transform can be used for the "analysis" of a signal. It is used for transformation from the time domain to frequency domain.

1.2 Fourier Series for Discrete Time Periodic Signals (DTFS) :

Consider $x(n)$ is a discrete time periodic signal i.e.

$$x(n) = x(n+N) \quad \text{... for all n.}$$

The FS representation for $x(n)$ consists of N harmonically related exponential function.

$$e^{j2\pi kn/N} \quad ...k = 0, 1, ...N-1$$

$$\therefore \quad x(n) = \sum_{k=0}^{N-1} C_k e^{j2\pi kn/N}$$

where C_k = coefficients in series expansion.

The equation for fourier coefficient is,

$$C_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad ...k = 0, ... N-1$$

This equation is called analysis equation

$\therefore$ synthesis equation,

$$x(n) = \sum_{k=0}^{N-1} C_k e^{j2\pi kn/N} \quad ...(1)$$

Analysis equation

$$C_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad ... k = 0,1...N-1 \quad ...(2)$$

Equation (1) provides discrete time fourier series. C_k provides amplitude and phase associated with frequency component of $x(n)$ in the frequency domain

1.2.1 Periodicity Condition :

The FS coefficients, C_k also satisfy periodicity condition,

$$\therefore \quad C_{k+N} = C_k$$

Proof:

$$C_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi(k+N)n}{N}}$$

$$\therefore \quad C_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\cdot e^{-j2\pi kn/N}\, e^{-j2\pi k}$$

$$\therefore \quad e^{-j2\pi k} = \cos 2\pi K - j \sin 2\pi K = 1 \text{ always}$$

$$\therefore \quad C_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}$$

$$\therefore \quad C_{k+N} = C_k$$

Note that fourier series is possible if f = k/N is rational number.

e.g. If $x(n) = \cos\sqrt{2}\,\pi n$

then $\omega = \sqrt{2}\,\pi$

$\therefore \quad 2\pi f = \sqrt{2}\,\pi$

$\therefore \quad f = \frac{1}{\sqrt{2}}$

$\therefore$ f is not rational number, the signal is not periodic.

$\therefore$ signal cannot be expanded in fourier series.

1.2.2 Properties of DTFS :

The properties of DTFS are summerized in Table E-1.

Table E-1

Properties of Discrete time fourier series		
Property	**Periodic signal**	**Fourier series coefficients**
	$x(n)$ periodic with period N and $y(n)$ fundamental frequency $\omega_0 = 2\pi / N$	a_k b_k periodic with period N
Linearity	$A\,x(n) + B\,y(n)$	$A\,a_k + B\,b_k$
Time shifting	$x(n - n_0)$	$a_k\, e^{-jk(2\pi/N)n_0}$

Properties of Discrete time fourier series		
Property	**Periodic signal**	**Fourier series coefficients**
Frequency shifting	$e^{jm(2\pi/N)n} x(n)$	
Conjugation	$x^*(n)$	a^*_{-k}
Time reversal	$x(-n)$	a_{-k}

1.2.3 Solved Problem on DTFS :

Ex. 1 : Determine the spectra of periodic signal $x(n) = \{1, 1, 1, 0\}$ with period $N = 4$.

Soln. : We have to determine the amplitude and phase spectra of given periodic sequence. This information is provided by coefficient C_k. We have,

$$C_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi k n / N} \quad .... \; k = 0, 1, ... N-1 \qquad ...(1)$$

Here period of sequence = N = 4

$$\therefore \quad C_k = \frac{1}{4} \sum_{n=0}^{3} x(n) e^{-j2\pi k n / N} \quad .... \; k = 0, 1, 2, 3 \qquad ...(2)$$

Expanding the summation we get,

$$C_k = \frac{1}{4}\left[x(0) e^{0} + x(1) e^{-j2\pi k/4} + x(2) e^{-j4\pi k/4} + x(3) e^{-j6\pi k/4} \right]$$

Now given sequence is,

$$x(n) = \{1, 1, 1, 0\}$$

$$\therefore \quad x(0) = 1, \; x(1) = 1, \; x(2) = 1 \text{ and } x(3) = 0$$

$$\therefore \quad C_k = \frac{1}{4}\left[1 + 1 e^{-j\pi k/2} + 1 \cdot e^{-j\pi k} + e^{-j3\pi k/2} \right] \qquad ...(3)$$

We know that range of k is k = 0 to 3.

For k = 0

$$C_0 = \frac{1}{4} [1 + e^0 + e^0 + 0]$$

$$\therefore \quad C_0 = \frac{1}{4} [1 + 1 + 1]$$

$$\therefore \quad C_0 = 3/4 \text{ That means magnitude } = \frac{3}{4} \text{ and phase } = 0$$

For k =1

$$C_1 = \frac{1}{4}\left[1 + e^{-j\pi/2} + e^{-j\pi} \right]$$

Using identity, $e^{-j\theta} = \cos\theta - j\sin\theta$ we get,

$$\therefore \quad C_1 = \frac{1}{4}\left[1 + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} + \cos\pi - j\sin\pi\right]$$

$$\therefore \quad C_1 = \frac{1}{4}[1 + 0 - 1j + (-1) - 0]$$

[**Note :** The calculations are done in radian mode]

$$\therefore \quad C_1 = \frac{1}{4}[-1j]$$

$$\therefore \quad C_1 = 0 - \frac{1}{4}j$$

Converting into polar form we get,

$$C_1 = 0.25 \angle -1.57$$

That means magnitude = 0.25 and angle = – 1.57 radians = $\frac{-\pi}{2}$.

For k = 2 :

$$C_2 = \frac{1}{4}\left[1 + 1\cdot e^{-j\pi} + 1\cdot e^{-j2\pi}\right]$$

$$\therefore \quad C_2 = \frac{1}{4}[1 + \cos\pi - j\sin\pi + \cos 2\pi - j\sin 2\pi]$$

$$\therefore \quad C_2 = \frac{1}{4}[1 - 1 - 0 + 1 + 0]$$

$$\therefore \quad C_2 = \frac{1}{4}$$

That means magnitude is $\frac{1}{4}$ and angle = 0

For k = 3 :

$$C_3 = \frac{1}{4}\left[1 + e^{-j3\pi/2} + 1\cdot e^{-j3\pi}\right]$$

$$\therefore \quad C_3 = \frac{1}{4}\left[1 + \cos\left(\frac{3\pi}{2}\right) - j\sin\left(\frac{3\pi}{2}\right) + \cos(3\pi) - j\sin(3\pi)\right]$$

$$\therefore \quad C_3 = \frac{1}{4}[1 + 0 + j - 1 - 0]$$

$$\therefore \quad C_3 = \frac{j}{4} = 0 + j\,0.25 = 0.25 \angle +1.57$$

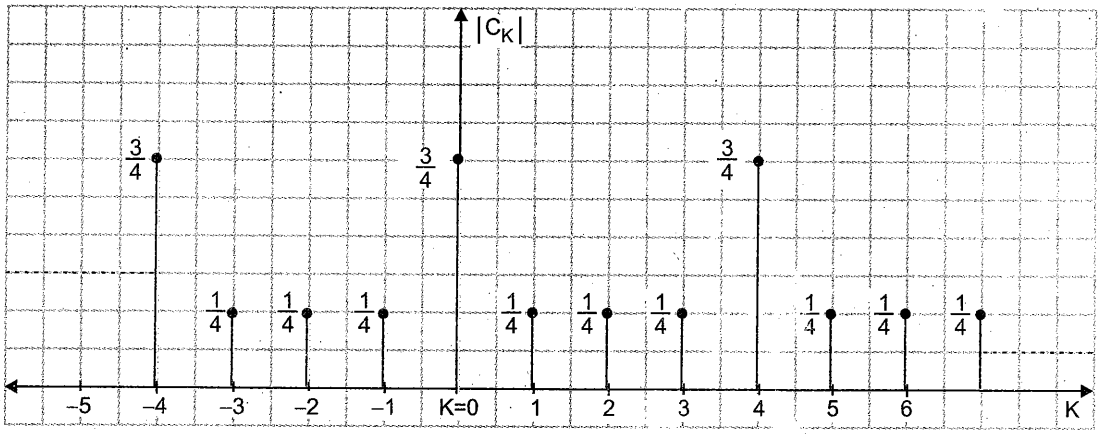
That means magnitude = 0.25 and angle = 1.57, radians = $\frac{\pi}{2}$

Table E-2 shows the summary of coefficients.

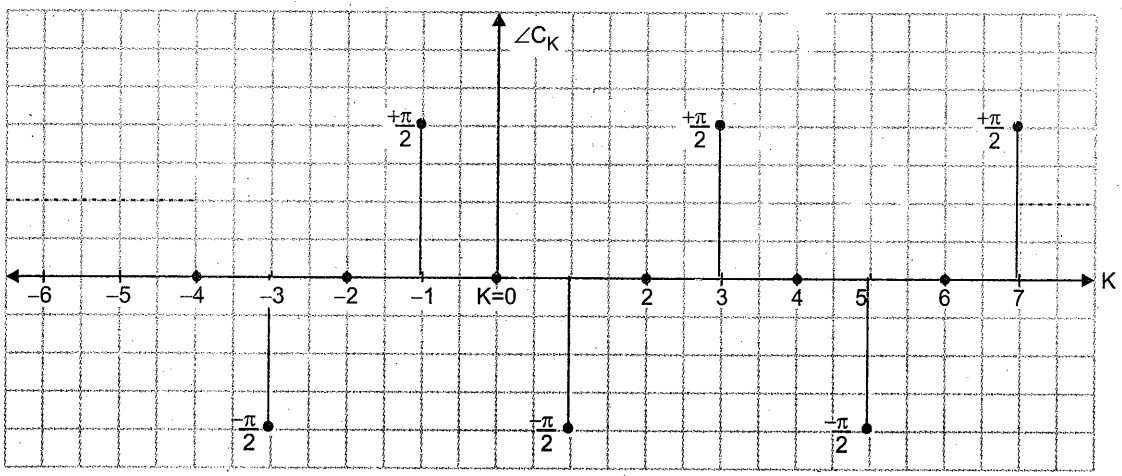
Table E-2

Sr. No.	Coefficient	Magnitude	Phase
1.	$C_0 = \frac{3}{4}$	$\frac{3}{4}$	0
2.	$C_1 = -\frac{1}{4}j$	$\frac{1}{4}$	$-\frac{\pi}{2}$
3.	$C_2 = \frac{1}{4}$	$\frac{1}{4}$	0
4.	$C_3 = \frac{j}{4}$	$\frac{1}{4}$	$\frac{\pi}{2}$

The magnitude response is shown in Fig. E-1(a) and the phase response is shown in Fig. E-1(b).



(a)



(b)

Fig. E-1

Since the sequence is periodic; we have calculated $|C_k|$ and $\angle C_k$ only for the range k = 0 to k = 3. Actual response is periodic in nature.

Ex. 2 : Sketch the magnitude and phase spectra of

$$x(n) = \cos\left(\frac{\pi}{8}n + \phi\right)$$

Soln. :

Step I : First we will check that the given sequence is periodic or not. Given,

$$x(n) = \cos\left(\frac{\pi}{8}n + \phi\right) \quad ...(1)$$

We have standard equation,

$$x(n) = \cos(2\pi fn + \phi) \quad ...(2)$$

Comparing Equations (1) and (2) we get,

$$2\pi f = \frac{\pi}{8}$$

$$\therefore \quad f = \frac{1}{16} \quad ...(3)$$

This is the ratio of two integer. If the frequency (f) is $\frac{k}{N}$, where k and N are integers then the sequence is periodic. Thus given sequence is periodic in nature with period N. That means from Equation (3), $F = \frac{k}{N} = \frac{1}{16}$; the period is N = 16.

Step II : We have the equation of as,

$$x(n) = \sum_{k=0}^{N-1} C_k e^{+j2\pi kn/N} \quad ...(4)$$

Now we can expressing given equation x(n) using Euler's identity as,

$$x(n) = e^{j\left(\frac{\pi}{8}n + \phi\right)} + e^{-j\left(\frac{\pi}{8}n + \phi\right)}$$

$$\therefore \quad x(n) = e^{-j\left(\frac{\pi}{8}n + \phi\right)} + e^{j\left(\frac{\pi}{8}n + \phi\right)}$$

$$\therefore \quad x(n) = \frac{1}{2}\left[e^{-\frac{j\pi n}{8}} \cdot e^{-j\phi}\right] + \frac{1}{2}\left[e^{\frac{j\pi n}{8}} \cdot e^{j\phi}\right] \quad ...(5)$$

We know that ϕ denotes the phase angle. Now comparing first term of Equation (5) with Equation (4) we get,

$$-\frac{j\pi}{8}n = \frac{j\,2\pi\,kn}{N}$$

$$\therefore \quad -\frac{j\,\pi}{8}n = \frac{j\,2\pi kn}{16}$$

$$\therefore \quad k = -1 \quad ...(6)$$

That means we will get term (1) when k = – 1. From Equations (4) and (5) we can write,

$$\therefore \quad C_k = C_{-1} = \frac{1}{2} e^{-j\phi} \qquad ...(7)$$

Similarly comparing term (2) of Equation (5) with Equation (4) we get,

$$e^{\frac{j\pi n}{8}} = e^{\frac{j2\pi kn}{N}}$$

$$\therefore \quad e^{\frac{j\pi n}{8}} = e^{\frac{j2\pi kn}{16}}$$

$$\therefore \quad k = 1 \qquad ...(8)$$

That means we will get term (2) when k = + 1

From Equations (4) and (5) we can write,

$$\therefore \quad C_k = C_1 = \frac{1}{2} e^{j\phi} \qquad ...(9)$$

Step III : Since C_k exists for negative value of k (k = – 1) and total period is N = 16; we can conclude that the range of k is k = – 7 to + 15. But from Equation (5) we can say that x (n) has only two terms.

That means C_k is present at k = – 1 and k = + 1; for other values of k; C_k is zero.

$$\therefore \quad C_k = \begin{cases} \frac{1}{2} e^{-j\phi} & , \ k = -1 \\ \frac{1}{2} e^{j\phi} & , \ k = +1 \\ 0 & , \ k = -7, -6, -5, -4, -3, -2, 0, 2, 3, 4, 5, 6, 7, 8 \end{cases}$$

That means magnitude is $\frac{1}{2}$ at k = – 1 and phase is – ϕ. While magnitude is $\frac{1}{2}$ at k = + 1 and phase is + ϕ. The magnitude and phase response are shown in Fig. E-2(a) and E-2(b) respectively.

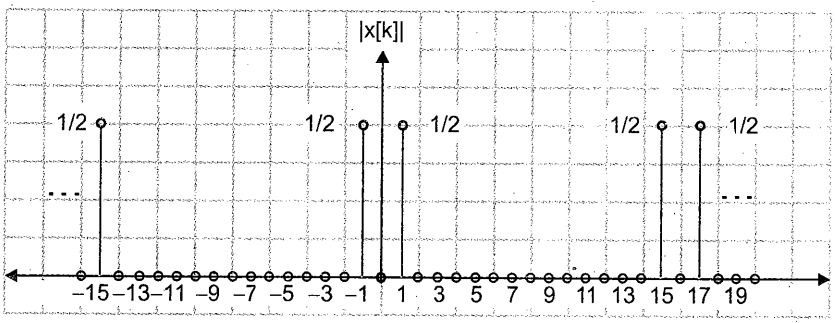
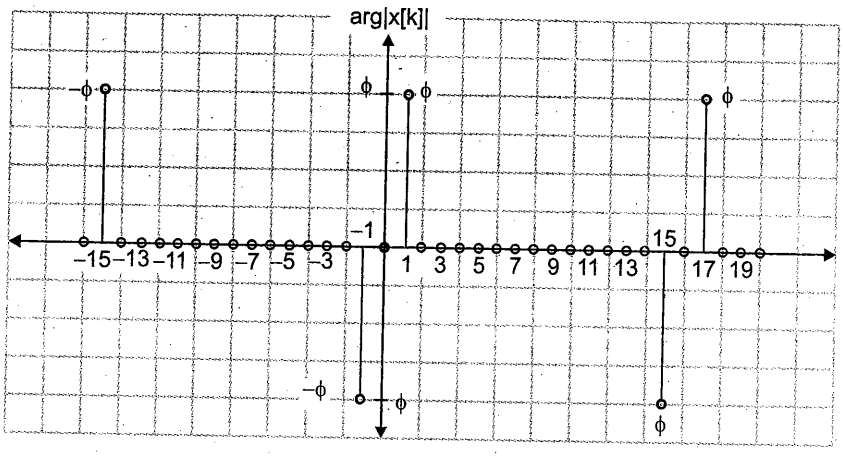


Fig. E-2(a)



(b)

Fig. E-2

1.3 Power Density :

In this section, we will discuss the power density spectrum of periodic signals.

Consider a discrete time periodic signal with period 'N'. The average power of such signal is given by,

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2 \quad \ldots(1)$$

Now we will derive expression of p_x in terms of fourier coefficients C_k.

We can express the term $|x(n)|^2$ as,

$$|x(n)|^2 = x(n) \cdot x^*(n) \quad \ldots(2)$$

Here $x^*(n)$ is complex conjugate of $x(n)$

We have the synthesis equation,

$$x(n) = \sum_{k=0}^{N-1} C_k e^{j2\pi kn/N}$$

Thus we can write,

$$x^*(n) = \sum_{k=0}^{N-1} C_k^* e^{-j2\pi kn/N} \quad \ldots(3)$$

Putting Equation (2) in Equation (1) we get,

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} x(n) \cdot x^*(n) \quad \ldots(4)$$

Putting Equation (3) in Equation (4) we get,

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} x(n) \left[\sum_{k=0}^{N-1} C_k^* e^{-j2\pi kn/N} \right] \quad ...(5)$$

Interchanging the order of summation and rearranging the terms we get,

$$P_x = \sum_{k=0}^{N-1} C_k^* \left[\frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \right] \quad ...(6)$$

Now recall analysis equation of DTFS,

$$C_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad ...(7)$$

Putting Equation (7) in Equation (6) we get,

$$P_x = \sum_{k=0}^{N-1} C_k^* \cdot C_k$$

But $C_k^* \cdot C_k = |C_k|^2$

$$\therefore \quad P_x = \sum_{k=0}^{N-1} |C_k|^2 \quad ...(8)$$

Thus from Equations (1) and (8) we can write,

$$P_x = \sum_{k=0}^{N-1} |C_k|^2 = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2 \quad ...(9)$$

Equation (9) gives the expression of average power in terms of fourier coefficients, C_k.

Meaning :

Average power in the signal is the sum of power of individual frequency components. The sequence $|C_k|^2$ for k = 0, 1, ... N – 1 is the distribution of power as a function of frequency and it is called as the power density spectrum of periodic signal. Now energy of discrete time signal in terms of C_k is given by,

$$E_N = \sum_{n=0}^{N-1} |x(n)|^2 = N \sum_{k=0}^{N-1} |C_k|^2$$

1.4 Fourier Transform of Discrete Time Aperiodic Signals (DTFT) :

Basically the fourier transform of aperiodic, finite energy discrete time signal is called as DTFT.

Mathematical Equation : Mathematically it is defined as,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \quad \ldots(1)$$

Here $X(\omega)$ = Fourier transform of $x(n)$

and $x(n)$ = Discrete time aperiodic signal

We can obtain discrete time signal $x(n)$ from $X(\omega)$ by taking inverse fourier transform. It is given by,

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega \quad \ldots(2)$$

Notation : The fourier transform of $x(n)$ is denoted by,

$$x(n) \overset{F}{\longleftrightarrow} X(\omega)$$

Equation (1) is called as Analysis Formula (FT) and Equation (2) is called as Synthesis Formula (IFT).

1.4.1 Periodic Nature of $X(\omega)$:

The frequency range of ω is from $-\pi$ to $+\pi$ or from 0 to 2π. So $X(\omega)$ becomes periodic with period 2π.

Proof : According to the definition of DTFT we have,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \quad \ldots(1)$$

Replacing ω by $(\omega + 2\pi k)$ we get,

$$X(\omega + 2\pi k) = \sum_{n=-\infty}^{\infty} x(n) e^{-j(\omega+2\pi k)n}$$

$$\therefore \quad X(\omega + 2\pi k) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \times e^{-j2\pi kn} \quad \ldots(2)$$

According to Euler's identity we have,

$$e^{-jx} = \cos x - j \sin x$$

$$\therefore \quad e^{-j2\pi kn} = \cos(2\pi kn) - j \sin(2\pi kn)$$

Now 'k' and 'n' both are integers

$$\therefore \quad \cos(2\pi kn) = 1 \qquad \text{and} \qquad \sin(2\pi kn) = 0$$

$$\therefore \quad e^{-j2\pi kn} = 1$$

Putting this value in Equation (2) we get,

$$X(\omega + 2\pi k) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \qquad ...(3)$$

Comparing Equations (1) and (3) we get,

$$\boxed{X(\omega) = X(\omega + 2\pi k)} \qquad ...(4)$$

Thus $X(\omega)$ is periodic with period 2π. So we can obtain $x(n)$ from $X(\omega)$ as, $x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega} d\omega$. Thus **$x(n)$ and $X(\omega)$ are inverse of each other**.

Condition for existence of FT : The fourier transform is convergent if,

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

That means fourier transform exists only if discrete time signal is absolutely summable. This is necessary and sufficient condition for the existence of fourier transform.

1.4.2 Convergence Effect :

We have, $\text{F.T.} = F\{x[n]\} = X[\omega] = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

Multiply both sides by $e^{j\omega m}$ and integrate over period $-\pi$ to π.

$\therefore$ We get,

$$N - 1 \int_{-\pi}^{\pi} X[\omega] e^{j\omega m} d\omega = \int_{-\pi}^{\pi} \left[\sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \right] e^{j\omega m} d\omega$$

We will interchange summation and integration sign. The interchange is possible only if,

$$X_N[\omega] = \sum_{n=-N}^{N} x[n] e^{-j\omega n}$$

converges uniformly to $X[\omega]$. This is achieved only if $N \to \infty$.

$$\therefore \quad X_N[\omega] \to X[\omega].$$

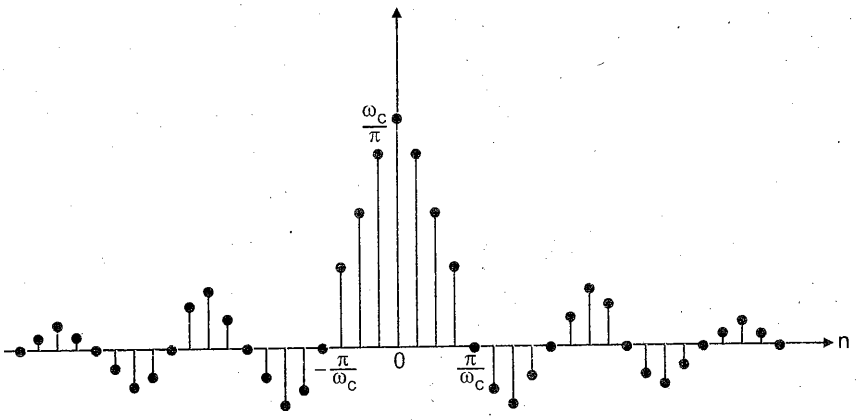
$$\therefore \quad \lim_{N \to \infty} \left| X_N[\omega] - X[\omega] \right| = 0.$$

Uniform convergence is guaranted if $x[n]$ is absolutely summable,

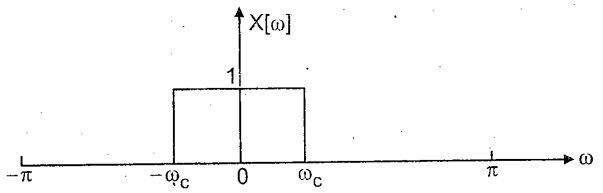
$$\text{i.e.} \quad \sum_{n=-\infty}^{\infty} |x[n]| < \infty.$$

Refer Fig. E-3(a).

Basic difference between $X_N[\omega]$ and $X[\omega]$ is, $X[\omega]$ has infinite limit that of $X_N[\omega]$ has finite [N] limit. In short we are *truncating* Fourier transform of the sequence by taking limited samples. But as we truncate we get slight difference between ideal and practical characteritic. If we want that both should match then extend $N \to \infty$.



(a)



(b)

Fig. E-3

Fig. E-3(a) shows $x[n]$ and Fig. E-3(b) shows its Fourier transform $X[\omega]$. (ideal)

By truncating Fourier transform we basically face one problem that is, we get oscillatory behaviour of the approximation $X_N[\omega]$ to the function $X[\omega]$ at the point of *discontinuity* of $X[\omega]$. This phenomena is called *Gibbs phenomena.*

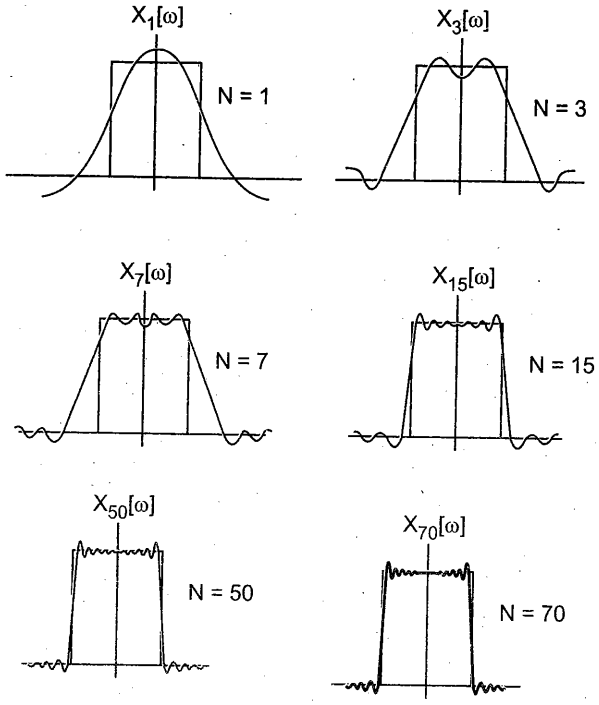


Fig. E-4

Fig. E-4 basically shows comparison of $X[\omega]$ and $X_N[\omega]$. $X[\omega]$ is ideal and $X_N[\omega]$ changes as we go on changing N value.

(a) N = 1 (b) N = 3 (c) N = 7

(d) N = 15 (e) N = 50 (f) N = 70

Thus as N increases it should exactly match (or converge) to $X[\omega]$. It does, but we get oscillations at discontinuties.

1.4.3 Energy Density :

Energy of discrete time signal, $x(n)$ is given by,

$$E_x = \sum_{n=\infty}^{\infty} |x(n)|^2 \quad ...(1)$$

We can write,

$$|x(n)|^2 = x(n) \cdot x^*(n) \quad ...(2)$$

$$\therefore \quad E_x = \sum_{n=\infty}^{\infty} x(n) \cdot x^*(n) \quad ...(3)$$

We have the equation of inverse DTFT (synthesis equation),

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

Thus we can write,

$$x^*(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\omega)\, e^{-j\omega n}\, d\omega \qquad ...(4)$$

Putting Equation (4) in Equation (3) we get,

$$E_x = \sum_{n=-\infty}^{\infty} x(n) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega)\, e^{-j\omega n}\, d\omega \right]$$

Interchanging the order of summation and integration we get,

$$E_x = \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\omega) \left[\sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \right] d\omega \qquad ...(5)$$

The summation term in Equation (5) is analysis equation. That means it is the equation for $X(\omega)$.

$$\therefore \quad E_x = \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\omega) \cdot X(\omega)$$

But $X^*(\omega) \cdot X(\omega) = |X(\omega)|^2$

$$\therefore \quad E_x = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 \qquad ...(6)$$

Using Equations (1) and (6) we can write,

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2\, d\omega \qquad ...(7)$$

Equation (7) is the Parseval's relation for discrete time periodic signal with finite energy.

Basically $X(\omega)$ is complex frequency function. So we can express it in terms of magnitude and angle as,

$$X(\omega) = |X(\omega)|\, e^{j\theta(\omega)} \qquad ...(8)$$

Here $|X(\omega)|$ = Magnitude spectrum

and $\theta(\omega)$ = phase spectrum

Now energy density spectrum of $x(n)$ is denoted by $S_{xx}(\omega)$

and $S_{xx}(\omega) = |X(\omega)|^2$ ...(9)

Here $S_{xx}(\omega)$ represents distribution of energy as a function of frequency.

If $x(n)$ is a real sequence then,

$$X^*(\omega) = X(-\omega)$$

This is called as even symmetry. Under this condition we can write,

$$S_{xx}(-\omega) = S_{xx}(\omega)$$

1.4.4 Fourier Transform of Standard Signals :

We will obtain the fourier transform of some standard signals with the help of examples.

Prob. 1 : Obtain DTFT of unit impulse $\delta(n)$.

Soln. : Here discrete time sequence is,

$$x(n) = \delta(n) \quad ...(1)$$

According to the definition of DTFT we have,

$$\text{DTFT}\{x(n)\} = X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$\therefore X(\omega) = \sum_{n=-\infty}^{\infty} \delta(n) \times e^{-j\omega n} \quad ...(2)$$

But we know that $\delta(n)$ exists only at $n = 0$. It is shown in Fig. E-5

$$\therefore \delta(n) = 1 \quad \text{for } n = 0$$
$$= 0 \quad \text{otherwise}$$

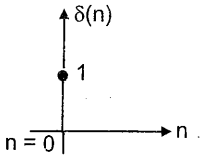


Fig. E-5

Thus Equation (2) becomes,

$$X(\omega) = \sum_{\text{at } n=0} 1 \times e^{-j\omega n} = 1 \times e^{0} = 1$$

$$\therefore \delta(n) \xleftrightarrow{\text{DTFT}} 1$$

This is the standard fourier transform pair.

Prob. 2 : Obtain DTFT of unit step $u(n)$.

Soln. :

Here $x(n) = u(n)$

According to definition of DTFT, $X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

$$\text{DTFT}\{u(n)\} = X(\omega) = \sum_{n=-\infty}^{\infty} u(n) e^{-j\omega n} \quad ...(1)$$

But we know that u (n) exists only for positive values of n.
It is shown in Fig E-6

$$\therefore \quad u(n) = 1 \quad \text{for } n \geq 0$$
$$= 0 \quad \text{for } n < 0$$

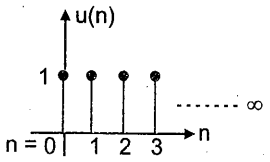


Fig. E-6

Thus we can write the limits of summation, from n = 0 to n = ∞.

$$\therefore \quad X(\omega) = \sum_{n=0}^{\infty} 1 \times e^{-j\omega n}$$

$$\therefore \quad X(\omega) = \sum_{n=0}^{\infty} \left[e^{-j\omega}\right]^n \quad ...(2)$$

Now, we have the geometric series formula,

$$\sum_{n=0}^{\infty} A^n = A^0 + A + A^2 + A^3 + \ldots = \frac{1}{1-A} \quad \textbf{if } |A| < 1$$

Let $A = e^{-j\omega}$. Thus Equation (2) can be written as,

$$X(\omega) = \frac{1}{1 - e^{-j\omega}} \quad ...(3)$$

Now, if we put ω = 0 in Equation (3) then we get, $X(\omega) = \frac{1}{0} = \infty$. That means at ω = 0; the DTFT obtained by Equation (3) is not valid. So we should obtain X (ω) for different values of 'ω' other than zero. Rearranging Equation (3) we get,

$$X(\omega) = \frac{1}{e^{-j\omega/2} \times e^{j\omega/2} - e^{-j\omega/2} \times e^{-j\omega/2}} \quad ...(4)$$

Note : $e^{-j\omega/2} \cdot e^{j\omega/2} = e^0 = 1$ and $e^{-j\omega/2} \cdot e^{-j\omega/2} = e^{-j\omega}$

Taking $e^{-j\omega/2}$ common from the denominator we get,

$$X(\omega) = \frac{1}{e^{-j\omega/2}\left[e^{j\omega/2} - e^{-j\omega/2}\right]} \quad ...(5)$$

According to Euler's identity we have,

$$\frac{e^{j\omega/2} - e^{-j\omega/2}}{2} = j \sin\frac{\omega}{2}$$

Thus Equation (5) becomes,

$$X(\omega) = \frac{1}{e^{-j\omega/2} \times 2j \sin\frac{\omega}{2}} \qquad \therefore \quad X(\omega) = \frac{e^{j\omega/2}}{2j \sin \omega/2}$$

$$\text{Thus} \quad u(n) \xleftrightarrow{\text{DTFT}} \frac{e^{j\omega/2}}{2j \sin \omega/2}$$

Prob. 3 : Obtain fourier transform of single sided exponential pulse and energy density spectrum for a = + 0.5 and a = – 0.5

$$x(n) = a^n u(n); \quad -1 < a < 1$$

Soln. : Here 'a' is some constant and $a^n u(n)$ is an exponential signal which exists from 0 to ∞. That means only for the positive values of 'n'. This is also called as right handed exponential sequence. It is shown in Fig. E-7

Note : When any signal is multiplied by unit step u (n) then the result of multiplication is positive signal. This is because u (n) exists from n = 0 to n = ∞, that means for positive values of 'n'. But since the amplitude of u (n) is 1; it will not change the amplitude of signal after multiplication. So while doing the calculations we will not consider u (n). It is just used to indicate the range of signal.

According to the definition of DTFT,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \qquad ...(1)$$

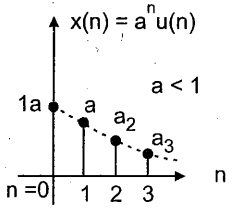


Fig. E-7

Here $x(n) = a^n u(n)$. The multiplication of a^n by u (n) indicates that the signal is present from n = 0 to n = ∞. So we will change the limits of summation as :

$$\therefore \quad X(\omega) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} \qquad ...(2)$$

Note that in Equation (2) we have not put $x(n) = a^n u(n)$. Because we have adjusted the limits accordingly.

$$\text{Now} \quad X(\omega) = \sum_{n=0}^{\infty} \left(a e^{-j\omega}\right)^n \qquad ...(3)$$

Recall the geometric series formula,

$$\sum_{n=0}^{\infty} A^n = A^0 + A^1 + A^2 + A^3 + \ldots = \frac{1}{1-A} \qquad \text{if } |A| < 1$$

Let $A = a e^{-j\omega}$, thus Equation (3) can be written as,

$$X(\omega) = \frac{1}{1 - a e^{-j\omega}}, \quad |a e^{-j\omega}| < 1$$

$$\therefore \quad a^n u(n) \xleftrightarrow{DTFT} \frac{1}{1 - a e^{-j\omega}}$$

Energy density spectrum :

Now we have,

$$X(\omega) = \frac{1}{1 - a e^{-j\omega}}, \quad |a e^{-j\omega}| < 1$$

That means $$X(\omega) = \frac{1}{1 - a e^{-j\omega}}, \quad |a| < 1$$

Note : $|e^{-j\omega}| = |\cos\omega - j\sin\omega| = \sqrt{\cos^2\omega + \sin^2\omega} = 1$

Energy density spectrum is given by,

$$S_{xx}(\omega) = |X(\omega)|^2$$

$$\therefore \quad S_{xx}(\omega) = X(\omega) \cdot X^*(\omega)$$

Here $X^*(\omega)$ is complex conjugate of $X(\omega)$.

That means, $$X^*(\omega) = \frac{1}{1 - a e^{+j\omega}}$$

$$\therefore \quad S_{xx}(\omega) = \frac{1}{1 - a e^{-j\omega}} \cdot \frac{1}{1 - a e^{+j\omega}}$$

$$\therefore \quad S_{xx}(\omega) = \frac{1}{1 - a e^{j\omega} - a e^{-j\omega} + a^2}$$

$$\therefore \quad S_{xx}(\omega) = \frac{1}{1 - a(e^{j\omega} - e^{-j\omega}) + a^2}$$

But $$e^{j\omega} - e^{-j\omega} = 2\cos\omega$$

$$\therefore \quad S_{xx}(\omega) = \frac{1}{1 - 2a\cos\omega + a^2}$$

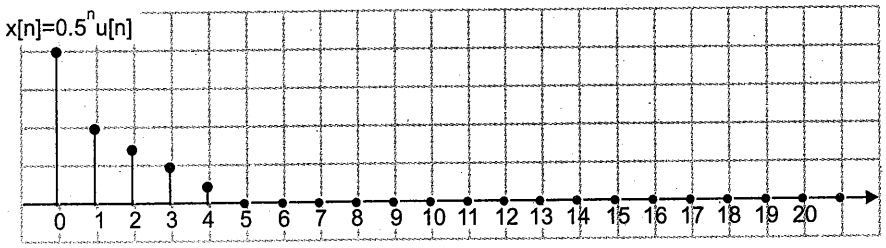
This is the equation of ESD.

While plotting the sequence, vary ω from $-\pi$ to $+\pi$.

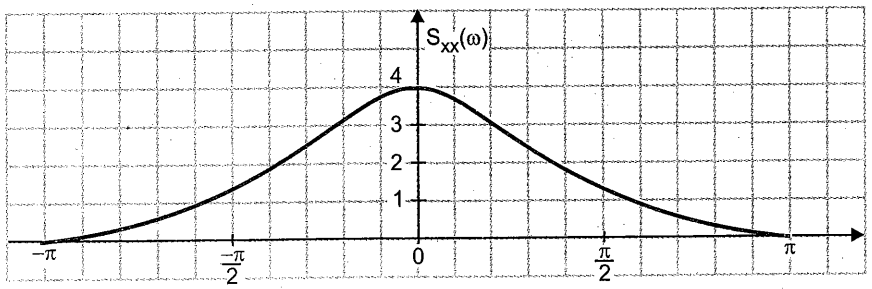
Case (i) : When $a = +0.5$

$$x(n) = 0.5^n u(n) \quad \text{and} \quad S_{xx}(\omega) = \frac{1}{1 - \cos\omega + 0.25}$$

The plot of $x(n) = (0.5)^n u(n)$ is shown in Fig. E-8(a) and its ESD is shown in Fig. E-8(b).



(a)



(b)

Fig. E-8

Case (ii) : When $a = -0.5$.

$$x(n) = (-0.5)^n u(n) \text{ and } S_{xx}(\omega) = \frac{1}{1 + \cos\omega + 0.25}$$

The plot of $x(n) = (-0.5)^n u(n)$ is shown in Fig. E-9(a) and its ESD is shown in Fig. E-9(b).

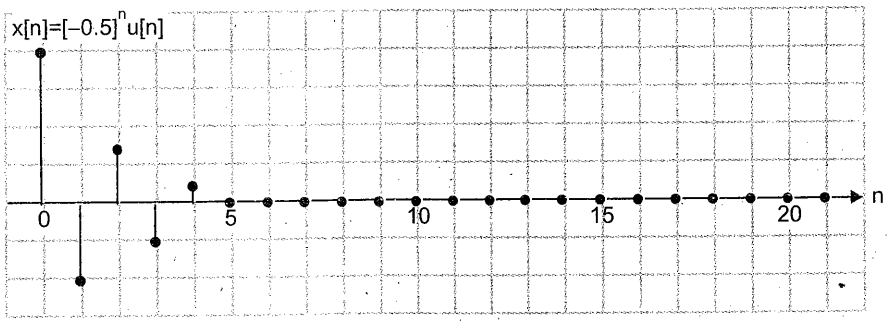


Fig. E-9 (a)

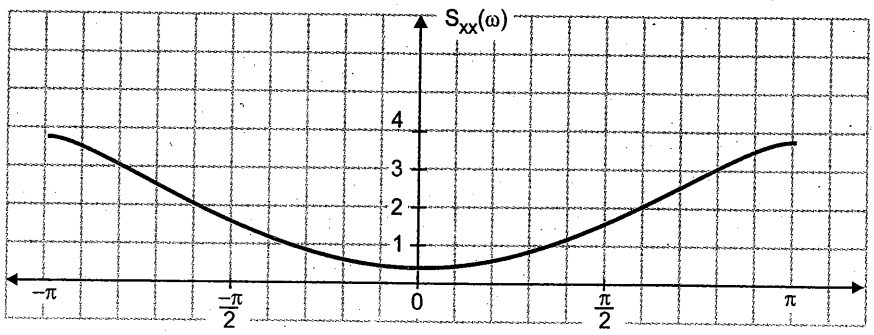


Fig. E-9 (b)

Prob. 4 : Obtain DTFT of left handed exponential signal, $x(n) = a^{-n} u(-n-1)$.

Soln. :

We know that $u(-n)$ is folded unit step and $u(-n-1)$ is folded unit step advanced by 1 sample. Similar to $u(n)$, the multiplication of signal by $u(-n-1)$ will not change the amplitude. But the range of signal becomes, $n = -1$ to $n = -\infty$; because $x(-n-1)$ is present from $n = 1$ to $n = -\infty$. It is shown in Fig. E-10.

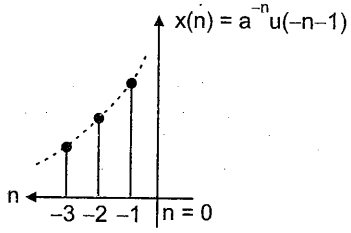


Fig. E-10

According to the definition of DTFT we have,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \qquad ...(1)$$

New limits of summation will be from $n = -1$ to $n = -\infty$.

$$\therefore \quad X(\omega) = \sum_{n=-1}^{n=-\infty} a^{-n} e^{-j\omega n} \qquad ...(2)$$

The limits are negative; to make it positive put $n = -l$. Then the limits will change as follows :

when $n = -1 \Rightarrow -1 = -l \quad \therefore \; l = 1$

and when $n = -\infty \Rightarrow -\infty = -l \quad \therefore \; l = \infty$

$$\therefore \quad X(\omega) = \sum_{l=1}^{\infty} a^{l} \cdot e^{j\omega l}$$

$$\therefore \quad X(\omega) = \sum_{l=1}^{\infty} (a\, e^{j\omega})^{l} \qquad ...(3)$$

We have the standard geometric series for the summation from '0' to '∞'. So we will adjust the summation as follows :

$$X(\omega) = \sum_{l=0}^{\infty} (a e^{j\omega})^l - 1$$

Using the formula of geometric series,

$$X(\omega) = \frac{1}{1 - a e^{j\omega}} - 1 = \frac{1 - 1 + a e^{j\omega}}{1 - a e^{j\omega}}$$

$$\therefore \quad X(\omega) = \frac{a e^{j\omega}}{1 - a e^{j\omega}}$$

$$\therefore \; a^{-n} u(-n-1) \xleftrightarrow{\text{DTFT}} \frac{a e^{j\omega}}{1 - a e^{j\omega}}$$

Prob. 5 : Obtain DTFT of double sided exponential signal.

$$x(n) = a^n u(n) + a^{-n} u(-n-1)$$

Soln. :

We have already obtained the DTFT of single sided exponential sequence separately. So by adding two DTFT's we can directly obtain the DTFT of double sided exponential signal.

$$\therefore \; a^n u(n) + a^{-n} u(-n-1) \xleftrightarrow{\text{DTFT}} \frac{1}{1 - a e^{-j\omega}} + \frac{a e^{j\omega}}{1 - a e^{j\omega}}$$

Prob. 6 : Obtain DTFT of rectangular pulse,

$$x(n) = A \quad \text{for} \quad 0 \le n \le L-1$$
$$= 0 \qquad \text{otherwise}$$

Soln. : This pulse is shown in Fig. E-11.

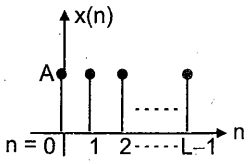


Fig. E-11

Here 'A' indicates the area of rectangular pulse and 'L' is the length of rectangular pulse.

According to the definition of DTFT we have,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \qquad \ldots(1)$$

But x (n) is present only in the range n = 0 to n = L – 1

$$\therefore \quad X(\omega) = \sum_{n=0}^{L-1} A e^{-j\omega n} \qquad \ldots(2)$$

Here area 'A' is constant and independent of limits of summation.

$$\therefore \quad X(\omega) = A \sum_{n=0}^{L-1} (e^{-j\omega})^n \qquad \ldots(3)$$

Now use the standard summation formula,

$$\sum_{k=N_1}^{N_2} A^k = \frac{A^{N_1} - A^{N_2+1}}{1-A}$$

Here $A = e^{-j\omega}$, $N_1 = 0$ and $N_2 = L-1$

Thus Equation (3) becomes,

$$X(\omega) = A\left[\frac{(e^{-j\omega})^0 - (e^{-j\omega})^{L-1+1}}{1-e^{-j\omega}}\right]$$

$$\therefore \quad X(\omega) = A\left[\frac{1-e^{-j\omega L}}{1-e^{-j\omega}}\right] \qquad \ldots(4)$$

Let us rearrange the terms to use Euler's identity.

For the numerator put $1 = e^{j\omega L/2} \times e^{-j\omega L/2}$ and $e^{-j\omega L} = e^{-j\omega L/2} \times e^{-j\omega L/2}$ and for the denominator put $1 = e^{j\omega/2} \times e^{-j\omega/2}$ and $e^{-j\omega} = e^{-j\omega/2} \times e^{-j\omega/2}$.

$$\therefore \quad X(\omega) = A\left[\frac{e^{j\omega L/2} \times e^{-j\omega L/2} - e^{-j\omega L/2} \times e^{-j\omega L/2}}{e^{j\omega/2} \times e^{-j\omega/2} - e^{-j\omega/2} \times e^{-j\omega/2}}\right]$$

$$\therefore \quad X(\omega) = A\frac{e^{-j\omega L/2}}{e^{-j\omega/2}}\left[\frac{e^{j\omega L/2} - e^{-j\omega L/2}}{e^{j\omega/2} - e^{-j\omega/2}}\right] \qquad \ldots(5)$$

According to Euler's identity we have,

$$\sin x = \frac{e^{jx} - e^{-jx}}{2} \qquad \therefore \quad e^{jx} - e^{-jx} = 2 \sin x$$

$$\therefore \quad X(\omega) = A \cdot e^{-j\omega L/2} \cdot e^{-j\omega/2}\left[\frac{2 \sin \omega L/2}{2 \sin \omega/2}\right]$$

$$\therefore \quad X(\omega) = A\, e^{-j\omega(L-1)/2}\left[\frac{\sin \omega L/2}{\sin \omega/2}\right]$$

$$\therefore \quad \text{'A'} \overset{\text{DTFT}}{\longleftrightarrow} A\, e^{-j\omega(L-1)/2}\left[\frac{\sin \omega L/2}{\sin \omega/2}\right]$$

Prob. 7 : Obtain DTFT of a signal

$$x(n) = r^n \cos(\omega_0 n)\, u(n),\ r<1$$

Soln. : According to the definition of DTFT,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \qquad ...(1)$$

The given sequence is multiplied by unit step, $u(n)$. So we can write the limits of summation from $n = 0$ to $n = \infty$.

$$\therefore \quad X(\omega) = \sum_{n=0}^{\infty} r^n \cos\omega_0 n\, e^{-j\omega n} \qquad ...(2)$$

According to Euler's identity we have,

$$\cos\omega_0 n = \frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}$$

$$\therefore \quad X(\omega) = \sum_{n=0}^{\infty} r^n \left(\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}\right) \times e^{-j\omega n}$$

$$\therefore \quad X(\omega) = \frac{1}{2}\left[\sum_{n=0}^{\infty} r^n e^{j\omega_0 n} \times e^{-j\omega n} + \sum_{n=0}^{\infty} r^n e^{-j\omega_0 n} \times e^{-j\omega n}\right]$$

$$\therefore \quad X(\omega) = \frac{1}{2}\left[\sum_{n=0}^{\infty} r^n e^{-j(\omega-\omega_0)n} + \sum_{n=0}^{\infty} r^n e^{-j(\omega+\omega_0)n}\right]$$

$$\therefore \quad X(\omega) = \frac{1}{2}\left\{\sum_{n=0}^{\infty} \left[r e^{-j(\omega-\omega_0)}\right]^n + \sum_{n=0}^{\infty} \left[r e^{-j(\omega+\omega_0)}\right]^n\right\} \qquad ...(3)$$

Use the standard geometric series formula,

$$\sum_{n=0}^{\infty} A^n = \frac{1}{1-A}$$

$$\therefore \quad X(\omega) = \frac{1}{2}\left[\frac{1}{1-re^{-j(\omega-\omega_0)}} + \frac{1}{1-re^{-j(\omega+\omega_0)}}\right] \qquad ...(4)$$

Now we will simplify Equation (4) as follows,

$$X(\omega) = \frac{1}{2}\left\{\frac{1-re^{-j(\omega+\omega_0)}+1-re^{-j(\omega-\omega_0)}}{\left[1-re^{-j(\omega-\omega_0)}\right]\left[1-re^{-j(\omega+\omega_0)}\right]}\right\} \quad ...(5)$$

Consider numerator term only. It can be simplified as,

$$\text{Numerator} = 2-re^{-j\omega}\times e^{-j\omega_0}-re^{-j\omega}\times e^{j\omega_0}$$

$$= 2-re^{-j\omega}\left(e^{-j\omega_0}+e^{j\omega_0}\right)$$

But we have, $\cos x = \frac{e^{jx}+e^{-jx}}{2}$

$$\therefore \quad \text{Numerator} = 2-re^{-j\omega}(2\cos\omega_0) = 2-2r\cos\omega_0 e^{-j\omega} \quad ...(6)$$

Now consider the denominator term,

$$\text{Denominator} = \left(1-re^{-j\omega}\times e^{j\omega_0}\right)\left(1-re^{-j\omega}\times e^{-j\omega_0}\right)$$

$$= 1-re^{-j\omega}\times e^{-j\omega_0}-re^{-j\omega}\times e^{j\omega_0}+r^2e^{-j\omega}\times e^{j\omega_0}\times e^{-j\omega}\times e^{-j\omega_0}$$

$$= 1-re^{-j\omega}\left(e^{-j\omega_0}+e^{j\omega_0}\right)+r^2\times e^{-j2\omega}$$

$$= 1-re^{-j\omega}\cdot 2\cos\omega_0+r^2e^{-j2\omega}$$

$$= 1-2r\cos\omega_0 e^{-j\omega}+r^2e^{-j2\omega} \quad ...(7)$$

Putting Equations (6) and (7) in Equation (5) we get,

$$X(\omega) = \frac{1}{2}\left[\frac{2-2r\cos\omega_0 e^{-j\omega}}{1-2r\cos\omega_0 e^{-j\omega}+r^2e^{-j2\omega}}\right]$$

$$\therefore \quad X(\omega) = \frac{1-r\cos\omega_0 e^{-j\omega}}{1-2r\cos\omega_0 e^{-j\omega}+r^2e^{-j2\omega}}$$

This is the required DTFT. From this we can write the standard DTFT pair as,

$$a^n\cos\omega_0 n\,u(n) \xleftrightarrow{\text{DTFT}} \frac{1-r\cos\omega_0 e^{-j\omega}}{1-2r\cos\omega_0 e^{-j\omega}+r^2e^{-j2\omega}}$$

Prob. 8 : Obtain DTFT and sketch the magnitude spectrum for $x(n) = u(n)-u(n-4)$.

Soln. :

We know that $u(n)$ is unit step and $u(n-4)$ is unit step delayed by 4 samples. Now sequence $x(n)$ is obtained by subtracting $u(n-4)$ from $u(n)$ as shown in Fig. E-12(a).

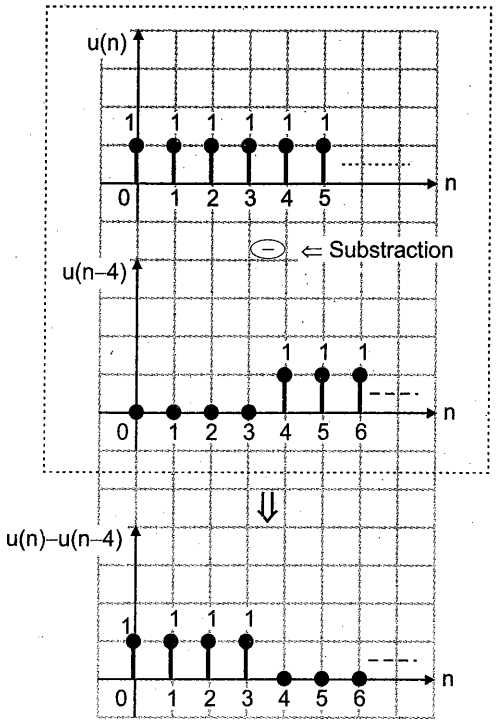


Fig. E-12 (a) : x (n) = u (n) – u (n – 4)

$$\therefore \quad x(n) = \{1, 1, 1, 1\} \qquad \text{..(1)}$$
$$\uparrow$$

Now according to the definition of DTFT,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \qquad \text{...(2)}$$

Here magnitude of x (n) is 1 and its range is from n = 0 to n = 3

$$\therefore \quad X(\omega) = \sum_{n=0}^{3} 1 \times e^{-j\omega n}$$

Expanding the summation we get,

$$X(\omega) = e^{0} + e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega}$$

$$\therefore \quad X(\omega) = 1 + e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega} \qquad \text{...(3)}$$

This is the required DTFT.

Magnitude plot : The magnitude of $X(\omega)$ is given by,

$$|X(\omega)| = \sqrt{[\text{Real part of } X(\omega)]^2 + [\text{Imaginary part of } X(\omega)]^2}$$

We have, $X(\omega) = 1 + e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega}$

Consider the term, $e^{-j\omega}$ using Euler's identity,

$$e^{-j\omega} = \cos\omega - j\sin\omega$$

$$\therefore \left| e^{-j\omega} \right| = \sqrt{\cos^2\omega + \sin^2\omega} = 1$$

similarly, $\left| e^{-j2\omega} \right| = \sqrt{\cos^2 2\omega + \sin^2 2\omega} = 1$

and $\left| e^{-j3\omega} \right| = \sqrt{\cos^2 3\omega + \sin^2 3\omega} = 1$

$$\therefore |X(\omega)| = 1 + 1 + 1 + 1$$

$$\therefore |X(\omega)| = 4 \quad \text{for all values of } \omega$$

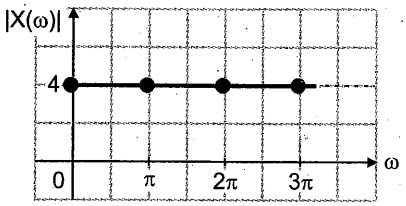


Fig. E-12 (b) : Magnitude spectrum

Now magnitude spectrum is plot of '$X(\omega)$' versus 'ω'. The sketch of magnitude spectrum is shown in Fig. E-12 (b).

Note : The phase spectrum is plot of $\angle X(\omega)$ versus 'ω'. And $\angle X(\omega)$ is given by,

$$\tan^{-1}\frac{\text{Im}(\omega)}{\text{Re}(\omega)}$$

Prob. 9 : Determine the discrete time signal $x(n)$ whose fourier transform is,

$$X(e^{j\omega}) = 1, \quad -\omega_c \le \omega \le \omega_c$$

$$= 0, \quad -\omega_c \le |\omega| \le \pi$$

Soln. : The given signal is plotted as shown in Fig. E-13

We have to obtain $x(n)$ which means we have to take inverse fourier transform (IFT) of given signal.

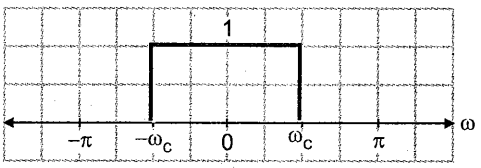


Fig. E-13 : Plot of $X(e^{j\omega})$

$$\therefore x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} cX(e^{j\omega}) \times e^{j\omega n}\, d\omega \quad ...(1)$$

But $X(e^{j\omega}) = 1$ for the range $-\omega_c$ to ω_c

$$\therefore x(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1 \times e^{j\omega n}\, d\omega$$

$$\therefore \quad x(n) = \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_c}^{\omega_c} = \frac{1}{2\pi jn}\left[e^{j\omega_c n} - e^{-j\omega_c n}\right]$$

According to Euler's identity,

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j} \qquad \therefore \quad x(n) = \frac{1}{2\pi n} \times 2 \sin \omega_c n$$

$$\therefore \quad x(n) = \frac{\sin \omega_c n}{\pi n}$$

Summary of standard DTFT pairs : Table E-3 shows DTFT of standard signals.

Table E-3

Sr. No.	Signal	Discrete time	DTFT
1.	Unit impulse	$\delta(n)$	1
2.	Unit step	$u(n)$	$\frac{e^{j\frac{\omega}{2}}}{2j \sin\frac{\omega}{2}}$
3.	Right handed exponential	$a^n u(n)$	$\frac{1}{1 - a e^{-j\omega}}$
4.	Left handed exponential	$a^{-n} u(-n-1)$	$\frac{a e^{j\omega}}{1 - a e^{j\omega}}$
5.	Double sided exponential	$a^n u(n) + a^{-n} u(-n-1)$	$\frac{1}{1 - a e^{-j\omega}} + \frac{a e^{j\omega}}{1 - a e^{j\omega}}$
6.	Rectangular pulse	$x(n) = A$	$A e^{\frac{-j\omega(L-1)}{2}} \left[\frac{\sin\frac{\omega L}{2}}{\sin\frac{\omega}{2}}\right]$

1.5 Symmetry Properties of F.T. :

Let us say $x(n)$ and $X(\omega)$ are complex valued functions. That means $x(n)$ and $X(\omega)$ can be represented in terms of real part and imaginary part.

$$\therefore \quad x(n) = x_R(n) + j\, x_I(n) \qquad \ldots(3)$$

Here suffix R indicates real part and I indicates imaginary part.

Similarly we can write $X(\omega)$ as,

$$X(\omega) = X_R(\omega) + X_I(\omega) \qquad \ldots(4)$$

Now according to the definition of DTFT we have,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \quad ...(5)$$

According to Euler's identity we have,

$$e^{-j\omega n} = \cos \omega n - j \sin \omega n \quad ...(6)$$

Putting Equations (6) and (3) in Equation (5),

$$X(\omega) = \sum_{n=-\infty}^{\infty} \left[x_R(n) + j\, x_I(n) \right] [\cos \omega n - j \sin \omega n]$$

$$\therefore \quad X(\omega) = \sum_{n=-\infty}^{\infty} \left[x_R(n) \cos \omega n - j\, x_R(n) \sin \omega n + j\, x_I(n) \cos \omega n - j^2 x_I(n) \sin \omega n \right]$$

Putting $j^2 = -1$ and rearranging the terms,

$$X(\omega) = \sum_{n=-\infty}^{\infty} \left[x_R(n) \cos \omega n + x_I(n) \sin \omega n \right] - j \left[x_R(n) \sin \omega n - x_I(n) \cos \omega n \right]$$

Now separating real and imaginary terms,

$$X_R(\omega) = \sum_{n=-\infty}^{\infty} x_R(n) \cos \omega n + x_I(n) \sin \omega n \quad ...(7)$$

and $$X_I(\omega) = -\sum_{n=-\infty}^{\infty} x_R(n) \sin \omega n - x_I(n) \cos \omega n \quad ...(8)$$

Equations (7) and (8) gives real and imaginary parts of $X(\omega)$. Similarly the real and imaginary parts of $x(n)$ can be calculated using definition of IFT.

$$\therefore \quad x_R(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[X_R(\omega) \cos \omega n - X_I(\omega) \sin \omega n \right] d\omega \quad ...(9)$$

and $$x_I(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[X_R(\omega) \sin \omega n + X_I(\omega) \cos \omega n \right] d\omega \quad ...(10)$$

Now we will consider different cases as follows :

Case (i) : Real signal :

If $x(n)$ is real then $x_R(n) = x(n)$ and $x_I(n) = 0$.

Thus Equation (7) becomes,

$$X_R(\omega) = \sum_{n=-\infty}^{\infty} x_R(n) \cos \omega n$$

but $x_R(n) = x(n)$

$$\therefore \quad X_R(\omega) = \sum_{n=-\infty}^{\infty} x(n) \cos \omega n \qquad ...(11)$$

Similarly Equation (8) becomes,

$$X_I(\omega) = -\sum_{n=-\infty}^{\infty} x(n) \sin \omega n \qquad ...(12)$$

Now we have $\cos(-\omega n) = \cos \omega n$. Thus from Equation (11),

$$X_R(-\omega) = X_R(\omega).$$

So X_R is even.

Now we have $\sin(-\omega n) = -\sin \omega n$. Thus from Equation (12),

$$X_I(-\omega) = -X_I(\omega).$$

So $X_I(\omega)$ is odd.

Thus combinely we can write,

$$X^*(\omega) = X(-\omega) \qquad ...(13)$$

Here '*' indicates complex conjugate.

Symmetry of magnitude and phase :

The spectrum of real discrete time signal has Hermitian symmetry.

Now magnitude, $|X(\omega)| = \sqrt{X_R^2(\omega) + X_I^2(\omega)}$ ...(14)

and phase, $\angle X(\omega) = \tan^{-1} \dfrac{X_I(\omega)}{X_R(\omega)}$ ...(15)

The magnitude spectrum has even symmetry and phase spectrum has odd symmetry.

Case (ii) : Real and Even signal :

When $x(n)$ is real and even then,

$$x(-n) = x(n)$$

Now we have $\cos(-\omega n) = \cos \omega n$ and $\sin(-\omega n) = -\sin \omega n$. Thus $x(n) \cos \omega n$ is even and $x(n) \sin \omega n$ is odd.

According to the definition of DTFT,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

Separating the summation,

$$X(\omega) = \sum_{n=-\infty}^{-1} x(n) e^{-j\omega n} + x(0) + \sum_{n=1}^{\infty} x(n) e^{-j\omega n}$$

Put $l = -n$ for first summation.

$$\therefore \quad X(\omega) = \sum_{l=1}^{\infty} x(-l) e^{j\omega l} + x(0) + \sum_{n=1}^{\infty} x(n) e^{-j\omega n}$$

$$\therefore \quad X(\omega) = x(0) + \sum_{l=1}^{\infty} x(-l) e^{j\omega l} + \sum_{n=1}^{\infty} x(n) e^{-j\omega n} \qquad ...(16)$$

In Equation (14), just for simplicity we have used different variables 'l' and 'n'. But the meaning remains the same. Now the two summation terms in Equation (16) are symmetric to each other since we have $x(-n) = x(n)$. Thus addition of these two terms can be written as,

$$X(\omega) = x(0) + \sum_{n=1}^{\infty} x(n) (e^{-j\omega n} + e^{j\omega n})$$

But $e^{-j\omega n} + e^{j\omega n} = 2 \cos \omega n$

$$\therefore \quad X(\omega) = x(0) + 2 \sum_{n=1}^{\infty} x(n) \cos \omega n \qquad ...(17)$$

Since imaginary term is zero $X_I(\omega) = 0$

$$\therefore \quad x(n) = \frac{1}{\pi} \int_{0}^{\pi} X_R(\omega) \cos \omega n \, d\omega \qquad ...(18)$$

Case (iii) : Real and odd signals :

When the sequence $x(n)$ is real and odd that means, $x(-n) = -x(n)$ then $x(n) \cos \omega n$ is odd and $x(n) \sin \omega n$ is even. That means $X_R(\omega) = 0$

$$\therefore \quad X_I(\omega) = -2 \sum_{n=1}^{\infty} x(n) \sin \omega n \qquad ...(19)$$

This is odd function.

$$\text{In this case,} \quad x(n) = -\frac{1}{\pi}\int_{0}^{\pi} X_I(\omega)\sin \omega n \, d\omega \qquad \ldots(20)$$

Case (iv) : Pure imaginary signal :

Here $x_R(n) = 0$ and $x(n) = j\,x_I(n)$

1. $$X_R(\omega) = \sum_{n=-\infty}^{\infty} \left[x_R(n)\cos \omega n + x_I(n)\sin \omega n \right]$$

But $x_R(n) = 0$

$$\therefore \quad X_R(\omega) = \sum_{n=-\infty}^{\infty} x_I(n)\sin \omega n \qquad \ldots(21)$$

This is odd function.

2. $$X_I(\omega) = -\sum_{n=-\infty}^{\infty} x_R(n)\sin \omega n - x_I(n)\cos \omega n$$

$$= \sum_{n=-\infty}^{\infty} x_I(n)\cos \omega n \qquad \ldots(22)$$

This is even function

3. $$x_I(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} [X_R(\omega)\sin \omega n + X_I(\omega)\cos \omega n]\, d\omega$$

As seen from Equation (21), if $x_I(n)$ is odd, sin ωn being odd, so, $X_R(\omega)$ will be also odd. On similar lines, as seen from Equation (22), if $X_I(n)$ is even, cos ωn being even, so, $X_I(\omega)$ will be also even.

Summary of symmetry property :

The summary of symmetry property is shown in Fig. E-14.

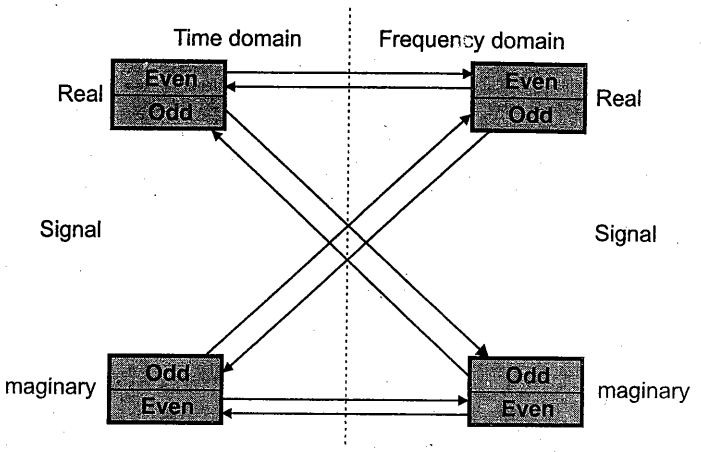


Fig. E-14 : Summary of symmetry property

This summary is shown in Table E-4.

Table E-4

Time domain signal x (n)	Frequency domain signal (DTFT)
1. Real and even	real and even
2. Real and odd	odd and imaginary
3. Pure imaginary	odd and real
4. Imaginary and even	Imaginary and even

1.6 F.T. Theorems (Properties of DTFT) :

In this section, we will study the properties of DTFT. We know that $x(n)$ and $X(\omega)$ are fourier transform pairs and denoted as,

$$x(n) \overset{DTFT}{\longleftrightarrow} X(\omega)$$

1.6.1 Linearity :

Statement :

If $x_1(n) \overset{DTFT}{\longleftrightarrow} X_1(\omega)$ and $x_2(n) \overset{DTFT}{\longleftrightarrow} X_2(\omega)$ then ,

$$a_1 x_1(n) + a_2 x_2(n) \overset{DTFT}{\longleftrightarrow} a_1 X_1(\omega) + a_2 X_2(\omega).$$

Proof : According to the definition of DTFT

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

Here input sequence $x(n) = a_1 x_1(n) + a_2 x_1(n)$

$$\therefore \quad X(\omega) = a_1 \sum_{n=-\infty}^{\infty} [a_1 x_1(n) + a_2 x_2(n)]\, e^{-j\omega n}$$

$$\therefore \quad X(\omega) = a_1 \sum_{n=-\infty}^{\infty} x_1(n)\, e^{-j\omega n} + a_2 \sum_{n=-\infty}^{\infty} x_2(n)\, e^{-j\omega n}$$

Comparing each summation term with the definition of DTFT we can write,

$$X(\omega) = a_1 X_1(\omega) + a_2 X_2(\omega)$$

Meaning : Fourier transform of a linear combination of two or more signals is equal to the same linear combination of fourier transform of individual signal.

1.6.2 Time Shifting :

Statement :

$$\text{if } x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\omega) \text{ then,}$$

$$x(n-k) \overset{\text{DTFT}}{\longleftrightarrow} e^{-j\omega k} X(\omega)$$

Proof : According to definition of DTFT

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

Here input $x(n) = x(n-k)$

$$\text{DTFT} = x(n-k) = \sum_{n=-\infty}^{\infty} x(n-k)\, e^{-j\omega n}$$

Put $n - k = m \Rightarrow n = m + k$

Limits will change as follows :

when $n = -\infty \quad \Rightarrow \quad -\infty - k = m \quad \therefore m = -\infty$

and when $n = +\infty \quad \Rightarrow \quad +\infty - k = m \quad \therefore m = +\infty$

$$\text{DTFT}\left\{x(m-k)\right\} = \sum_{m=-\infty}^{\infty} x(m)\,e^{-j\omega(m+k)}$$

$$= \sum_{m=-\infty}^{\infty} x(m)\,e^{-j\omega m}\cdot e^{-j\omega k}$$

$$= e^{-j\omega k} \sum_{m=-\infty}^{\infty} x(m)\,e^{-j\omega n}$$

Comparing the summation term with definition of DTFT;

$$\text{DTFT}\left\{x(n-k)\right\} = e^{-j\omega k}\cdot X(\omega)$$

$$\therefore \quad x(n-k) \overset{\text{DTFT}}{\longleftrightarrow} e^{-j\omega k} X(\omega)$$

Meaning : If a signal is shifted in time domain by k samples then the magnitude spectrum is unchanged but the phase spectrum is changed by amount $(-\omega k)$.

1.6.3 Frequency Shifting :

Statement : If $x(n) \overset{FT}{\longleftrightarrow} X(\omega)$ then $e^{j\omega_o n}\,x(n) \overset{FT}{\longleftrightarrow} X(\omega-\omega_o)$

Proof : From the definition of DTFT.

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\,e^{-j\omega n}$$

$$\therefore \quad \text{DTFT}\left\{e^{j\omega_o n}\,x(n)\right\} = \sum_{n=-\infty}^{\infty} e^{-j\omega_o n}\,x(n)\,e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\,e^{-j(\omega-\omega_o)n}$$

Comparing the summation term with the definition of DTFT

$$\text{DTFT}\left\{ e^{j\omega_o n} x(n) \right\} = X(\omega - \omega_o)$$

$$\therefore \quad e^{j\omega_o n} \cdot x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\omega - \omega_o)$$

Meaning : Multiplying sequence $x(n)$ by $e^{j\omega_o n}$ in time domain corresponds to frequency shift by ω_o in frequency domain.

1.6.4 Time Reversal :

Statement : If $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\omega)$ then,

$$x(-n) \overset{\text{DTFT}}{\longleftrightarrow} X(-\omega)$$

Proof : According to definition of DTFT,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$\therefore \text{DTFT}\left\{ x(-n) \right\} = \sum_{n=-\infty}^{\infty} x(-n) e^{-j\omega n}$$

Put $-n = m$

$$\therefore \text{DTFT}\left\{ x(-n) \right\} = \sum_{m=-\infty}^{\infty} x(m) e^{j\omega m}$$

$$= \sum_{n=-\infty}^{\infty} x(m) e^{-j(-\omega)m}$$

Comparing summation term with the definition we get,

$$\text{DTFT}\left\{ x(-n) \right\} = X(-\omega)$$

$$\therefore \quad x(-n) \overset{\text{DTFT}}{\longleftrightarrow} X(-\omega)$$

Meaning : If we fold the sequence in time domain then the magnitude spectrum is unchanged but the polarity of phase spectrum is changed.

1.6.5 Differentiation :

Statement : If $x(n) \xleftrightarrow{DTFT} X(\omega)$ then ,

$$n\,x(n) \xleftrightarrow{DTFT} j\frac{d}{d\omega}X(\omega)$$

Proof : According to the definition of DTFT we have,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\,e^{-j\omega n} \qquad ...(1)$$

Taking differentiation on both sides,

$$\frac{dX(\omega)}{d\omega} = \frac{d}{d\omega}\left[\sum_{n=-\infty}^{\infty} x(n)\,e^{-j\omega n}\right] = \sum_{n=-\infty}^{\infty} x(n)\frac{d}{d\omega}e^{-j\omega n}$$

$$\therefore \quad \frac{dX(\omega)}{d\omega} = -j\sum_{n=-\infty}^{\infty} n\,x(n)\,e^{-j\omega n}$$

Multiplying both sides by j,

$$j\frac{dX(\omega)}{d\omega} = -j\times j\sum_{n=-\infty}^{\infty} n\,x(n)\,e^{-j\omega n}$$

$$\therefore \quad j\frac{dX(\omega)}{d\omega} = -j^2\sum_{n=-\infty}^{\infty} n\,x(n)\,e^{-j\omega n} \qquad ...(2)$$

Here $j^2 = -1$ and comparing R.H.S. with Equation (1) we get,

$$j\frac{dX(\omega)}{d\omega} = FT\{n\,x(n)\}$$

$$\therefore \quad n\,x(n) \xleftrightarrow{DTFT} j\frac{dX(\omega)}{d\omega}$$

Hence proved.

1.6.6 Parseval's Theorem :

Statement : If $x_1(n) \overset{FT}{\longleftrightarrow} X_1(\omega)$ and $x_2(n) \overset{FT}{\longleftrightarrow} X_2(\omega)$ then,

$$\sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\omega)\, X_2^*(\omega)\, d\omega.$$

Proof : We have the definition of inverse DTFT

$$x_2(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_2(\omega)\, e^{-j\omega n}\, d\omega$$

$$\therefore \quad x_2^*(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_2^*(\omega)\, e^{-j\omega n}\, d\omega \qquad \ldots(1)$$

Now consider L.H.S. term that means the term $\sum_{n=-\infty}^{\infty} x_1(n) \cdot x_2^*(n)$ and put Equation (1) in place of $x_2^*(n)$

$$\therefore \sum_{n=-\infty}^{\infty} x_1(n) \cdot x_2^*(n) = \sum_{n=-\infty}^{\infty} x_1(n) \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} x_2^*(\omega)\, e^{-j\omega n}\, d\omega$$

Changing the order of summation and integration,

$$\sum_{n=-\infty}^{\infty} x_1(n) \cdot x_2^*(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_2^*(\omega)\, d\omega \sum_{n=-\infty}^{\infty} x_1(n)\, e^{-j\omega n}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X_2^*(\omega) \cdot X(\omega) \cdot d\omega \qquad \ldots(2)$$

Meaning : The energy of discrete signal is given by,

$$E = \sum_{n=-\infty}^{\infty} x_1(n) \cdot x_2^*(n) = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

Thus from Equation (2) energy can also be expressed as,

$$E = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2\, d\omega$$

1.6.7 Convolution Theorem :

Statement :

If $x_1(n) \overset{DTFT}{\longleftrightarrow} X_1(\omega)$ and $x_2(n) \overset{DTFT}{\longleftrightarrow} X_2(\omega)$ then,

$$x_1(n) * x_2(n) \overset{DTFT}{\longleftrightarrow} X_1(\omega) \cdot X_2(\omega)$$

Proof : According to the definition of DTFT.

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$\therefore \quad DTFT\left\{x_1(n) * x_2(n)\right\} = \sum_{n=-\infty}^{\infty} \left\{x_1(n) * x_2(n)\right\} e^{-j\omega n} \qquad ...(1)$$

But according to the definition of linear convolution,

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k) \qquad ...(2)$$

Putting this value in Equation (1) we get,

$$DTFT\left\{x_1(n) * x_2(n)\right\} = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} [x_1(k)\, x_2(n-k)]\, e^{-j\omega n}$$

Changing the order of summations,

$$DTFT\left\{x_1(n) * x_2(n)\right\} = \sum_{k=-\infty}^{\infty} x_1(k) \sum_{n=-\infty}^{\infty} x_2(n-k)\, e^{-j\omega n}$$

Put $n - k = m$ i.e. $n = m + k$ in the second summation

$$DTFT\left\{x_1(n) * x_2(n)\right\} = \sum_{k=-\infty}^{\infty} x_1(k) \sum_{m=-\infty}^{\infty} x_2(m)\, e^{-j\omega(m+k)}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) \sum_{m=-\infty}^{\infty} x_2(m)\, e^{-j\omega m} \cdot e^{-j\omega k}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)\, e^{-j\omega k} \sum_{m=-\infty}^{\infty} x_2(m)\, e^{-j\omega m}$$

Comparing each summation with the definition of DTFT;

$$\text{DTFT}\left\{x_1(n) * x_2(n)\right\} = X_1(\omega)\cdot X_2(\omega)$$

$$\therefore \quad x_1(n) * x_2(n) \overset{\text{DTFT}}{\longleftrightarrow} X_1(\omega)\cdot X_2(\omega)$$

Meaning : Convolution of two signals in time domain is equivalent to multiplication in frequency domain.

1.6.8 Windowing (Modulation) Theorem :

Statement :

$$\text{If} \quad x_1(n) \overset{\text{FT}}{\longleftrightarrow} X_1(\omega) \quad \text{and} \quad x_2(n) \overset{\text{FT}}{\longleftrightarrow} X_2(\omega) \text{ then,}$$

$$x_1(n)\cdot x_2(n) \overset{\text{FT}}{\longleftrightarrow} X_3(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)\, X_2(\omega-\lambda)\, d\lambda$$

Proof : From the definition of DTFT

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \qquad \ldots(1)$$

$$\therefore \text{DTFT}\left\{x_1(n)\cdot x_2(n)\right\} = \sum_{n=-\infty}^{\infty} [x_1(n)\cdot x_2(n)]\, e^{-j\omega n} \qquad \ldots(2)$$

Now according to the definition of inverse DTFT,

$$x_1(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)\, e^{j\omega n}\, d\omega$$

Let us use variable λ in place of ω

$$\therefore \quad x_1(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)\, e^{j\lambda n}\, d\lambda \qquad \ldots(3)$$

Putting this value in place of $x_1(n)$ we get,

$$\text{DTFT}\left\{x_1(n)\cdot x_2(n)\right\} = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)\, e^{j\lambda n}\, d\lambda \cdot x_2(n)\, e^{-j\omega n}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda) \left[\sum_{n=-\infty}^{\infty} x_2(n) e^{-j(\omega-\lambda)n} \right] \cdot d\lambda$$

Comparing the summation term with the definition of DTFT we get,

$$\text{DTFT}\left\{ x_1(n) \cdot x_2(n) \right\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda) \cdot X_2(\omega - \lambda)\, dX$$

Hence proof.

Meaning : Multiplication of two discrete time sequences is equivalent to convolution of their fourier transforms.

This meaning is valid for continuous time signals. But for discrete time signals, the fourier transform is a sum while inverse fourier transform is an integral with a periodic integrand. Thus discrete time convolution of sequences is equivalent to multiplication of corresponding periodic fourier transforms and multiplication of sequences is equivalent to transforms and multiplication of sequencs is equivalent to periodic convolution of corresponding fourier transform.

1.6.9 Summary of Properties of DTFT :

Table E-5 shows summary of properties of DTFT.

Table E-5

Name of property	Time domain representation	Frequency domain representation
1. Linearity	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(\omega) + a_2 X_2(\omega)$
2. Time shifting	$x(n-k)$	$e^{-j\omega k} X(\omega)$
3. Time reversal	$x(-n)$	$X(-\omega)$
4. Convolution theorem	$x_1(n) * x_2(n)$	$X_1(\omega) \cdot X_2(\omega)$
5. Correlation theorem	$r_{x_1 x_2}(n)$	$X_1(\omega) \cdot X_2(-\omega)$
6. Frequency shifting	$e^{j\omega_0 n} \cdot x(n)$	$X(\omega - \omega_0)$
7. Multiplication of two sequences	$x_1(n) \cdot x_2(n)$	$X_3(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda) X_2(\omega - \lambda)\, d\lambda$
8. Parseval's theorem	$x_1(n) \cdot x_2^*(n)$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\omega) X_2^*(\omega)\, d\omega$
9. Differentiation in Frequency domain	$nx(n)$	$j\frac{d}{d\omega} X(\omega)$

1.7 Relationship between Fourier Transform and Z Transform :

The fourier transform of discrete time signal can be expressed as,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \quad \ldots(1)$$

While Z-transform of sequence x(n) is given by,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \quad \ldots(2)$$

As 'Z' is a complex variable; it can be represented as,

$$Z = r e^{j\omega} \quad \ldots(3)$$

Here 'r' is the magnitude of Z; that means $r = |Z|$ and 'ω' is the angle of Z, that means $\omega = \angle Z$

Substitute $Z = r e^{j\omega}$ in Equation (2),

$$\therefore X(Z) = \sum_{n=-\infty}^{\infty} x(n) \left[r e^{j\omega} \right]^{-n}$$

$$\therefore X(Z) = \sum_{n=-\infty}^{\infty} \left[x(n) r^{-n} \right] e^{-j\omega n} \quad \ldots(4)$$

This Equation (4) shows that X(Z) is a fourier transform (FT) of a sequence $[x(n) r^{-n}]$. Now if we evaluate Equation (4) on the unit circle that means $|Z| = 1$ then we an write,

$$r = |Z| \Rightarrow r = 1 \qquad \therefore r^{-n} = (1)^{-n} = 1$$

Thus if X(Z) is evaluated on unit circle then Equation (4) becomes,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \quad \ldots \text{ if } Z = e^{j\omega} \text{ and } |Z| = 1 \quad \ldots(5)$$

The R.H.S. of Equation (5) is nothing else but the fourier transform of x(n). Thus *fourier transform of a signal is basically equal to the Z-transform of a signal evaluated on a unit circle.*

Frequency scale on the unit circle :

We know that $Z = r e^{j\omega}$ ...(1)

Equation (1) represents a phasor with magnitude 'r' and angle 'ω'. Now on the unit circle $r = |Z| = 1$. The magnitude is expressed as :

$$|Z| = \sqrt{R_e^2\{Z\} + I_m^2\{Z\}} \quad \ldots(2)$$

and angle is expressed as, $\angle Z = \tan^{-1}\left[\frac{I_m\{Z\}}{R_e\{Z\}}\right]$

$$\therefore \quad \omega = \tan^{-1}\left[\frac{I_m\{Z\}}{Re\{Z\}}\right] \qquad ...(3)$$

On the unit circle, 'ω' that means $\angle Z$ is going to take values from 0 to 2π radians. However because FT = ZT on the unit circle; these values of 'ω' correspond to frequencies. Thus unit circle becomes frequency axis.

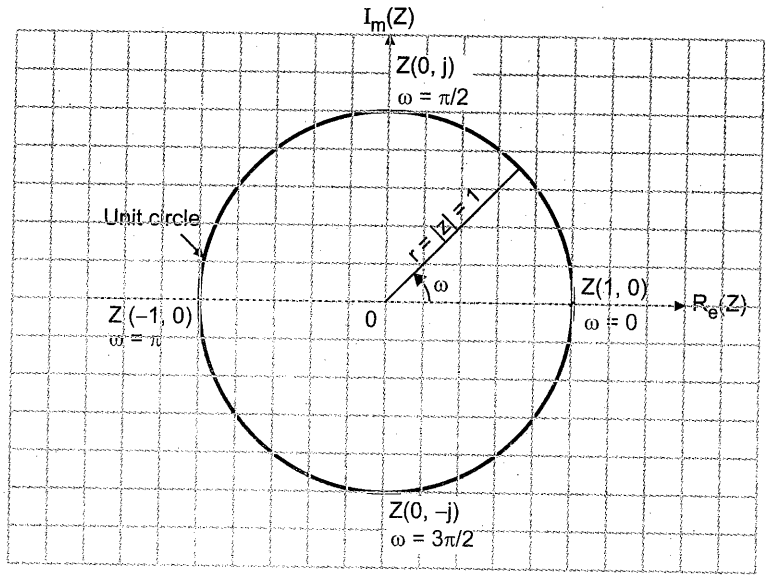


Fig. E-15 : Frequency scale on unit circle

The range of frequency 'ω' is from 0 to 2π or $-\pi$ to π. Note that 'zero' and '2π' points correspond to the same point on the unit circle. Also π and $-\pi$ points correspond to the same point $(-1, 0j)$ on the unit circle.

If the maximum frequency is π then the upper part of unit circle represents the actually used frequency scale that is 0 to π. This frequency scale on unit circle is shown in Fig. E-15.

Theory Questions

Q. 1 State and prove differentiation property of fourier transform.

Q. 2 State and prove convolution property of fourier transform.

Q. 3 State the analysis and synthesis relationships for discrete time fourier transform and show that they are inverse of each other.

Q. 4 State and prove symmetry properties of discrete time fourier transform (DTFT).

Q. 5 State and explain the sufficient conditions for the existence of a fourier transform of a discrete-time singal.

Q. 6 State and explain the convolution property of fourier transform of a DT signal.

Q. 7 State and prove following properties of fourier transform :

(i) Periodicity

(ii) Time shifting

Q. 8 State and prove windowing theorem of fourier transform (DTFT).

Unsolved Numericals

Q. 1 Find fourier transform of $x(n) = \begin{cases} \frac{1}{3} & \text{for } -1 \le n \le 1 \\ 0 & \text{otherwise} \end{cases}$

Ans. :

$$X(\omega) = \frac{1}{3}[1 + 2\cos\omega]$$

Q. 2 Obtain fourier transform of $x(n) = a^{|n|}$, $-1 < a < 1$

Hint : $a^{|n|} = \begin{cases} a^n u(n) & \text{For } n \ge 0 \\ a^n u(-n-1) & \text{for } n < 0 \end{cases}$

Ans. :

$$X(\omega) = \frac{1 - a^2}{1 - 2a\cos\omega + a^2}$$

Q. 3 Using DTFT obtain convolution of,

$$x(n) = \{1, \underset{\uparrow}{1}, 1\} \text{ and } h(n) = \{1, \underset{\uparrow}{1}, 1\}$$

Ans. :

$$Y(n) = \{1, 2, \underset{\uparrow}{3}, 2, 1\}$$

Q. 4 Obtain fourier transform of $h(n) = \left\{\frac{1}{2}, 1, \frac{1}{2}\right\}$

Ans. :

$$H(\omega) = (1 + \cos\omega)\, e^{-j\omega}$$

Q. 5 Using DTFT obtain magnitude and phase spectrum of,

$$\delta(n-k) = \begin{cases} 1 & \text{for } n = k \\ 0 & \text{otherwise} \end{cases}$$

Ans. :

$$|D(\omega)| = 1, \quad \theta = -\omega k$$

❑❑❑

Discrete Fourier Transform

Section No.	Name of the Topic
1.1	Introduction
1.2	Definition of DFT
1.3	Properties of DFT
1.4	Linear Filtering Methods based on DFT
1.5	Frequency Analysis of Signals using DFT
1.6	Relationship between DFT and Z-Transform

• Code : DSP6

1.1 Introduction :

What is DFT ?

According to the definition of DTFT,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \qquad ...(1)$$

We know that $X(\omega)$ is fourier transform of discrete time signal $x(n)$. The range of 'ω' is from 0 to 2π or $-\pi$ to π. Thus it is not possible to compute $X(\omega)$ on digital computer. Because in Equation (1) the range of summation is from $-\infty$ to $+\infty$. But if we make this range finite then it is possible to do these calculations on digital computer.

When a fourier transform is calculated only at discrete points then it is called as discrete fourier transform (DFT).

1.1.1 Sampling the F.T. :

If we have aperiodic time domain signal then discrete time fourier transform (DTFT) is obtained. But DTFT is continuous in nature and its range is from $-\infty$ to $+\infty$. Then a finite range sequence is obtained by extracting a particular portion from such infinite sequence.

Now $X(\omega)$ is a continuous signal. A discrete signal is obtained by sampling $X(\omega)$. A particular sequence which is extracted from infinite sequence is called as windowed sequence. A windowed signal is considered as periodic signal. We can obtain periodic extension of this signal. This periodic extension in frequency domain is called as Discrete Fourier Transform (DFT). From this original sequence, x(n) is obtained by performing inverse process which is Inverse Discrete Fourier Transform (IDFT).

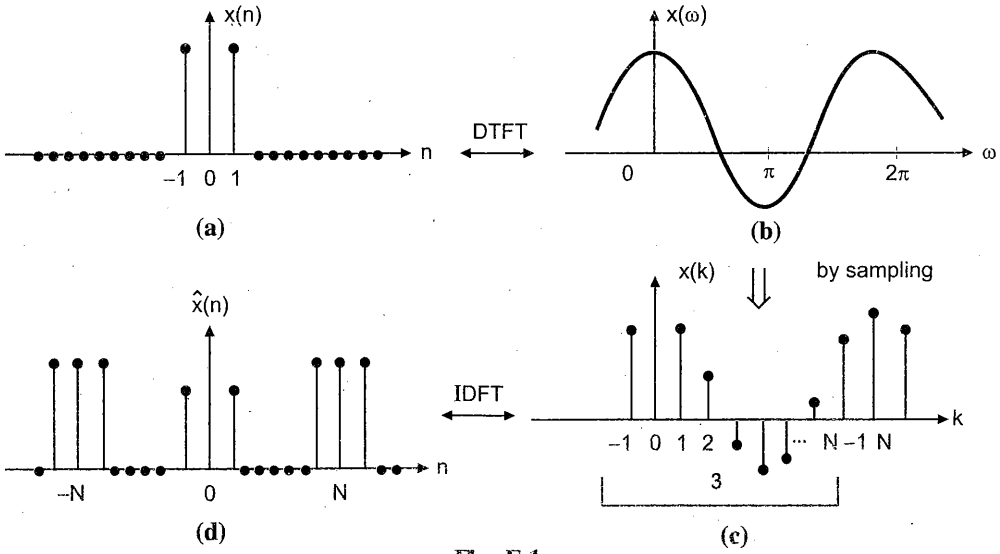


Fig. F-1

This process is explained graphically as shown in Fig. F-1. Fig. F-1(a) shows discrete time signal $x(n)$. By taking DTFT of $x(n)$; $X(\omega)$ is obtained as shown in Fig. F-1(b). The sampled version of $X(\omega)$ is denoted by $X(k)$ which is called as DFT. It is shown in Fig. F-1(c). By performing IDFT; original is obtained. It is denoted by $\hat{x}(n)$. It is shown in Fig. F-1(d). It is periodic extension of sequence $x(n)$.

Here 'N' denotes the number of samples of input sequence and the number of frequency points in the DFT output.

1.2 Definition of DFT :

The Discrete Fourier Transform :

Definition of DFT :

It is a finite duration discrete frequency sequence which is obtained by sampling one period of fourier transform. Sampling is done at 'N' equally spaced points over the period extending from $\omega = 0$ to $\omega = 2\pi$.

Mathematical Equations :

The DFT of discrete sequence $x(n)$ is denoted by $X(k)$. It is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j2\pi kn/N} \quad \ldots(1)$$

Here $k = 0, 1, 2 \ldots N-1$

Since this summation is taken for 'N' points; it is called as 'N' point DFT.

We can obtain discrete sequence $x(n)$ from its DFT. It is called as inverse discrete fourier transform (IDFT). It is given by,

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} \quad \ldots(2)$$

Here $n = 0, 1, 2, \ldots N-1$

This is called as 'N' point IDFT.

Now we will define the new term 'W' as,

$$W_N = e^{-j2\pi/N} \quad \ldots(3)$$

This is called as twiddle factor. Twiddle factor makes the computation of DFT a bit easy and fast.

Using twiddle factor we can write equations of DFT and IDFT as follows :

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad \ldots(4)$$

Here $n = 0, 1, 2 \ldots N-1$

$$\text{and} \quad x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn} \qquad ...(5)$$

Here $n = 0, 1, 2, \ldots N-1$

1.2.1 Relationship between DTFT and DFT :

The DTFT is discrete time fourier transform and is given by,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \qquad ...(6)$$

The range of ω is from $-\pi$ to π or 0 to 2π.

Now we know that discrete fourier transform (DFT) is obtained by sampling one cycle of fourier transform. And DFT of $x(n)$ is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j 2\pi kn / N} \qquad ...(7)$$

Comparing Equations (6) and (7), we can say that DFT is obtained from DTFT by putting $\omega = \frac{2\pi k}{N}$

$$\therefore X(k) = X(\omega) \Big|_{\omega = \frac{2\pi k}{N}}$$

By comparing DFT with DTFT we can write,

1. The continuous frequency spectrum $X(\omega)$ is replaced by discrete fourier spectrum $X(k)$.
2. Infinite summation in DTFT is replaced by finite summation in DFT.
3. The continuous frequency variable is replaced by finite number of frequencies located at $\frac{2\pi k}{NT_s}$; where T_s is sampling time.

1.2.2 DFT of Standard Signals :

In this sub-section, we will obtain DFT of some standard signals with the help of solved examples as follows :

Prob. 1 : Obtain DFT of unit impulse $\delta(n)$.

Soln. :

Here $x(n) = \delta(n)$ ...(1)

According to the definition of DFT we have,

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad ...(2)$$

But $\delta(n) = 1$ only at $n = 0$,

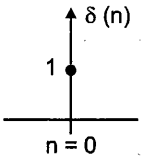


Fig. F-2

It is shown in Fig. F-2

Thus Equation (2) becomes,

$$X(k) = \delta(0) e^{0} = 1$$

$$\therefore \delta(n) \xleftrightarrow{DFT} 1$$

This is the standard DFT pair.

Prob. 2 : Obtain DFT of delayed unit impulse $\delta(n - n_0)$.

Soln. : We know that $\delta(n - n_0)$ indicates unit impulse delayed by 'n_0' samples.

Here $x(n) = \delta(n - n_0)$ ...(1)

Now we have,

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad ...(2)$$

But $\delta(n - n_0) = 1$ only at $n = n_0$.

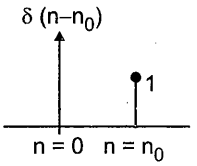


Fig. F-3

It is shown in Fig. F-3

Thus Equation (2) becomes,

$$X(k) = 1 \cdot e^{-j2\pi kn_0/N}$$

$$\therefore \delta(n - n_0) \xleftrightarrow{DFT} e^{-j2\pi kn_0/N}$$

Similarly we can write,

$$\delta(n + n_0) \xleftrightarrow{DFT} e^{j2\pi kn_0/N}$$

Prob. 3 : Obtain N-point DFT of exponential sequence :

$$x(n) = a^n u(n) \text{ for } 0 \le n \le N-1$$

Soln. : According to the definition of DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad ...(1)$$

Here $x(n) = a^n u(n)$

The multiplication of a^n with $u(n)$ indicates sequence is positive. Putting $x(n) = a^n$ in Equation (1) we get,

$$X(k) = \sum_{n=0}^{N-1} a^n e^{-j2\pi kn/N}$$

$$\therefore \quad X(k) = \sum_{n=0}^{N-1} (ae^{-j2\pi k/N})^n \qquad ...(2)$$

Now use the standard summation formula,

$$\sum_{k=N_1}^{N_2} A^k = \frac{A^{N_1} - A^{N_2+1}}{1-A}$$

Here $N_1 = 0$, $N_2 = N-1$ and $A = ae^{-j2\pi k/N}$

$$\therefore \quad X(k) = \frac{(ae^{-j2\pi k/N})^0 - (ae^{-j2\pi k/N})^{N-1+1}}{1-ae^{-j2\pi k/N}}$$

$$\therefore \quad X(k) = \frac{1-a^N e^{-j2\pi k}}{1-ae^{-j2\pi k/N}} \qquad ...(3)$$

Using Euler's identity to the numerator term, we get,

$$e^{-j2\pi k} = \cos 2\pi k - j \sin 2\pi k$$

But k is an integer

$$\therefore \quad \cos 2\pi k = 1 \text{ and } \sin 2\pi k = 0$$

$$\therefore \quad e^{-j2\pi k} = 1 - j0 = 1$$

$$\therefore \quad X(k) = \frac{1-a^N}{1-ae^{-j2\pi k/N}}$$

$$\therefore a^n u(n) \overset{DFT}{\longleftrightarrow} \frac{1-a^N}{1-ae^{-j2\pi k/N}}$$

Prob. 4 : Find the DFT of following window function,

$$w(n) = u(n) - u(n-N)$$

Soln. : According to the definition of DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \qquad ...(1)$$

The given equation is $x(n) = w(n) = 1$ for $0 \le n \le N-1$. We will assume some value of N. Let N = 4; so we will get 4-point DFT.

$$\therefore \quad X(k) = \sum_{n=0}^{3} 1 \cdot e^{-j2\pi kn/4} \qquad ...(2)$$

The range of k is from '0' to N – 1. So in this case 'k' will vary from 0 to 3.

For $k = 0 \Rightarrow X(0) = \sum_{n=0}^{3} 1 \cdot e^{0} = \sum_{n=0}^{3} 1 = 1+1+1+1 = 4$

For $k = 1 \Rightarrow X(1) = \sum_{n=0}^{3} e^{-j2\pi n/4}$

$$\therefore \quad X(1) = e^{0} + e^{-j2\pi/4} + e^{-j4\pi/4} + e^{-j6\pi/4}$$

$$\therefore \quad X(1) = 1 + \left(\cos\frac{2\pi}{4} - j\sin\frac{2\pi}{4}\right) + \left(\cos\frac{4\pi}{4} - j\sin\frac{4\pi}{4}\right) + \left(\cos\frac{6\pi}{4} - j\sin\frac{6\pi}{4}\right)$$

$$\therefore \quad X(1) = 1 + (0 - j) + (-1 - 0) + (0 + j)$$

$$\therefore \quad X(1) = 1 - j - 1 + j = 0$$

For $k = 2 \Rightarrow X(2) = \sum_{n=0}^{3} e^{-j2\pi \times 2n/4} = \sum_{n=0}^{3} e^{-j\pi n}$

$$\therefore \quad X(2) = e^{0} + e^{-j\pi} + e^{-j2\pi} + e^{-j3\pi}$$

$$\therefore \quad X(2) = 1 + (\cos\pi - j\sin\pi) + (\cos 2\pi - j\sin 2\pi) + (\cos 3\pi - j\sin 3\pi)$$

$$\therefore \quad X(2) = 1 + (-1 - 0) + (1 - 0) + (-1 - 0) = 1 - 1 + 1 - 1 = 0$$

For $k = 3 \Rightarrow X(3) = \sum_{n=0}^{3} e^{-j2\pi \times 3n/4} = \sum_{n=0}^{3} e^{-j6\pi n/4}$

$$\therefore \quad X(3) = e^{0} + e^{-j6\pi/4} + e^{-j3\pi} + e^{-j9\pi/2}$$

$$\therefore \quad X(3) = 1 + \left(\cos\frac{6\pi}{4} - j\sin\frac{6\pi}{4}\right) + (\cos 3\pi - j\sin 3\pi) + \left(\cos\frac{9\pi}{2} - j\sin\frac{9\pi}{2}\right)$$

$$\therefore \quad X(3) = 1 + (0 + j) + (-1 - 0) + (0 - j) = 1 + j - 1 - j = 0$$

$$\therefore \quad X(k) = \{4, 0, 0, 0\}$$

1.2.3 Cyclic Property of Twiddle Factor :

The twiddle factor is denoted by W_N and is given by,

$$W_N = e^{-j2\pi/N} \qquad ...(1)$$

Now the discrete time sequence x (n) can be denoted by x_N. Here 'N' stands for 'N' point DFT.

While in case of 'N' point DFT; the range of 'n' is from 0 to N – 1. Now the sequence x_N is represented in the matrix form as follows :

$$x_N = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}_{N \times 1} \quad \ldots(2)$$

This is a "N × 1" matrix and 'n' varies from 0 to N – 1. Now the DFT of x (n) is denoted by X (k). We have denoted x (n) by x_N; similarly we can denote X (k) by x_N. In the matrix form X_k can be represented as follows,

$$X_k = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix}_{N \times 1} \quad \ldots(3)$$

This is also "N × 1" matrix and 'k' varies from 0 to N – 1. Now recall the definition of DFT.

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad \ldots(4)$$

We can also represent W_N^{kn} in the matrix form. Remember that 'k' varies from 0 to N – 1 and 'n' also varies from 0 to N – 1.

$$\therefore \quad W_N^{kn} = \begin{array}{c} \\ k=0 \\ k=1 \\ k=2 \\ \vdots \\ k=N-1 \end{array} \begin{array}{c} \begin{array}{cccccc} n=0 & n=1 & n=2 & \ldots\ldots & n=N-1 \end{array} \\ \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ W_N^0 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ W_N^0 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix}_{N \times N} \end{array} \quad \ldots(5)$$

Note that each value is obtained by taking multiplication of k and n.

For example if k = 2, n = 2, then we get $W_N^{kn} = W_N^4$.

Thus DFT can be represented in the matrix form as,

$$X_N = [W_N]\, x_N \qquad ...(6)$$

Similarly, IDFT can be represented in the matrix form as,

$$x_N = \frac{1}{N}\left[W_N^*\right] X_N \qquad ...(7)$$

Here W_N^* is complex conjugate of W_N.

Now we will show that W_N possess the periodicity property. That means, after some period the value of W_N repeats. Let us consider 8-point DFT, that means N = 8.

We have, $W_N = e^{-\frac{j2\pi}{N}}$ $\quad\therefore\quad W_N^{kn} = e^{-\frac{j2\pi}{N}\times kn}$

But N = 8 $\quad\therefore\quad W_8^{kn} = e^{-\frac{j2\pi}{8}\times kn} = e^{-\frac{j\pi}{4}\times kn}$ ...(8)

Now we will obtain W_8^{kn} by putting different values of kn. This is shown in Table F-1.

Note that all calculations are done by using Euler's identity. For example when kn = 1, Equation (8) becomes,

$$W_8^1 = e^{-j\frac{\pi}{4}} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} = 0.707 - j\,0.707$$

Table F-1

Value of kn	$W_8^{kn} = e^{-j\frac{\pi}{4}\times kn}$	Value of the phasor
0	$W_8^0 = e^0$	1
1	$W_8^1 = e^{-j\frac{\pi}{4}\times 1} = e^{-j\frac{\pi}{4}}$	$0.707 - j\,0.707$
2	$W_8^2 = e^{-j\frac{\pi}{4}\times 2} = e^{-j\frac{\pi}{2}}$	$0 - j\,1$
3	$W_8^3 = e^{-j\frac{\pi}{4}\times 3} = e^{-j\frac{3\pi}{4}}$	$-0.707 - j\,0.707$
4	$W_8^4 = e^{-j\frac{\pi}{4}\times 4} = e^{-j\pi}$	-1
5	$W_8^5 = e^{-j\frac{\pi}{4}\times 5} = e^{-j\frac{5\pi}{4}}$	$-0.707 + j\,0.707$
6	$W_8^6 = e^{-j\frac{\pi}{4}\times 6} = e^{-j\frac{3\pi}{2}}$	$0 + j\,1$
7	$W_8^7 = e^{-j\frac{\pi}{4}\times 7} = e^{-j\frac{7\pi}{4}}$	$0.707 + j\,0.707$

Value of kn	$W_8^{kn} = e^{-j\frac{\pi}{4}\times kn}$	Value of the phasor
8	$W_8^8 = e^{-j\frac{\pi}{4}\times 8} = e^{-j2\pi}$	1
9	$W_8^9 = e^{-j\frac{\pi}{4}\times 9} = e^{-j\frac{9\pi}{4}}$	$0.707 - j\,0.707$
10	$W_8^{10} = e^{-j\frac{\pi}{4}\times 10} = e^{-j\frac{5\pi}{4}}$	$0 - j1$
11	$W_8^{11} = e^{-j\frac{\pi}{4}\times 11} = e^{-j\frac{11\pi}{4}}$	$-0.707 - j\,0.707$

Now observe this table. The value of W_8^0 is same as W_8^8. Similarly W_8^1 is same as W_8^9 and W_8^2 is same as W_8^{10}. Since this is 8-point DFT (N = 8); after 8 points the value repeats.

$$\therefore \quad W_8^0 = W_8^8 = W_8^{16} \ldots.$$

$$W_8^1 = W_8^9 = W_8^{17} \ldots.$$

$$W_8^2 = W_8^{10} = W_8^{18} \ldots.$$

$$\vdots$$

$$W_8^7 = W_8^{15} = W_8^{23} \ldots.$$

This property of twiddle factor is called as periodicity property of cyclic property.

In Table F-1, every value of W_N^{kn} can be represented in terms of magnitude and angle. For example, we have $W_8^0 = e^0$. We know that magnitude and angle can be expressed as, magnitude $e^{j\,angle}$. Thus for $W_8^0 = 1\,e^0$; magnitude is 1 and angle is zero. Similarly, we have $W_8^1 = 1 \cdot e^{-j\frac{\pi}{4}}$, so magnitude is 1 and angle is $-\frac{\pi}{4}$. Likewise, we can write the magnitude and angle of each value.

The cyclic property of twiddle factor is shown in Fig. F-4.

In Fig. F-4, we have drawn unit circle that means a circle having radius equal to 1. Every point is in the clockwise direction because we have negative angles. Here we have considered 8-point DFT; so the circle is divided into 8 points. This spacing of DFT or the resolution of DFT is also called as the bin spacing of DFT output.

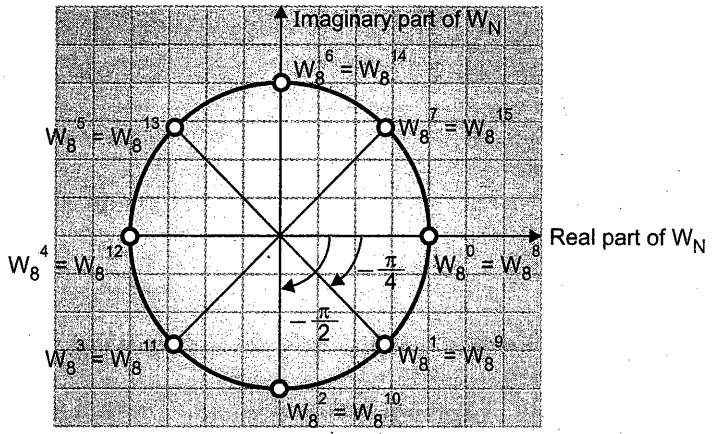


Fig. F-4 : Cyclic property of twiddle factor

1.2.4 Solved Problems on DFT :

Prob. 1 : Determine 2-point and 4-point DFT of a sequence,

$$x(n) = u(n) - u(n-2)$$

Sketch the magnitude of DFT in both the cases.

Soln. : First we will obtain the sequence x (n). It is represented as shown in Fig. F-5.

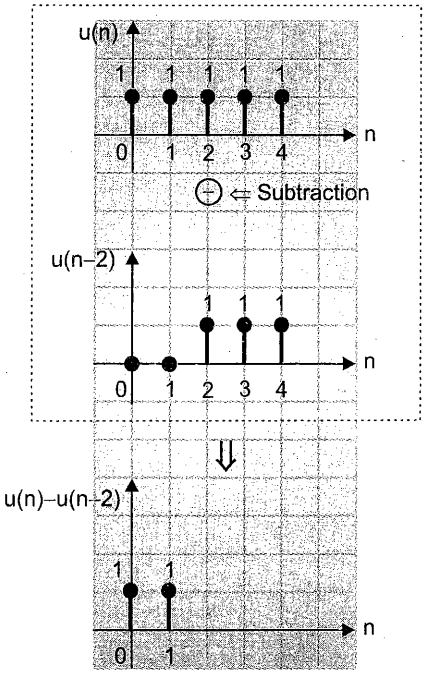


Fig. F-5 : x (n) = u (n) – u (n – 2)

$$\therefore \quad x(n) = \{1, 1\} \qquad \ldots(1)$$

Determination of 2-point DFT :

For 2-point DFT, $N = 2$

We have, $W_N = e^{-j\frac{2\pi}{N}}$ $\qquad \therefore \quad W_2 = e^{-j\frac{2\pi}{2}} = e^{-j\pi}$

$$\therefore \quad W_2^{kn} = e^{-j\pi kn} \qquad \ldots(2)$$

We know that 'n' is from 0 to $N-1$. In this case, 'n' is from 0 to 1. Similarly, 'k' is from 0 to $N-1$. In this case 'k' is from 0 to 1.

Now the matrix $W_N = W_2^{kn} = e^{-j\pi kn}$ can be written as,

$$W_2^{kn} = \begin{matrix} \\ k=0 \\ k=1 \end{matrix} \begin{matrix} \begin{matrix} n=0 & n=1 \end{matrix} \\ \begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix} \end{matrix} \qquad \ldots(3)$$

According to Equation (2) we have,

$$W_2^{kn} = e^{-j\pi kn}$$

For $kn = 0 \Rightarrow W_2^0 = e^{-j\pi \times 0} = e^0 = 1$

For $kn = 1 \Rightarrow W_2^1 = e^{-j\pi \times 1} = e^{-j\pi} = \cos\pi - j\sin\pi = -1$

Putting these values in Equation (3),

$$W_2^{kn} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \ldots(4)$$

Now given sequence $x(n) = \{1, 1\}$. In the matrix form it can be written as,

$$\therefore \quad x_N = x(n) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \ldots(5)$$

Now DFT matrix is given by,

$$X_N = [W_N]\, x_N$$

$$\therefore \quad X_N = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} (1\times 1)+(1\times 1) \\ (1\times 1)+(1\times -1) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Thus 2-point DFT is,

$$\boxed{X(k) = \{2, 0\}} \qquad \ldots(6)$$

Magnitude plot :

We have, magnitude = $\sqrt{(\text{Real part})^2 + (\text{Imaginary part})^2}$

In Equation (6), the imaginary part is zero.

Thus magnitude at k = 0 is 2 and magnitude at k = 1 is 0. This magnitude plot is shown in Fig. F-6.

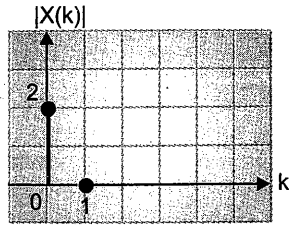


Fig. F-6 : Magnitude plot

Determination of 4-point DFT :

For 4-point DFT, N = 4.

$$\text{We have,} \quad W_N = W_4 = e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}}$$

$$\therefore \quad W_N^{kn} = e^{-j\frac{\pi}{2}kn} \qquad \ldots(7)$$

The range of K and n is from 0 to N – 1. That means 0 to 3.

$$[W_4] = W_4^{kn} = \begin{array}{c} \\ k=0 \\ k=1 \\ k=2 \\ k=3 \end{array} \begin{array}{c} \begin{array}{cccc} n=0 & n=1 & n=2 & n=3 \end{array} \\ \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \end{array} \qquad \ldots(8)$$

Now using Equation (7) we get,

$$W_4^0 = e^{-j\frac{\pi}{2}\times 0} = e^0 = 1$$

$$W_4^1 = e^{-j\frac{\pi}{2}\times 1} = e^{-j\frac{\pi}{2}} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

$$W_4^2 = e^{-j\frac{\pi}{2}\times 2} = e^{-j\pi} = \cos\pi - j\sin\pi = -1$$

$$W_4^3 = e^{-j\frac{\pi}{2}\times 3} = e^{-j\frac{3\pi}{2}} = \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} = +j$$

According to cyclic property of DFT.

$$W_4^0 = W_4^4 = 1$$

$$W_4^1 = W_4^5 = W_4^9 = -j$$

$$W_4^2 = W_4^6 = W_4^{10} = -1$$

$$\text{and} \quad W_4^3 = W_4^7 = W_4^{11} = +j$$

Putting these values in Equation (8) we get,

$$[W_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \quad \ldots(9)$$

Now given sequence is,

$x(n) = \{1, 1\}$. We want the length of this sequence equal to 4. It is obtained by adding zeros at the end of sequence. This is called as zero padding.

$$\therefore \quad x(n) = \{1, 1, 0, 0\}$$

$$\therefore \quad x_N = x_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \ldots(10)$$

Now the DFT is given by, $X_N = [W_N]\, x_N$

$$\therefore \quad X_N = X_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad X_N = X_4 = \begin{bmatrix} 1+1+0+0 \\ 1-j+0+0 \\ 1-1+0+0 \\ 1+j+0+0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1-j \\ 0 \\ 1+j \end{bmatrix}$$

$$\therefore \quad X_4 = \{2, \quad 1-j, \quad 0, \quad 1+j\}$$

This DFT sequence can also be written as,

$$X_4 = \{2 + j\,0, \; 1-j, \; 0 + j\,0, \; 1 + j\}$$

$$\uparrow$$

$$k = 0$$

Magnitude plot : The magnitude at different values is obtained as follows,

$$\text{For } k = 0 \Rightarrow |X(k)| = \sqrt{(2)^2 + (0)^2} = 2$$

$$\text{For } k = 1 \Rightarrow |X(k)| = \sqrt{(1)^2 + (-1)^2} = \sqrt{2} = 1.414$$

$$\text{For } k = 2 \Rightarrow |X(k)| = \sqrt{0+0} = 0$$

$$\text{For } k = 3 \Rightarrow |X(k)| = \sqrt{(1)^2 + (1)^2}$$

$$= \sqrt{2} = 1.414$$

This magnitude plot is shown in Fig. F-7.

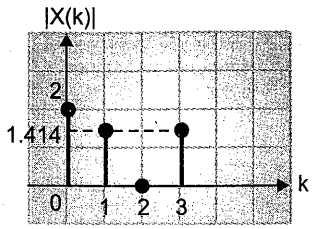


Fig. F-7 : Magnitude plot

Prob. 2 : Compute the DFT of four point sequence x (n) = {0, 1, 2, 3}

Soln. : The four point DFT in the matrix form is given by,

$$X_4 = [W_4] \cdot x(n)$$

$$\therefore \quad X_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\therefore \quad X_4 = \begin{bmatrix} 0+1+2+3 \\ 0-j-2+3j \\ 0-1+2-3 \\ 0+j-2-3j \end{bmatrix} = \begin{bmatrix} 6 \\ 2j-2 \\ -2 \\ -2j-2 \end{bmatrix}$$

$$\therefore \quad X_4 = \{6, 2j-2, -2, -2j-2\}$$
$$\uparrow$$

Prob. 3 : Calculate 8 point DFT of :

$$x(n) = \{1, 2, 1, 2\}$$

Soln. : First we will make length of given sequence '8' by doing zero padding.

$$\therefore \quad x(n) = \{1, 2, 1, 2, 0, 0, 0, 0\} \qquad ...(1)$$

We have, $W_N = e^{-j\frac{2\pi}{N}}$

$$\therefore \quad W_8^{kn} = e^{-j\frac{2\pi}{8}} = e^{-j\frac{\pi}{4}kn} \qquad ...(2)$$

Here the range of K and n is from 0 to N – 1 that means from 0 to 7.

Now the matrix W_8^{kn} is as follows,

$$[W_8] = \begin{array}{c} \\ k=0 \\ k=1 \\ k=2 \\ k=3 \\ k=4 \\ k=5 \\ k=6 \\ k=7 \end{array} \begin{array}{c} \begin{array}{cccccccc} n=0 & n=1 & n=2 & n=3 & n=4 & n=5 & n=6 & n=7 \end{array} \\ \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^8 & W_8^{10} & W_8^{12} & W_8^{14} \\ W_8^0 & W_8^3 & W_8^6 & W_8^9 & W_8^{12} & W_8^{15} & W_8^{18} & W_8^{21} \\ W_8^0 & W_8^4 & W_8^8 & W_8^{12} & W_8^{16} & W_8^{20} & W_8^{24} & W_8^{28} \\ W_8^0 & W_8^5 & W_8^{10} & W_8^{15} & W_8^{20} & W_8^{25} & W_8^{30} & W_8^{35} \\ W_8^0 & W_8^6 & W_8^{12} & W_8^{18} & W_8^{24} & W_8^{30} & W_8^{36} & W_8^{42} \\ W_8^0 & W_8^7 & W_8^{14} & W_8^{21} & W_8^{28} & W_8^{35} & W_8^{42} & W_8^{49} \end{bmatrix} \end{array} \quad ...(3)$$

In Table F-1 we have already obtained different values of W_8^{kn}.

$$\therefore \quad W_8^0 = W_8^8 = W_8^{16} = W_8^{24} = W_8^{32} = W_8^{40} = ... = 1$$

$$W_8^1 = W_8^9 = W_8^{17} = W_8^{25} = W_8^{33} = W_8^{41} = W_8^{49} = ... = 0.707 - j\,0.707$$

$$W_8^2 = W_8^{10} = W_8^{18} = W_8^{26} = W_8^{34} = W_8^{42} = ... = -j$$

$$W_8^3 = W_8^{11} = W_8^{19} = W_8^{27} = W_8^{35} = W_8^{43} = ... = -0.707 - j\,0.707$$

$$W_8^4 = W_8^{12} = W_8^{20} = W_8^{28} = W_8^{36} = W_8^{44} = ... = -1$$

$$W_8^5 = W_8^{13} = W_8^{21} = W_8^{29} = W_8^{37} = W_8^{45} = ... = -0.707 + j\,0.707$$

$$W_8^6 = W_8^{14} = W_8^{22} = W_8^{30} = W_8^{38} = W_8^{46} = ... = j$$

$$W_8^7 = W_8^{15} = W_8^{23} = W_8^{31} = W_8^{39} = W_8^{47} = ... = 0.707 + j\,0.707$$

Now the DFT is given by,

$$X_8 = [W_8]\, x_n \quad ...(4)$$

Putting values of W_8^{kn} in Equation (3) and write x_n in matrix form we get,

$$X_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0.707-j0.707 & -j & -0.707-j0.707 & -1 & -0.707+j0.707 & j & 0.707+j0.707 \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & -0.707-j0.707 & j & 0.707-j0.707 & -1 & 0.707+j0.707 & -j & -0.707+j0.707 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -0.707+j0.707 & -j & 0.707+j0.707 & -1 & 0.707-j0.707 & j & -0.707-j0.707 \\ 1 & j & -1 & -j & 1 & j & -1 & -j \\ 1 & 0.707+j0.707 & j & -0.707+j0.707 & -1 & -0.707-j0.707 & -j & 0.707-j0.707 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad X_8 = \begin{bmatrix} 1+2+1+2+0+0+0+0 \\ 1+1.414-j\,1.414-j-1.414-j\,1.414+0+0+0+0 \\ 1-j\,2-1+j\,2+0+0+0+0 \\ 1-1.414-j\,1.414+j+1.414-j\,1.414+0+0+0+0 \\ 1-2+1-2+0+0+0+0 \\ 1-1.414+j\,1.414-j+1.414+j\,1.414+0+0+0+0 \\ 1+j\,2-1-j\,2+0+0+0+0 \\ 1+1.414+j\,1.414+j-1.414+j\,1.414+0+0+0+0 \end{bmatrix}$$

$$\therefore \quad X_8 = \begin{bmatrix} 6 \\ 1-j\,2.414 \\ 0 \\ 1-j\,1.828 \\ -2 \\ 1+j\,1.828 \\ 0 \\ 1+j\,3.828 \end{bmatrix}$$

This is the required DFT.

Prob. 4 : Determine the length-4 sequence from its DFT.

$$X(k) = \{4, 1-j, -2, 1+j\}$$

Soln. : The IDFT in matrix form is given by,

$$\text{IDFT} = x(n) = x_N = \frac{1}{N}\left[W_N^*\right] \cdot X_N \qquad \ldots(1)$$

Here X_N is the given DFT matrix. '*' indicates complex conjugate. To obtain the complex conjugate, we have to change the sign of j term. For example, complex conjugate of $1-j1$ is $1+j1$.

Now we have already obtained the matrix W_4 in problem (1). It is,

$$[W_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \qquad \ldots(2)$$

$$\therefore \quad [W_4^*] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \qquad \ldots(3)$$

Given matrix of DFT is,

$$X_N = X_4 = \begin{bmatrix} 4 \\ 1-j \\ -2 \\ 1+j \end{bmatrix} \qquad \ldots(4)$$

Putting Equations (3) and (4), and putting N = 4 in Equation (1) we get,

$$x_N = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 4 \\ 1-j \\ -2 \\ 1+j \end{bmatrix}$$

$$\therefore \quad x_N = \frac{1}{4}\begin{bmatrix} 4+1-j-2+1+j \\ 4+j-j^2+2-j-j^2 \\ 4-1+j-2-1-j \\ 4-j+j^2+2+j+j^2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ 4+2+1+1 \\ 4-4 \\ 4+2-2 \end{bmatrix} \qquad \ldots\text{as } j^2 = -1$$

$$\therefore \quad x_N = \frac{1}{4}\begin{bmatrix} 4 \\ 8 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore x(n) = \{1, 2, 0, 1\}$$

Prob. 5 : Calculate the DFT of a sequence $x(n) = \{1, 1, 0, 0\}$ and check the validity of your answer by calculating its IDFT.

Soln. : We will compute 4 point DFT. We have already obtained the matrix for $[W_4]$. It is,

$$[W_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix}$$

Now given sequence is $x(n) = \{1, 1, 0, 0\}$

The DFT of this sequence is calculated in problem (1). It is $X_N = X(k) = \{2, 1-j, 0, 1+j\}$

Now we will check this answer by using the formulas for IDFT.

The IDFT is given by,

$$x(n) = \frac{1}{N}\left[W_N^*\right] \cdot X_N$$

$$\text{Here } \left[W_N^*\right] = \left[W_4^*\right] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

$$\text{and } X_N = X_4 = \begin{bmatrix} 2 \\ 1-j \\ 0 \\ 1+j \end{bmatrix}$$

$$\therefore x(n) = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 2 \\ 1-j \\ 0 \\ 1+j \end{bmatrix}$$

$$\therefore x(n) = \frac{1}{4}\begin{bmatrix} 2+1-j+0+1+j \\ 2+j+1+0-j+1 \\ 2-1+j+0-1-j \\ 2-j-1+0+j-1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ 4 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore x(n) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

That means $x(n) = \{1, 1, 0, 0\}$

This is same as the given sequence; so calculated DFT is correct.

Prob. 6 : If $y(n) = \dfrac{[x(n)+x(-n)]}{2}$

Find $Y(k)$ if $X(k) = \{0.5, 2+j, 3+j2, j, 3, -j, 3-j2, 2-j\}$

Soln. : We have,

$$y(n) = \frac{[x(n)+x(-n)]}{2} \qquad \ldots(1)$$

Taking DFT of both sides we get,

$$Y(k) = \frac{[X(k)+X(-k)]}{2} \quad \ldots(2)$$

Given $X(k) = \{0.5, 2+j, 3+j2, j, 3, -j, 3-j2, 2-j\}$

$\therefore$ $X(-k) = \{0.5, 2-j, 3-j2, -j, +3, j, 3+j2, 2+j\}$

Putting these values in Equation (2) we get,

$$Y(k) = \frac{1}{2}\{1, 4, 6, 0, 6, 0, 6, 4\}$$

$$\therefore Y(k) = \{0.5, 2, 3, 0, 6, 0, 3, 2\}$$

1.3 Properties of DFT :

In this section we will study some important properties of DFT. We know that, the DFT of discrete time sequence, $x(n)$ is denoted by $X(k)$. And the DFT and IDFT pair is denoted by,

$$x(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X(k)$$

Here 'N' indicates 'N' point DFT.

1.3.1 Linearity :

Statement : If $x_1(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_1(k)$ and $x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_2(k)$ then,

$$a_1 x_1(n) + a_2 x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} a_1 X_1(k) + a_2 X_2(k)$$

Here a_1 and a_2 are some constants.

Proof : According to the definition of DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad \ldots(1)$$

Here $x(n) = a_1 x_1(n) + a_2 x_2(n)$

$$\therefore X(k) = \sum_{n=0}^{N-1} [a_1 x_1(n) + a_2 x_2(n)] W_N^{kn}$$

$$= \sum_{n=0}^{N-1} a_1 x_1(n) W_N^{kn} + \sum_{n=0}^{N-1} a_2 x_2(n) W_N^{kn}$$

Since a_1 and a_2 are constants; we can take it out of the summation sign.

$$\therefore \quad X(k) = a_1 \sum_{n=0}^{N-1} x_1(n) W_N^{kn} + a_2 \sum_{n=0}^{N-1} x_2(n) W_N^{kn} \qquad ...(2)$$

Comparing Equation (2) with the definition of DFT,

$$X(k) = a_1 X_1(k) + a_2 X_2(k)$$

Meaning : DFT of linear combination of two or more signals is equal to the sum of linear combination of DFT of individual signal.

1.3.2 Periodicity :

Statement : If $x(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X(k)$ then

$$x(n+N) = x(n) \qquad \text{for all n}$$

$$\text{and} \quad X(k+N) = X(k) \qquad \text{for all k.}$$

Proof : According to the definition of DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad ...(1)$$

Replacing k by k + N we get,

$$X(k+N) = \sum_{n=0}^{N-1} x(n) W_N^{(k+N)n}$$

$$\therefore \quad X(k+N) = \sum_{n=0}^{N-1} x(n) W_N^{kn} W_N^{Nn} \qquad ...(2)$$

We know that W_N is a twiddle factor and it is given by,

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$\therefore \quad W_N^{Nn} = \left(e^{-j\frac{2\pi}{N}}\right)^{Nn} = e^{-j\frac{2\pi}{N} \cdot Nn} = e^{-j2\pi n}$$

$$\therefore \quad W_N^{Nn} = \cos 2\pi n - j \sin 2\pi n \qquad ....(3)$$

But 'n' is an integer $\therefore \cos 2\pi n = 1$ and $\sin 2\pi n = 0$

$$\therefore \quad W_N^{Nn} = `1 \qquad ...(4)$$

Putting this value in Equation (2)

$$X(k+N) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad \ldots(5)$$

Comparing Equations (1) and (5),

$$\boxed{X(k+N) = X(k)}$$ Hence proved

Meaning : DFT of a finite length sequence results in a periodic sequence.

1.3.3 Circular Symmetries of a Sequence :

We have studied the periodicity property of DFT. Suppose input discrete time sequence is $x(n)$ then, the periodic sequence is denoted by $x_p(n)$. The period of $x_p(n)$ is 'N' which means after 'N' the sequence $x(n)$ repeats itself. Now we can write the sequence $x_p(n)$ as,

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN) \quad \ldots(1)$$

We will consider one example. Let $x(n) = \{1, 2, 3, 4\}$. This sequence is shown in Fig. F-8(a). The periodic sequence $x_p(n)$ is shown in Fig. F-8(b).

We will delay the periodic sequence $x_p(n)$ by two sample as shown in Fig. F-8(c). This sequence is denoted by $x_p(n-2)$. Now the original signal is present in the range n = 0 to n = 3. In the same range we will write the shifted signal as shown in Fig. F-8(d). This signal is denoted by $x'(n)$.

Now from Fig. F-8 we can write every sequence as follows :

$$x(n) = \underset{\uparrow}{\{1, 2, 3, 4\}} \quad \ldots(2)$$

$$x_p(n) = \{ \ldots. 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, \ldots.\} \quad \ldots(3)$$
$$\uparrow$$

$$x_p(n-2) = \{\ldots 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, \ldots..\} \quad \ldots.(4)$$
$$\uparrow$$

$$\text{and} \quad x'(n) = \underset{\uparrow}{\{3, 4, 1, 2\}} \quad \ldots(5)$$

Now observe Equations (2) and (5). We can say that the sequence $x'(n)$ is obtained by circularly shifting sequence $x(n)$, by two samples. That means $x'(n)$ is related to $x(n)$ by circular shift.

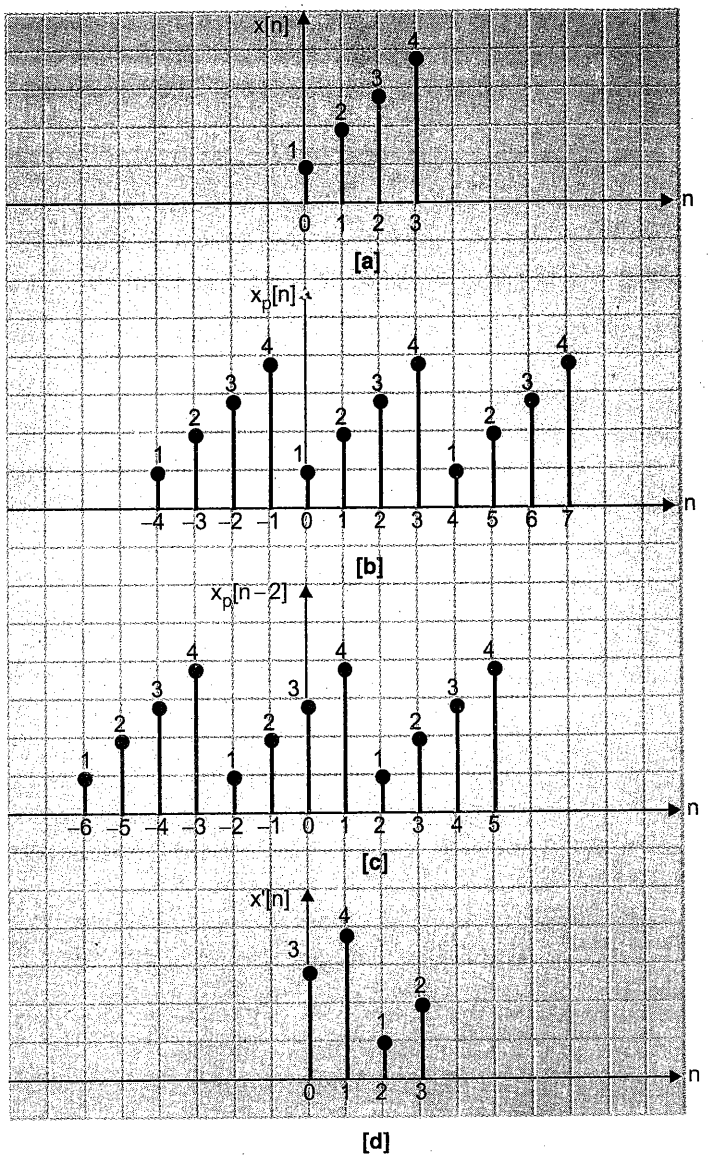


Fig. F-8 : Shifting of sequence x (n)

Notation :

This relation of circular shift is denoted by,

$$x'(n) = x(n-k, \text{modulo } N) \quad \ldots(6)$$

It means that divide $(n-k)$ by N and retain the remainder only. We can also use the short hand notation as follows :

$$x'(n) = x((n-k))_N \quad \ldots(7)$$

Here 'k' indicates the number of samples by which $x(n)$ is delayed and 'N' indicates N-point DFT. In the present example, the sequence $x(n)$ is delayed by two samples; thus $k = 2$. Since there are four samples in $x(n)$; this is four point DFT. Thus $N = 4$. Now for this example Equation (6) becomes,

$$x'(n) = x((n-2))_4 \qquad ...(8)$$

Graphical Representation :

The circular shifting of a sequence is plotted graphically as follows :

(1) Circular plot of s equence $x(n)$: Here we have considered,

$$x(n) = \{1, \underset{\uparrow}{2}, 3, 4\}$$

Circular plot of $x(n)$ is denoted by $x((n))_4$. This plot is obtained by writing the samples of $x(n)$ circularly anticlockwise. It is shown in Fig. F-9(a).

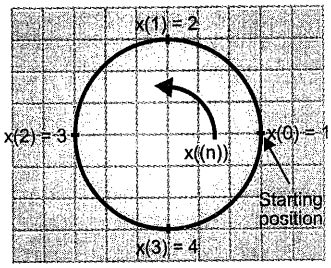


Fig. F-9(a) : $x((n))_4$ – The samples of $x(n)$ are plotted circularly anticlockwise

(2) Circular delay by one sample : To delay sequence $x(n)$ circularly by one sample; shift every sample circularly in anticlockwise direction by 1. This is shown in Fig. F-9(b). This operation is denoted by $x((n-1))$.

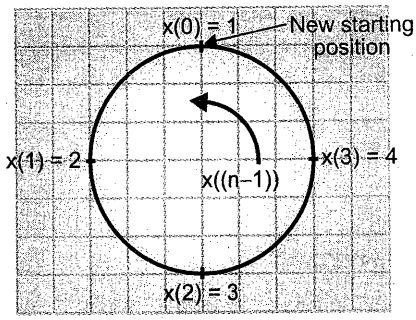


Fig. F-9(b) : $x((n-1))$ shift every sample by 1 in anticlockwise direction

Remember that delay by 'k' samples means shift the sequence circularly in anticlockwise direction by k.

(3) Circular advance by one sample : To advance sequence x (n) circularly by one sample; shift every sample circularly in clockwise direction by 1 sample. This sequence is denoted by x ((n + 1)). It is shown in Fig. F-9(c).

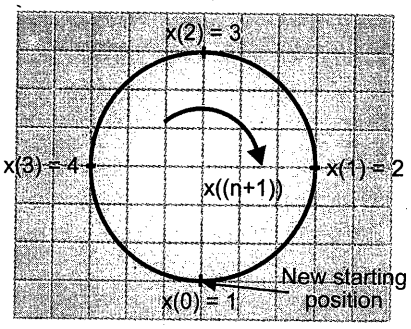


Fig. F-9(c) : x ((n + 1)) shift every sample by one in clockwise direction

Remember that advance by k samples means shift the sequence circularly in clockwise direction by k.

(4) Circularly folded sequence : A circularly folded sequence is denoted by x ((− n)). We have plotted sequence x ((n)) in anticlockwise direction. So folded sequence x ((− n)) is plotted in clockwise direction. It is shown in Fig. F-9(d).

Remember that circular folding means plot the samples in clockwise direction.

Now recall Equation (7) it is,

$$x'(n) = x((n-2))_4$$

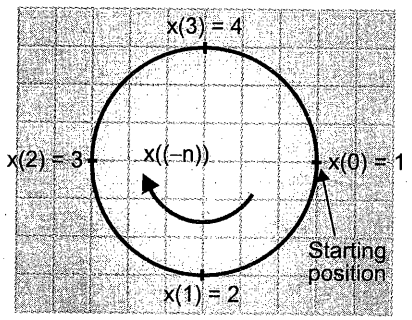


Fig. F-9(d) : x ((− n)) samples are plotted circularly clockwise

It indicates delay of sequence x (n) by two samples. It is obtained by rotating samples of Fig. F-9(a) in anticlockwise direction by two samples. This sequence is shown in Fig. F-9(e).

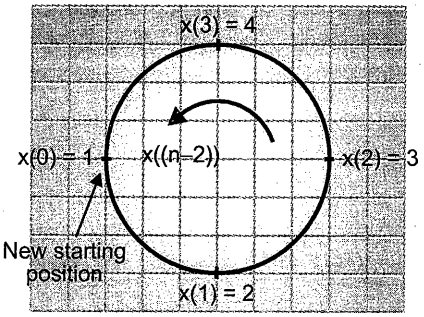


Fig. F-9(e) : Plot of $x((n-2))_4$

(5) Circularly even sequence :

The N-point discrete time sequence is circularly even if it is symmetric about the point zero on the circle.

That means,

$$x(N-n) = x(n), \quad 1 \leq n \leq N-1$$

Consider the sequence,

$x(n) = \{1, 4, 3, 4\}$. It is plotted as shown in Fig. F-9(f).

Note that this sequence is symmetric about point zero on the circle. So it is circularly even sequence. We can also verify it using mathematical equation,

The sequence is $x(n) = \{1, 4, 3, 4\}$

$\therefore x(0) = 1, \quad x(1) = 4, \quad x(2) = 3$ and $x(3) = 4$

We have the condition for circularly even sequence,

$$x(N-n) = x(n) \qquad ...(1)$$

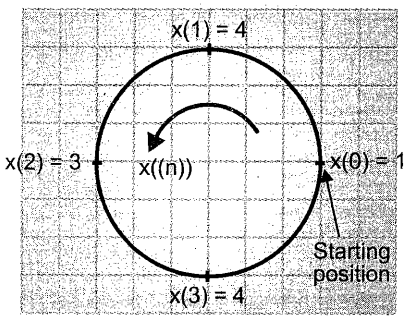


Fig. F-9(f) : $x(n) = \{1, 4, 3, 4\}$

Here N = 4. We will check this condition by putting different values of n as follows :

For $n = 1 \Rightarrow x(4-1) = x(1)$ that means $x(3) = x(1) = 4$

For n = 2 $\Rightarrow$ x (4 – 2) = x (2) that means x (2) = x (2) = 3

For n = 3 $\Rightarrow$ x (4 – 3) = x (3) that means x (1) = x (3) = 4.

Since for all values of 'n', Equation (1) is satisfied; the given sequence is circularly even.

(6) Circularly odd sequence :

A N-point sequence is called circularly odd if it is antisymmetric about point zero on the circle.

That means,

$$x(N-n) = -x(n), \quad 1 \le n \le N-1$$

Consider the sequence, x (n) = {2, – 3, 0, 3}

This sequence is plotted as shown in Fig. F-9.(g)

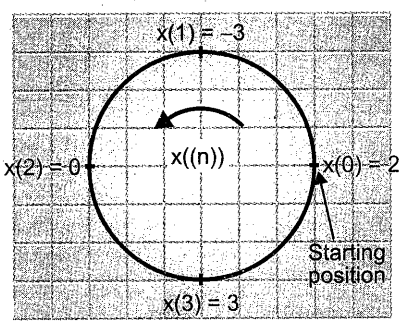


Fig. F-9(g) : Plot of x (n) = {2, – 3, 0, 3}

Here x (0) = 2, x (1) = – 3, x (2) = 0 and x (3) = 3.

We have the condition for circularly odd sequence,

$$x(N-n) = -x(n), \qquad \text{for } 1 \le n \le N-1 \qquad \ldots(1)$$

For n = 1 $\Rightarrow$ x (4 – 1) = – x (1) that means x (3) = – x (1)

For n = 2 $\Rightarrow$ x (4 – 2) = – x (2) that means x (2) = – x (2)

For n = 3 $\Rightarrow$ x (4 – 3) = – x (3) that means x (1) = – x (3)

Thus for all values of 'n', Equation (1) is satisfied. Hence the sequence is circularly odd.

Summary of circular properties :

The summary of circular property is shown in Table F-2.

Table F-2

Sr. No.	Sequence	Equation	Meaning
1.	Input sequence	$x((n))$	Plot the samples of $x(n)$ in anticlockwise direction. Anticlockwise means positive direction.
2.	Circular delay	$x((n-k))$	Shift sequence $x(n)$ in anticlockwise direction by K samples.
3.	Circular advance	$x((n+k))$	Shift sequence $x(n)$ in clockwise direction by K samples.
4.	Circular folding	$x((-n))$	Plot the samples of $x(n)$ in clockwise direction. Clockwise means negative direction.
5.	Circularly even	$x(N-n) = x(n)$	Sequence is symmetric about the point zero on the circle.
6.	Circularly odd	$x(N-n) = -x(n)$	Sequence is antisymmetric about the point zero on the circle.

1.3.4 Symmetry Properties of DFT :

The symmetry properties of DFT are derived in the similar way of DTFT symmetry properties. We know that DFT of sequence $x(n)$ is denoted by $X(k)$. Now if $x(n)$ and $X(k)$ are complex valued sequence then it can be represented as follows :

$$x(n) = x_R(n) + j\,x_I(n), \quad 0 \le n \le N-1 \qquad ...(1)$$

and $$X(k) = X_R(k) + j\,X_I(k), \quad 0 \le k \le N-1 \qquad ...(2)$$

Here 'R' stands for real part and 'I' stands for imaginary part.

According to the definition of DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\,2\pi\,kn/N} \qquad ...(3)$$

Putting Equation (1) in Equation (3)

$$X(k) = \sum_{n=0}^{N-1} [x_R(n) + j\,x_I(n)]\, e^{-j\,2\pi kn/N} \qquad ...(4)$$

But according to Euler's identify,

$$e^{-j\,2\pi kn/N} = \cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right)$$

Putting this value in Equation (4) we get,

$$X(k) = \sum_{n=0}^{N-1} [x_R(n) + j\,x_I(n)]\left[\cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right)\right]$$

$$\therefore \quad X(k) = \sum_{n=0}^{N-1} \left[x_R(n)\cdot\cos\left(\frac{2\pi kn}{N}\right) - j\,x_R(n)\sin\left(\frac{2\pi kn}{N}\right) + j\,x_I(n)\cdot\cos\left(\frac{2\pi kn}{N}\right) - j^2\,x_I(n)\sin\left(\frac{2\pi kn}{N}\right)\right]$$

Here $j^2 = -1$; and writing summation for real and imaginary parts separately we get,

$$X(k) = \sum_{n=0}^{N-1}\left[x_R(n)\cos\left(\frac{2\pi kn}{N}\right) + x_I(n)\sin\left(\frac{2\pi kn}{N}\right)\right] - j\sum_{n=0}^{N-1}\left[x_R(n)\sin\left(\frac{2\pi kn}{N}\right) - x_I(n)\cos\left(\frac{2\pi kn}{N}\right)\right] \quad ...(5)$$

Comparing Equations (5) and (2) we can write,

$$X_R(k) = \sum_{n=0}^{N-1}\left[x_R(n)\cos\left(\frac{2\pi kn}{N}\right) + x_I(n)\sin\left(\frac{2\pi kn}{n}\right)\right] \quad ...(6)$$

$$\text{and} \quad X_I(k) = -\sum_{n=0}^{N-1}\left[x_R(n)\sin\left(\frac{2\pi kn}{N}\right) - x_I(n)\cos\left(\frac{2\pi kn}{N}\right)\right] \quad ...(7)$$

Equations (6) and (7) are obtained by using definition of DFT. Similarly we can obtain real and imaginary parts of x (n) using definition of IDFT.

$$\therefore \quad x_R(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left[X_R(k)\cos\left(\frac{2\pi kn}{N}\right) - X_I(k)\sin\left(\frac{2\pi kn}{N}\right)\right] \quad ...(8)$$

$$\text{and} \quad x_I(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left[X_R(k)\sin\left(\frac{2\pi kn}{N}\right)+X_I(k)\cos\left(\frac{2\pi kn}{N}\right)\right] \qquad ...(9)$$

Now we will consider different cases as follows :

Case (i) : When x (n) is real valued :

Statement : If x (n) is real valued then

$$X(N-k) = X(-k) = X^*(k)$$

Proof : According to the definition of DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n)\,W_N^{kn} \qquad ...(10)$$

Replacing k by N – k, $X(N-k) = \sum_{n=0}^{N-1} x(n)\,W_N^{(N-k)n}$

$$\therefore \quad X(N-k) = \sum_{n=0}^{N-1} x(n)\,W_N^{Nn}\,W_N^{-kn} \qquad ...(11)$$

Now we have, twiddle factor $W_N = e^{-\frac{j2\pi}{N}}$

$$\therefore \quad W_N^{Nn} = \left(e^{-\frac{j2\pi}{N}}\right)^{Nn} = e^{-j2\pi n} = \cos 2\pi n - j\sin 2\pi n$$

Since n is an integer, cos 2 πn = 1 and sin 2 πn = 0

$$\therefore \quad W_N^{Nn} = 1$$

Thus Equation (11) becomes,

$$X(N-k) = \sum_{n=0}^{N-1} x(n)\,W_N^{-kn} \qquad ...(12)$$

Comparing Equation (12) with definition of DFT (Equation (10)) we get,

$$X(N-k) = X(-k) \qquad ..(13)$$

Now using Equation (10) we can write,

$$X^*(k) = \sum_{n=0}^{N-1} x(n)\,W_N^{-kn} \qquad ...(14)$$

Thus, from Equations (13) and (14) we get,

$$X(N-k) = X(-k) = X^*(k)$$

Case (ii) : When x (n) is real and even :

Statement : When x (n) is real and even which means,

$$x(n) = x(N-n) \text{ then DFT becomes,}$$

$$X(k) = X_R(k)$$

Proof : Since imaginary part is zero; putting $x_I(n) = 0$ in Equation (6) we get,

$$X_R(k) = \sum_{n=0}^{N-1} X_R(n) \cos\left(\frac{2\pi kn}{N}\right)$$

Similarly IDFT can be written by putting $X_I(k) = 0$ in Equation (8).

$$\therefore \quad x_R(n) = \frac{1}{N} \sum_{n=0}^{N-1} X_R(k) \cos\left(\frac{2\pi kn}{N}\right)$$

Case (iii) : When x (n) is real and odd :

Statement : When x (n) is real and odd which means,

$$x(n) = -x(N-n) \text{ then the DFT becomes,}$$

$$X(k) = -j \sum_{n=0}^{N-1} x_R(n) \sin\left(\frac{2\pi kn}{N}\right)$$

Proof : Since x (n) is real, we will put $x_I(n) = 0$ in Equation (5). Similarly x (n) is odd and 'cos' is even function so we can write, $\cos\left(\frac{2\pi kn}{N}\right) = 0$. Thus first summation in Equation (5) becomes zero. In the second summation of Equation (5), putting $x_I(n) = 0$ we get,

$$X(k) = -j \sum_{n=0}^{N-1} x_R(n) \sin\left(\frac{2\pi kn}{N}\right)$$

Similarly IDFT can be written as

$$x(n) = j\frac{1}{N} \sum_{k=0}^{N-1} X_R(k) \sin\left(\frac{2\pi kn}{N}\right)$$

Case (iv) : When x (n) is purely imaginary sequence : When x (n) is purely imaginary which means $x_R(n) = 0$ and $x(n) = j\,x_I(n)$ then putting $x_R(n) = 0$ in Equation (6) we get,

$$X_R(k) = \sum_{n=0}^{N-1} x_I(n) \sin\left(\frac{2\pi kn}{N}\right)$$

And putting $x_R(n) = 0$ in Equation (7) we get,

$$X_I(k) = \sum_{n=0}^{N-1} x_I(n) \cos\left(\frac{2\pi kn}{N}\right)$$

Symmetry properties can be summarized as shown in Fig. F-10.

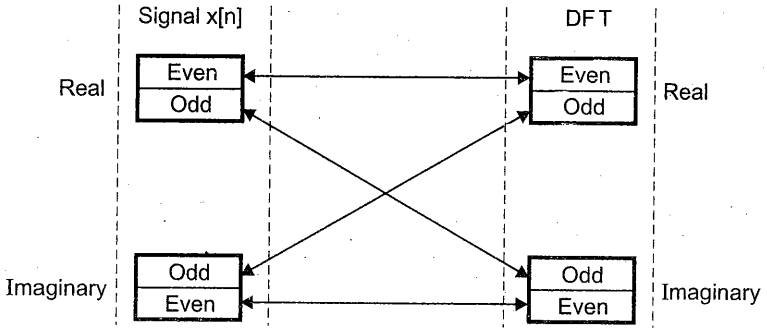


Fig. F-10 : Summary of symmetry property

This summary is shown in Table F-3.

Table F-3

N point sequence $x[n]$ $0 \le n \le N-1$	N point DFT
$x^*[n]$	$X^*[N-k]$
$x^*[N-n]$	$X^*[k]$
$x_R[n]$	$X_{ce}[k] = \frac{1}{2}\left[X[k] + X^*[N-k]\right]$
$j\,x_I[n]$	$X_{ce}[k] = \frac{1}{2}\left[X[k] - X^*[N-k]\right]$
$x_{ce}[n] = \frac{1}{2}\left[x[n] + x^*[N-n]\right]$	$X_R[k]$
$x_{co}[n] = \frac{1}{2}\left[x[n] + x^*[N-n]\right]$	$j\,X_I[k]$

1.3.5 Solved Problems using Properties of DFT :

Prob. 1 : The first five points of the 8 point DFT of a real valued sequence are {0.25, 0.125 – j0.3018, 0, 0.125 – j 0.0518, 0}. Determine the remaining three points.

Soln. : Given DFT points are :

$$X(0) = 0.25$$
$$X(1) = 0.125 - j\,0.3018$$
$$X(2) = 0$$
$$X(3) = 0.125 - j\,0.0518$$
$$X(4) = 0$$

Given sequence is a real valued sequence. According to the symmetry property we have,

$$X^*(k) = X(N-k)$$

$$\text{or } X(k) = X^*(N-k) \quad \ldots(1)$$

This is 8 point DFT. Thus N = 8

$$\therefore X(k) = X^*(8-k) \quad \ldots(2)$$

Now we want remaining three samples namely X(5) X(6) and X(7). Putting k = 5 in Equation (2),

$$X(5) = X^*(8-5) = X^*(3)$$

We have $X(3) = 0.125 - j\,0.0518$

$$\therefore X^*(3) = 0.125 + j\,0.0518$$

$$\therefore X(5) = 0.125 + j\,0.0518$$

Putting k = 6 in Equation (2),

$$X(6) = X^*(8-6) = X^*(2)$$

We have $X(2) = 0$, Thus $X^*(2) = 0$

$$\therefore X(6) = 0$$

Similarly putting k = 7 in Equation (2) we get,

$$X(7) = X^*(8-7) = X^*(1)$$

We have $X(1) = 0.125 - j\,0.3018$

$$\therefore X(7) = 0.125 + j\,0.3018$$

Prob. 2 : The first five DFT points of real and even sequence x(n) of length eight are given below. Find remaining three points. X(k) = {5, 1, 0, 2, 3, ...}

Soln. : Given DFT points are :

$$X(0) = 5,\ X(1) = 1,\ X(2) = 0,\ X(3) = 2 \quad \text{and} \quad X(4) = 3.$$

According to symmetry property we have,

$$X^*(k) = X(N-k)$$

$$\therefore \quad X(k) = X^*(N-k)$$

This is 8 point DFT. Thus N = 8

$$\therefore \quad X(k) = X^*(8-k)$$

$$\therefore \quad X(5) = X^*(8-5) = X^*(3)$$

$$\therefore X(5) = 2$$

$$X(6) = X^*(8-6) = X^*(2)$$

$$\therefore X(6) = 0$$

$$X(7) = X^*(8-7) = X^*(1)$$

$$\therefore X(7) = 1$$

1.3.6 Duality Property :

Statement : If $x(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X(k)$

then $x(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} N\,x[((-k))_N]$

Proof : Consider a discrete time sequence $x(n)$. Its periodic extension is denoted by $x_p(n)$. Now DFT of $x(n)$ is $X(k)$ and the periodic extension of $X(k)$ is denoted by $X_p(k)$.

That means,

$$x_p(n) = x((n))_N \qquad ...(1)$$

and $$X_p(k) = X((k))_N \qquad ...(2)$$

Thus we can write,

$$x_p(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X_p(k) \qquad ...(3)$$

Now we will define periodic sequence $x_{1p}(n) = X_p(n)$. One period of this sequence is a finite duration sequence $x_1(n) = x(n)$.

The discrete fourier series coefficients of $x_{1p}(n)$ are denoted by $X_{1p}(k)$ and $X_{1p}(k) = N\,x_p(-k)$.

Thus DFT of $x_1(n)$ which is denoted by $X_1(k)$ is,

$$X_1(k) = \begin{cases} N x_p(-k), & 0 \le k \le N-1 \\ 0 & \text{elsewhere} \end{cases} \quad ...(4)$$

Equation (4) can also be written as,

$$X_1(k) = \begin{cases} Nx[((-k))]_N, & 0 \le k \le N-1 \\ 0 & \text{elsewhere} \end{cases} \quad ...(5)$$

Now $X_1(k)$ is DFT of $x_1(n)$ and $x_1(n)$ is a finite duration sequence denoted by $X(n)$.

$$\therefore \quad X(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} N x[((-k))_N]$$

1.3.7 Circular Convolution :

Statement : The multiplication of two DFTs is equivalent to the circular convolution of their sequences in time domain.

Mathematical equation :

$$\text{If } x_1(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X_1(k) \text{ and } x_2(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X_2(k) \text{ then,}$$

$$x_1(n) \textcircled{N} x_2(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X_1(k) \cdot X_2(k) \quad ...(1)$$

Here $\textcircled{N}$ indicates circular convolution.

Let the result of circular convolution of $x_1(n)$ and $x_2(n)$ be $y(m)$ then the circular convolution can also be expressed as,

$$y(m) = \sum_{n=0}^{N-1} x_1(n) x_2((m-n))_N, \; m = 0, 1, ... N-1 \quad ...(2)$$

Here the term $x_2((m-n))_N$ indicates the circular convolution.

Proof : Consider two discrete time sequences $x_1(n)$ and $x_2(n)$. The DFT of $x_1(n)$ can be expressed as,

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n) e^{\frac{-j2\pi kn}{N}}, \; k = 0, 1, ... N-1 \quad ..(3)$$

To avoid the confusion we will write the DFT of $x_2(n)$ using different index of summation.

$$\therefore \quad X_2(k) = \sum_{l=0}^{N-1} x_2(n) e^{\frac{-j2\pi kl}{N}}, \; k = 0, 1, ... N-1 \quad ...(4)$$

Note that in Equation (4); instead of 'n' we have used '*l*'.

We will denote the multiplication of two DFTs; $X_1(k)$ and $X_2(k)$ by $Y(k)$.

$$\therefore \quad Y(k) = X_1(k) \cdot X_2(k) \qquad ...(5)$$

Let IDFT of $Y(k)$ be $y(m)$. Then using definition of IDFT,

$$y(m) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) e^{\frac{j 2\pi km}{N}} \qquad ...(6)$$

Putting Equation (5) in Equation (6),

$$y(m) = \frac{1}{N} \sum_{k=0}^{N-1} X_1(k) \cdot X_2(k) e^{\frac{j 2\pi km}{N}} \qquad ...(7)$$

Putting the values of $X_1(k)$ and $X_2(k)$ from Equations (3) and (4) in Equation (7) we get,

$$y(m) = \frac{1}{N} \sum_{k=0}^{N-1} \left[\sum_{n=0}^{N-1} x_1(n) e^{-\frac{j 2\pi kn}{N}} \right] \left[\sum_{l=0}^{N-1} x_2(l) e^{-\frac{j 2\pi kl}{N}} \right] e^{\frac{j 2\pi km}{N}} \qquad ...(8)$$

Rearranging the summations and terms in Equation (8) we get,

$$y(m) = \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \left[\sum_{k=0}^{N-1} e^{\frac{-j2\pi kl}{N}} \cdot e^{\frac{-j 2\pi kn}{N}} \cdot e^{j 2\pi km} \right]$$

$$\therefore \quad y(m) = \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \left[\sum_{k=0}^{N-1} e^{+j 2\pi k (m-n-l)/N} \right] \qquad ..(9)$$

Consider the last term of Equation (9); it can be written as,

$$e^{j 2\pi k (m-n-l)/N} = \left[e^{j 2\pi (m-n-l)/N} \right]^k \qquad ...(10)$$

Now use the standard summation formula,

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N & \text{for } a = 1 \\ \dfrac{1-a^N}{1-a} & \text{for } a \neq 1 \end{cases} \qquad ..(11)$$

$$\text{Let here,} \quad a = e^{+j 2\pi \frac{(m-n-l)}{N}} \qquad ...(12)$$

Now according to Equation (11) we will consider two cases :

Case (i) : When a = 1

If $(m-n-l)$ is multiple of N which means,

$(m-n-l)$ = N, 2N, 3N, ... then Equation (12) becomes,

$$a = e^{+j2\pi} = e^{+j 2\pi (2)} = e^{+j 2\pi (3)} \ldots = 1$$

Thus when $(m-n-l)$ is multiple of N (that means a = 1) then according to Equation (11);

the third summation in Equation (9) becomes equal to N.

Case (ii) : When a ≠ 1 :

If a ≠ 1 that means if $m - n - l$ is not multiple of N then according to Equation (11),

$$\sum_{k=0}^{N-1} a^k = \frac{1 - a^N}{1 - a} \qquad ..(13)$$

Putting Equation (12) in Equation (13) we get,

$$\sum_{k=0}^{N-1} \left(e^{+j 2\pi (m-n-l)} \right)^k = \frac{1 - e^{+j 2\pi (m-n-l)}}{1 - e^{\frac{+j 2\pi (m-n-l)}{N}}} \qquad ..(14)$$

Here m, n and l are integers; thus $e^{+j 2\pi (m-n-l)} = 1$ always. Thus R.H.S. of Equation (14) becomes zero when a ≠ 1. So to get the result of Equation (11) we have to consider the condition a = 1. That means when $m - n - l$ is multiple of N. For this condition, we have the result of summation equals to N. Thus Equation (9) becomes,

$$y(m) = \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \cdot N$$

$$\therefore \quad y(m) = \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \qquad ...(15)$$

We have obtained Equation (15) for the condition; $(m - n - l)$ is multiple of N. This condition can be expressed as,

$$m - n - l = -pN \qquad ...(16)$$

Here 'p' is an integer and an integer can be positive or negative. For the simplicity we have considered negative integer. Now from Equation (16) we get,

$$l = m - n + pN \qquad ...(17)$$

Putting this value in Equation (15) we get,

$$y(m) = \sum_{n=0}^{N-1} x_1(n) \cdot x_2(m - n + pN) \qquad ...(18)$$

Here we have not considered the second summation of Equation (15). Because this summation is in terms of 'l' and exponential term is absent in Equation (18).

Now the term $x_2(m - n + pN)$ indicates a periodic sequence with period N; this is because 'p' is an integer. This term also indicates that the periodic sequence is delayed by 'n' samples. We have studied that, if a sequence is periodic and delayed then it can be expressed as,

$$x_2(m - n + pN) = x_2((m - n))_N \qquad ...(19)$$

Here the R.H.S. term indicates circular shifting of $x_2(n)$. Putting this value in Equation (18). we get,

$$y(m) = \sum_{n=0}^{N-1} x_1(n) \cdot x_2((m-n))_N \qquad m = 0, 1, \ldots N-1 \qquad \ldots(20)$$

This is the equation of circular convolution. Since in this equation, the sequence $x_2(n)$ is shifted circularly; this type of convolution is called as CircularConvolution.

1.3.8 Solved Problems on Circular Convolution :

We can obtain the circular convolution of two sequences using two methods as follows :

A) Using Graphical Method B) Using Matrix Method.

A) Circular convolution using Graphical Method :

Prob. 1 : Given the two sequence of length 4 are :

$$x(n) = \{0, 1, 2, 3\}$$
$$h(n) = \{2, 1, 1, 2\}$$

Find the circular convolution.

Soln. : According to the definition of circular convolution,

$$y(m) = \sum_{n=0}^{N-1} x_1(n) \cdot x_2((m-n))_N \qquad \ldots(1)$$

Here given sequences are $x(n)$ and $h(n)$. The length of sequence is 4 that means N = 4. Thus Equation (1) becomes,

$$y(m) = \sum_{n=0}^{3} x(n)\, h((m-n))_4 \qquad \ldots(2)$$

Step I : Draw $x(n)$ and $h(n)$ as shown in Fig. F-11 (a) and (b).

Note that $x(n)$ and $h(n)$ are plotted in anticlockwise direction.

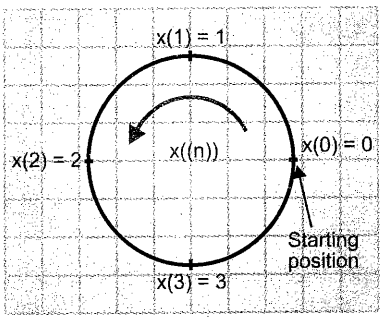


Fig. F-11 (a) : x(n) = {0, 1, 2, 3}

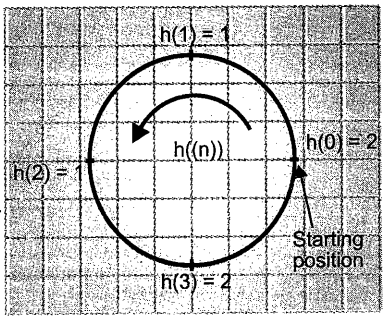


Fig. F-11 (b) : h (n) = {2, 1, 1, 2}

Now we will calculate different values of y (m) by putting m = 0 to m = 3 in Equation (2).

Step II : Calculation of y (0) :

Putting m = 0 in Equation (2) we get,

$$y(0) = \sum_{n=0}^{3} x(n)\, h((-n))_4 . \qquad ...(3)$$

Equation (3) shows that we have to obtain the product of x (n) and h ((– n)), and then we have to take the summation of product elements. Using graphical method this calculation is done as follows. The sequence $h((-n))_4$ indicates circular folding of h (n). This sequence is obtained by plotting h (n) in a clockwise direction as shown in Fig. F-11(c).

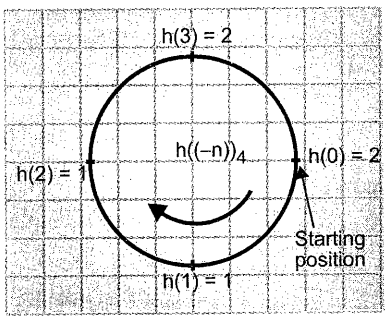


Fig. F-11(c) : h (n) is plotted in clockwise direction

To do the calculations; plot x (n) a nd h ((– n)) on two concentric circles as shown in Fig. F-11(d). x (n) is plotted on the inner circle and h ((– n)) is plotted on the outer circle. Now according to Equation (2); individual values of product x (n) and h ((– n) are obtained by multiplying two sequences point by point. Then y (0) is obtained by adding all product terms.

$$\therefore \quad y(0) = (0\times 2)+(1\times 2)+(1\times 2)+(3\times 1) = 0 + 2 + 2 + 3$$

$$\therefore \quad y(0) = 7$$

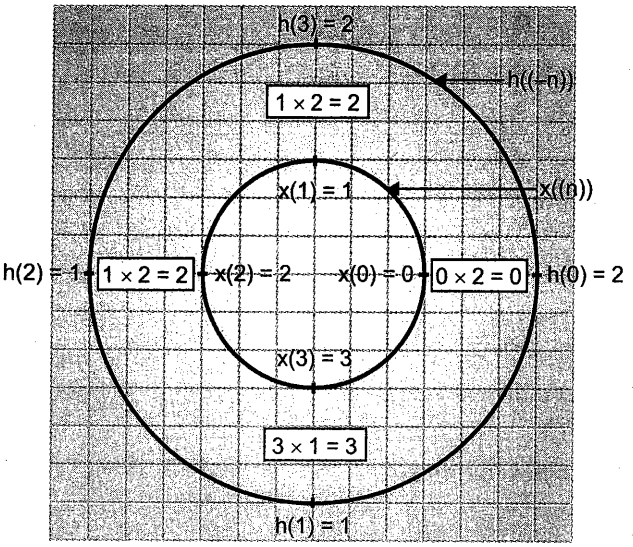


Fig. F-11(d) : $\sum_{n=0}^{3} x(n)\, h((-n))_4$

Step III : Calculation of y (1) : Putting m = 1 in Equation (2),

$$y(1) = \sum_{n=0}^{3} x(n)\, h((1-n))_4 \qquad ...(4)$$

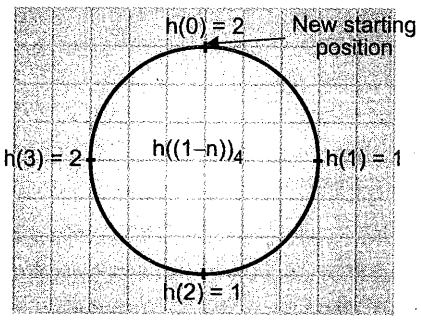


Fig. F-11(e) : $h((-n+1))_4$

Here $h((1-n))_4$ is same as $h((-n+1))_4$. This indicates delay of $h((-n))$ by 1 sample. This is obtained by shifting $h(-n))$ in anticlockwise direction by 1 sample, as shown in Fig. F-11(e).

We have already drawn the sequence x (n) as shown in Fig. F-11(a). To do the calculations, according to Equation (4), two sequences x (n) and $h((1-n))_4$ are plotted on two concentric circles as shown in Fig. F-11(f). y (1) is obtained by adding the product of individual terms.

$$\therefore \quad y(1) = (0\times1)+(3\times1)+(2\times2)+(1\times2) = 0+3+4+2$$

$$\therefore \quad y(1) = 9$$

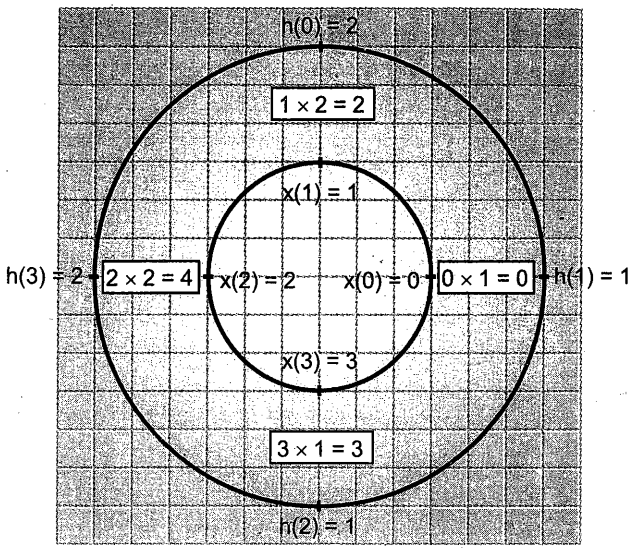


Fig. F-11(f) : $y(1) = \sum_{n=0}^{3} x(n)\, h((1-n))_4$

Step IV : Calculation of y (2) : Putting m = 2 in Equation (2) we get,

$$y(2) = \sum_{n=0}^{3} x(n)\, h((2-n))_4 \qquad ..(5)$$

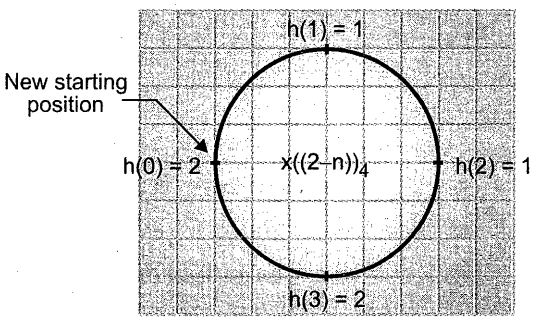


Fig. F-11(g) : $h((-n+2))_4$

Here $h((2-n))_4$ is same as $h((-n+2))_4$. It indicates delay of $h((-n))_4$ by 2 samples. It is obtained by shifting $h((-n))_4$ by two samples in anticlockwise direction as shown in Fig. F-11(g).

According to Equation (4) the value of y (2) is obtain by adding individual product terms as shown in Fig. F-11(h).

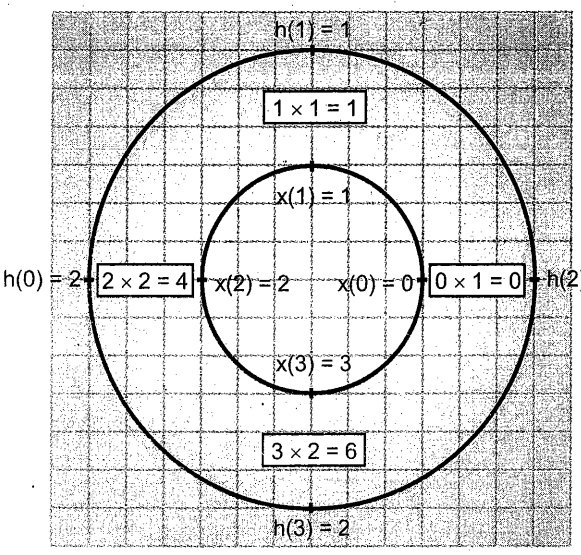


Fig. F-11(h) : $y(2) = \sum_{n=0}^{3} x(n)\, h((2-n))_4$

$$\therefore \quad y(2) = (0\times1)+(3\times2)+(2\times2)+(1\times1) = 0+6+4+1$$

$$\therefore \quad y(2) = 11$$

Step V : Calculation of y (3) :

Putting m = 3 in Equation (2) we get,

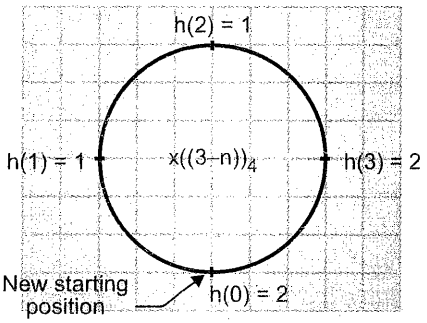


Fig. F-11(i) : $h((-n+3))_4$

$$y(3) = \sum_{n=0}^{3} x(n)\, h((3-n))_4 \qquad ...(6)$$

Here $h((3-n))_4$ is same as $h((-n+3))_4$. It indicates delay of $h((-n))_4$ by 3 samples. It is obtained by shifting $h((-n)_4$ by 3 samples in anticlockwise direction as shown in Fig. F-11(i).

According to Equation (5), y (3) is obtained by adding individual product terms as shown in Fig. F-11(j).

$$\therefore \quad y(3) = (0\times2)+(3\times2)+(2\times1)+(1\times1) = 0+6+2+1$$

$$\therefore \quad y(3) = 9$$

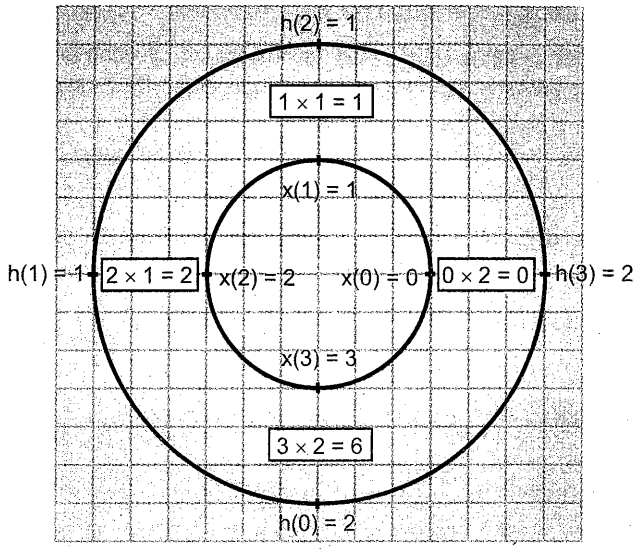


Fig. F-11(j) : $y(3) = \sum_{n=0}^{3} x(n)\, h((3-n))_4$

Now the resultant sequence y (m) can be written as,

$$y(m) = \{y(0), y(1), y(2), y(3)\}$$

$$\therefore \quad y(m) = \{7, 9, 11, 9\}$$

Prob. 2 : Using graphical method, obtain a 5-point circular convolution of two DT signals defined as,

$$x(n) = (1.5)^n, \qquad 0 \le n \le 2$$
$$y(n) = 2n - 3, \qquad 0 \le n \le 3$$

Does the circular convolution obtained is same to that of linear convolution ?

Soln. : First we will obtain the sequences x (n) and y (n) by putting values of n as follows :

Given, $x(n) = (1.5)^n, \quad 0 \le n \le 2$

For $n = 0 \Rightarrow x(0) = (1.5)^0 = 1$

For $n = 1 \Rightarrow x(1) = (1.5)^1 = 1.5$

For $n = 2 \Rightarrow x(2) = (1.5)^2 = 2.25$

$\therefore \quad x(n) = \{1, 1.5, 2.25\}$ ...(1)

Now $y(n) = 2n - 3, \; 0 \le n \le 3$

For $n = 0 \Rightarrow y(0) = 0 - 3 = -3$

For $n = 1 \Rightarrow y(1) = 2 - 3 = -1$

For $n = 2 \Rightarrow y(2) = 4 - 3 = 1$

For $n = 3 \Rightarrow y(3) = 6 - 3 = 3$

$$\therefore \quad y(n) = \{-3, -1, 1, 3\} \qquad ...(2)$$

It is asked to calculate 5-point DFT. That means length of each sequence should be 5. This length is adjusted by adding zeros at the end of each sequence as follows (zero padding) :

$$x(n) = \{1, 1.5, 2.25, 0, 0\} \qquad ...(3)$$

$$\text{and} \quad y(n) = \{-3, -1, 1, 3, 0\} \qquad ...(4)$$

Now according to the definition of circular convolution we have,

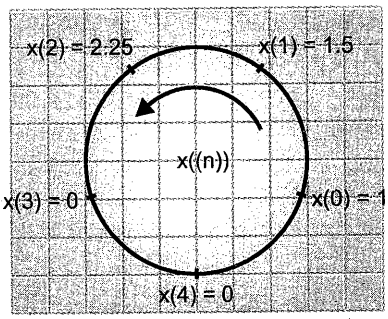
$$y(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2((m-n))_N \qquad ...(5)$$

Here the given sequences are x (n) and y (n) and length N = 5.

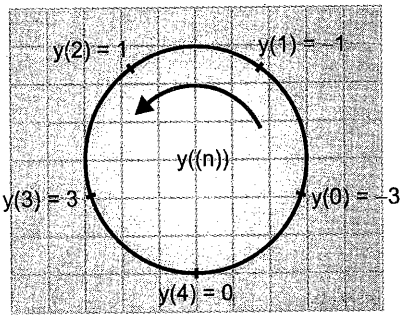
$$\therefore \quad y(m) = \sum_{n=0}^{4} x(n)\, y((m-n))_5 \qquad ...(6)$$

Step I : Draw x (n) and y (n) as shown in Fig. F-12(a) and (b) respectively.

Here x (n) and y (n) are plotted in anticlockwise direction.



(a) x (n) = {1, 1.5, 2.25, 0, 0}



(b) y (n) = {– 3, – 1, 1, 3, 0}

Fig. F-12

Now we will obtain values of y (m) by putting m = 0 to m = 4 in Equation (6).

Step II : Calculation of y (0) :

Putting m = 0 in Equation (6) we get, $$y(0) = \sum_{n=0}^{4} x(n)\, y((-n))_5 \quad ...(7)$$

Here y ((– n)) indicates circular 'folding of y (n). It is obtained by plotting y (n) in clockwise direction as shown in Fig. F-12(c).

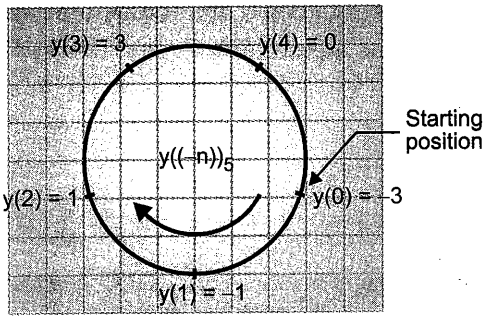


Fig. F-12(c) : y ((– n))

According to Equation (7), y (0) is obtained by adding all product terms as shown in Fig. F-12(d). Here x (n) and y ((– n))$_5$ is drawn on two concentric circles.

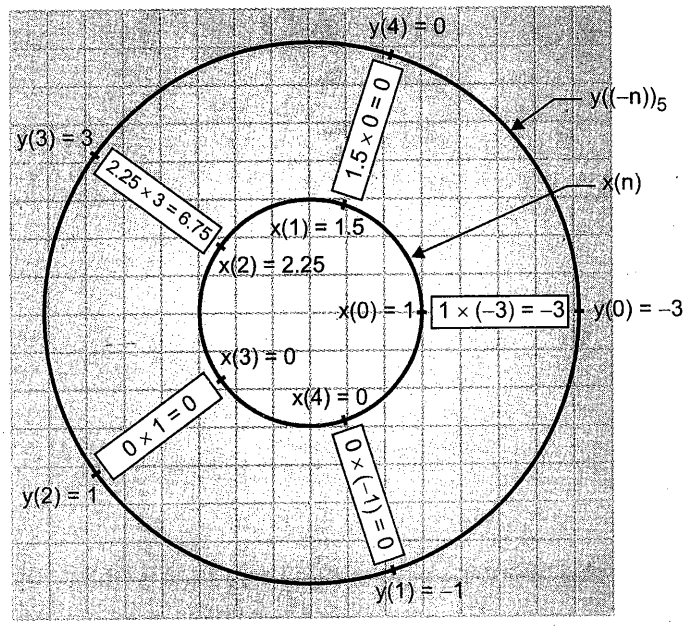
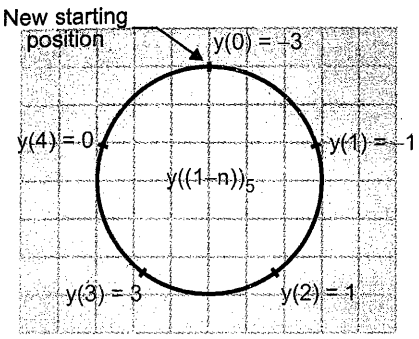


Fig. F-12(d) : $y(0) = \sum_{n=0}^{4} x(n)\, y((-n))$

$$\therefore \quad y(0) = [1\times(-3)] + [0\times(-3)] + [0\times 1] + [2.25\times 3] + [1.5\times 0]$$
$$= 3 + 0 + 0 + 6.75 + 0$$

$\therefore$ y (0) = 3.75

Step III : Putting m = 1 in Equation (6),



$$y(1) = \sum_{n=0}^{4} x(n)\, y((1-n))_5 \qquad ...(8)$$

Here $y((1-n))_5$ indicates delay of $y((-n))$ by 1 sample. It is obtained by shifting $y((-n))$ by 1 sample in anticlockwise direction as shown in Fig. F-12(e).

Fig. F-12(e) : $y((1-n))_5$

According to Equation (8), value of y (1) is obtained a shown in Fig. F-12(f).

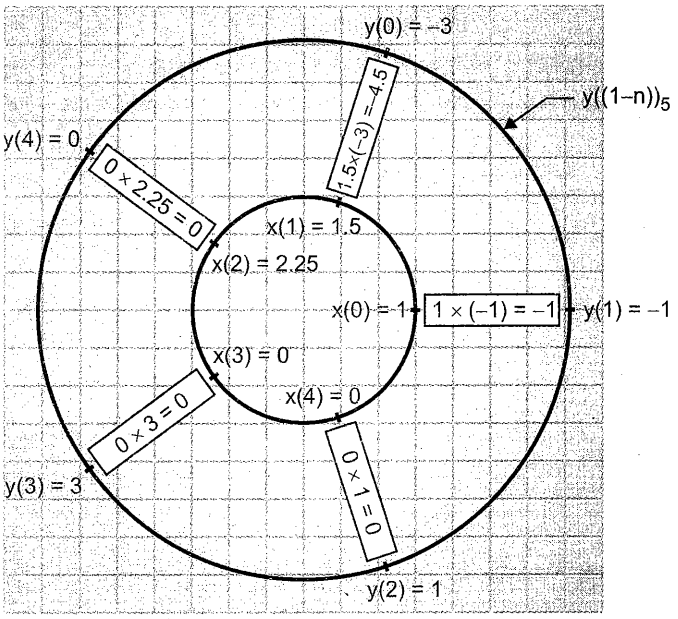


Fig. F-12(f) : $y(1) = \sum_{n=0}^{4} x(n)\, y((1-n))_5$

$$\therefore \quad y(1) = [1\times(-1)] + [0\times1] + [0\times3] + [0\times2.25] + [1.5\times(-3)] = -1+0+0+0-4.5$$

$$\therefore \quad y(1) = -5.5$$

Step IV : Putting m = 2 in Equation (6),

$$y(2) = \sum_{n=0}^{4} x(n)\, y((2-n))_5 \qquad ...(9)$$

Here $y((2-n))_5$ is delay of $y((-n))_5$ by 2 samples. It is obtained by shifting $y((-n))_5$ by 2 samples in anticlockwise direction as shown in Fig. F-12(g).

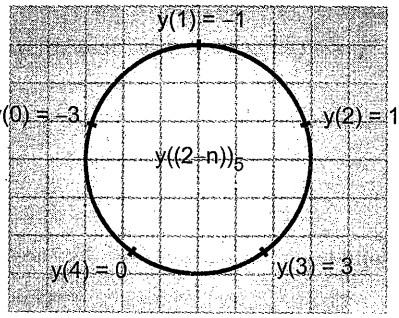


Fig. F-12(g) : $y((2-n))_5$

y (2) is obtained by adding all product terms as shown in Fig. F-12(h).

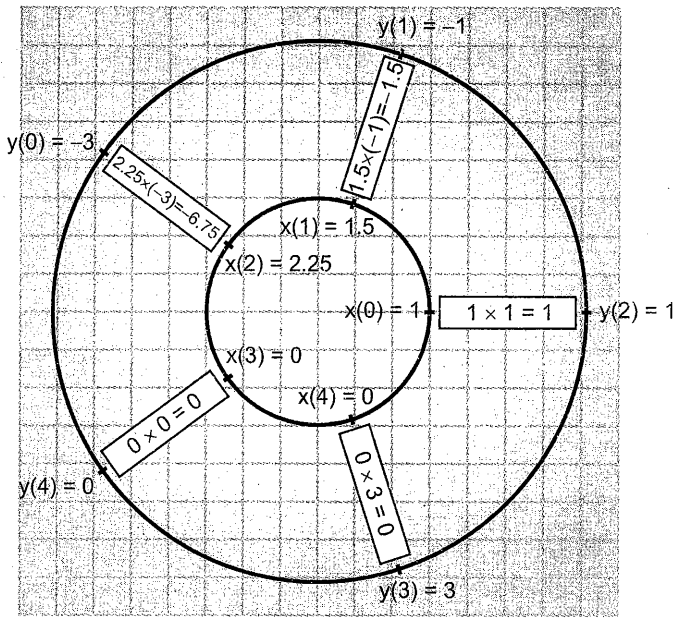


Fig. F-12(h) : $y(2) = \sum_{n=0}^{4} x(n)\, y((2-n))_5$

$$\therefore \quad y(2) = [1\times 1] + [0\times 3] + [0\times 0] + [2.25\times(-3)] + [1.5\times(-1)] = 1 + 0 + 0 - 6.75 - 1.5$$

$$\therefore \quad y(2) = -7.25$$

Step V : Calculation of **y (3)** :

Putting m = 3 in Equation (6) we get,

$$y(3) = \sum_{n=0}^{4} x(n)\, y((3-n))_5 \qquad ...(10)$$

Here $y((3-n))_5$ is delay of $y((-n))_5$ by 3 samples. It is obtained by shifting $y((-n))_5$ by 3 samples in anticlockwise direction as shown in Fig. F-12(i).

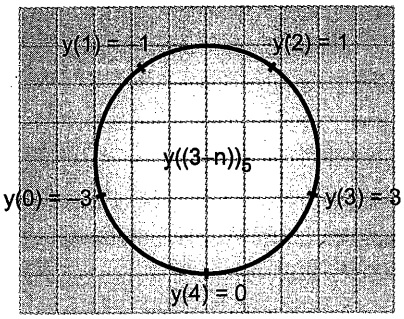


Fig. F-12(i) : $y((3-n))_5$

y (3) is obtained by adding all product terms as shown in Fig. F-12(j).

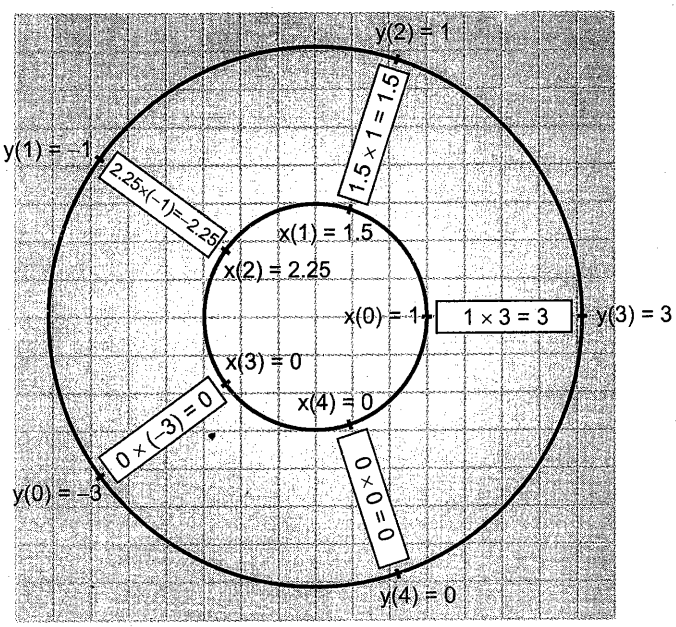


Fig. F-12(j) : $y(3) = \sum_{n=0}^{4} x(n)\, y((3-n))_5$

$$\therefore \quad y(3) = [1\times 3] + [0\times 0] + [0\times(-3)] + [2.25\times(-1)] + [1.5\times 1]$$

$$= 3 + 0 + 0 - 2.25 + 1.5$$

$$\therefore y(3) = 2.25$$

Step VI : Calculation of y (4) :

Putting m = 4 in Equation (6) we get,

$$y(4) = \sum_{n=0}^{4} x(n)\, y((4-n))_5 \qquad ...(11)$$

Here $y((4-n))_5$ is delay of $y((-n))_5$ by 4 samples. It is obtained by shifting $y((-n))_5$ by 4 samples in anticlockwise direction as shown in Fig. F-12(k).

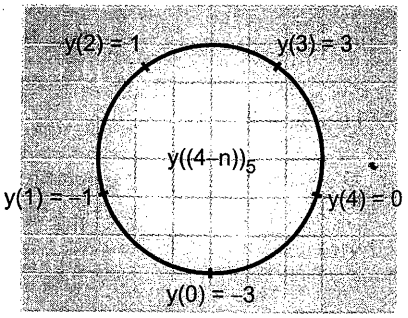


Fig. F-12(k) : $y((4-n))_5$

y (4) is obtained as shown in Fig. F-12(*l*)

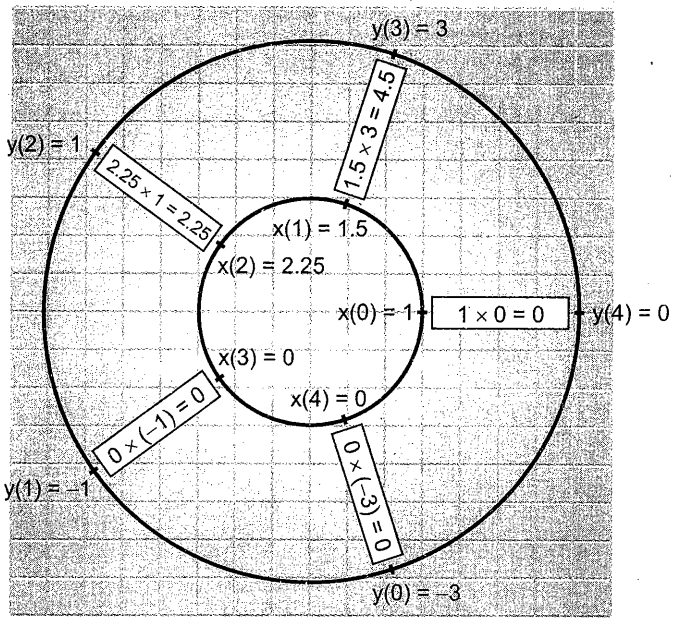


Fig. F-12(*l*) : $y(4) = \sum_{n=0}^{4} x(n)\, y((4-n))_5$

$$\therefore \quad y(4) = [1\times 0] + [0\times(-3)] + [0\times(-1)] + [2.25\times 1] + [1.5\times 3]$$
$$= 0 + 0 + 0 + 2.25 + 4.5$$

$$\therefore y(4) = 6.75$$

Now the result of circular convolution can be expressed as,

$$y(m) = \{y(0), y(1), y(2), y(3), y(4)\}$$

$$\therefore \quad y(m) = \{3.75, -5.5, -7.25, 2.25, 6.75\} \qquad ...(12)$$

A) Comparison with linear convolution :

We will obtain linear convolution of two sequences using tabular method. We have,

$$x(n) = \{1, 1.5, 2.25\} = \{1, 1.5, 2.25, 0\}$$

$$\text{and} \quad y(n) = \{-3, -1, 1, 3\}$$

$$\text{Let} \quad y_1(n) = x(n) * y(n)$$

The linear convolution of x (n) and y (n) is shown in Fig. F-12(m).

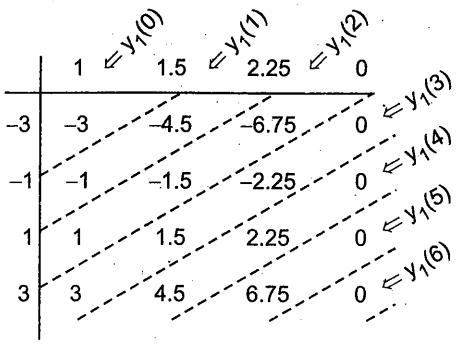


Fig. F-12(m) : x (n) * y (n)

From Fig. F-12(m),

$$y_1(0) = -3$$

$$y_1(1) = -1 - 4.5 = -5.5$$

$$y_1(2) = 1 - 1.5 - 6.75 = -7.25$$

$$y_1(3) = 3 + 1.5 - 2.25 = 2.25$$

$$y_1(4) = 4.5 + 2.25 = 6.75$$

$$y_1(5) = 6.75 + 0 = 6.75$$

$$y_1(6) = 0$$

$$\text{Thus} \quad x(n) * y(n) = \{-3, -5.5, -7.25, 2.25, 6.75, 6.75, 0\} \qquad ..(13)$$

Equations (12) and (13) show that circular convolution and linear convolution are not same.

B) Circular convolution using matrix method :

The graphical method which we have studied is tedious, especially when many samples are present. While the matrix method is more convenient. In the matrix method, one sequence is repeated via circular shifting of samples. It is represented as follows :

we have $y(m) = x(n) \textcircled{N} h(n) = h(n) \textcircled{N} Nx(n)$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \cdot \\ \cdot \\ \cdot \\ y(N-2) \\ y(N-1) \end{bmatrix} = \begin{bmatrix} h(0) & h(N-1) & h(N-2) & \cdots & h(2) & h(1) \\ h(1) & h(0) & h(N-1) & \cdots & h(3) & h(2) \\ h(2) & h(1) & h(0) & \cdots & h(4) & h(3) \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ h(N-2) & h(N-3) & h(N-4) & \cdots & h(0) & h(N-1) \\ h(N-1) & h(N-2) & h(N-3) & \cdots & h(1) & h(0) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \cdot \\ \cdot \\ \cdot \\ x(N-2) \\ x(N-1) \end{bmatrix}$$

Prob. 3 : Determine the sequence

$$y(n) = x(n) \textcircled{N} h(n)$$

where $x(n) = \{1, 2, 3, 1\}$ (↑ at 1)

and $h(n) = \{4, 3, 2, 2\}$ (↑ at 1)

Soln. :

We have, $y(m) = x_2(n) \textcircled{N} x_1(n) = h(n) \textcircled{N} x(n)$

Using matrix method,

Here $x(0) = 1$, $x(1) = 2$, $x(3) = 3$, $x(3) = 1$

and $h(0) = 4$, $h(1) = 3$, $h(2) = 2$, $h(3) = 2$

Here $N = 4$

In the matrix form we have

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} h(0) & h(3) & h(2) & h(1) \\ h(1) & h(0) & h(3) & h(2) \\ h(2) & h(1) & h(0) & h(3) \\ h(3) & h(2) & h(1) & h(0) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 & 3 \\ 3 & 4 & 2 & 2 \\ 2 & 3 & 4 & 2 \\ 2 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} (4\times1)+(2\times2)+(2\times3)+(3\times1) \\ (3\times1)+(4\times2)+(2\times3)+(2\times1) \\ (2\times1)+(3\times2)+(4\times3)+(2\times1) \\ (2\times1)+(2\times2)+(3\times3)+(4\times1) \end{bmatrix} = \begin{bmatrix} 4+4+6+3 \\ 3+8+6+2 \\ 2+6+12+2 \\ 2+4+9+4 \end{bmatrix}$$

$$\therefore \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} 17 \\ 19 \\ 22 \\ 19 \end{bmatrix}$$

$$\therefore \quad y(m) = x(n) \textcircled{N} h(n) = \{17, 19, 22, 19\}$$

Prob. 4 : Use the four point DFT and IDFT to determine the circular convolution of sequences

$x_1(n) = (1, 2, 3, 1)$
$\uparrow$
$x_2(n) = (4, 3, 2, 2)$
$\uparrow$

Soln. : The four point DFT of $x_1(n)$ is $X_1(k)$ and it is given by,

$$X_1(k) = [W_4]\, x_{1N}$$

We have, $$[W_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$\therefore \quad X_1(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

$$\therefore \quad X_1(k) = \begin{bmatrix} 1+2+3+1 \\ 1-2j-3+j \\ 1-2+3-1 \\ 1+2j-3-j \end{bmatrix} = \begin{bmatrix} 7 \\ -2-j \\ 1 \\ -2+j \end{bmatrix}$$

$$\therefore \quad X_1(k) = \{7, -2-j, 1, -2+j\} \qquad ...(1)$$

Similarly, $$X_2(k) = [W_4]\, x_{2N}$$

$$\therefore \quad X_2(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 2 \end{bmatrix}$$

$$\therefore \quad X_2(k) = \begin{bmatrix} 4+3+2+2 \\ 4-3j-2+2j \\ 4-3+2-2 \\ 4+3j-2-2j \end{bmatrix} = \begin{bmatrix} 11 \\ 2-j \\ 1 \\ 2+j \end{bmatrix}$$

$$\therefore \quad X_2(k) = \{11, 2-j, 1, 2+j\}$$

Now according to property of circular convolution,

$$x_1(n) \textcircled{N} x_2(n) = X_1(k) \cdot X_2(k) = X_3(k)$$

$$\therefore \quad X_3(k) = \{7, -2-j, 1, -2+j\} \cdot \{11, +2-j, 1, 2+j\}$$

$$\therefore \quad X_3(k) = \{77, -5, 1, -5\}$$

Let the result of $x_1(n) \textcircled{N} x_2(n)$ be sequence $x_3(n)$. It is obtained by computing IDFT of $X_3(k)$. According to the definition of IDFT we have,

$$x_3(n) = \frac{1}{N}[W_N^*] \cdot X_N$$

$$\therefore \quad x_3(n) = \frac{1}{4}[W_4^*] \cdot X_{3N}$$

$$\therefore \quad x_3(n) = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & +j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & +j \end{bmatrix} \begin{bmatrix} 77 \\ -5 \\ 1 \\ -5 \end{bmatrix}$$

$$\therefore \quad x_3(n) = \frac{1}{4}\begin{bmatrix} 77-5+1-5 \\ 77-5j-1+5j \\ 77+5+1+5 \\ 77+5j-1-5j \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 68 \\ 76 \\ 88 \\ 76 \end{bmatrix} = \begin{bmatrix} 17 \\ 19 \\ 22 \\ 19 \end{bmatrix}$$

Considering real part only, approximately sequence $x_3(n)$ can be written as,

$$x_3(n) = \{17, 19, 22, 19\}$$

Prob. 5 : Compute the 8-point circular convolution for following sequences

$$x_1(n) = \{1, 1, 1, 1, 0, 0, 0, 0\}$$

$$x_2(n) = \sin\left(\frac{3\pi n}{8}\right) \quad 0 \le n \le 7$$

Soln. : First we will obtain sequence $x_2(n)$ by putting values of n from n = 0 to n = 7.

For $n = 0 \Rightarrow x_2(0) = \sin\left(\frac{3\pi \times 0}{8}\right) = 0$

For $n = 1 \Rightarrow x_2(1) = \sin\left(\frac{3\pi \times 1}{8}\right) = 0.92$

For $n = 2 \Rightarrow x_2(2) = \sin\left(\frac{3\pi \times 2}{8}\right) = 0.707$

For $n = 3 \Rightarrow x_2(3) = \sin\left(\frac{3\pi \times 3}{8}\right) = -0.38$

For $n = 4 \Rightarrow x_2(4) = \sin\left(\frac{3\pi \times 4}{8}\right) = -1$

For $n = 5 \Rightarrow x_2(5) = \sin\left(\frac{3\pi \times 5}{8}\right) = -0.38$

For $n = 6 \Rightarrow x_2(6) = \sin\left(\frac{3\pi \times 6}{8}\right) = 0.707$

For $n = 7 \Rightarrow x_2(7) = \sin\left(\frac{3\pi \times 7}{8}\right) = 0.92$

Thus sequence $x_2(n)$ is,

$$x_2(n) = \{0, 0.92, 0.707, -0.38, -1, -0.38, 0.707, 0.92\}$$

Now we have, $y(m) = x_1(n) \textcircled{N} x_2(n)$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ y(6) \\ y(7) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.92 \\ 0.707 \\ -0.38 \\ -1 \\ -0.38 \\ 0.707 \\ 0.92 \end{bmatrix}$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ y(6) \\ y(7) \end{bmatrix} = \begin{bmatrix} 0+0+0+0+0-0.38+0.707+0.92 \\ 0+0.92+0+0+0+0+0.707+0.92 \\ 0+0.92+0.707+0+0+0+0+0.92 \\ 0+0.92+0.707-0.38+0+0+0+0 \\ 0+0.92+0.707-0.38-1+0+0+0 \\ 0+0+0.707-0.38-1-0.38+0+0 \\ 0+0+0-0.38-1-0.38+0.707+0 \\ 0+0+0+0-1-0.38+0.707+0.92 \end{bmatrix} = \begin{bmatrix} 1.247 \\ 2.547 \\ 2.547 \\ 1.247 \\ 0.247 \\ -1.053 \\ -1.053 \\ 0.247 \end{bmatrix}$$

$$\therefore \quad y(m) = \{1.247, 2.547, 2.547, 1.247, 0.247, -1.053, -1.053, 0.247\}$$
$$\uparrow$$

Prob. 6 : Compute the circular convolution of following sequences and compare the results with linear convolution.

$$x(n) = \{1, 1, 1, 1, -1, -1, -1, -1\}$$
$$\text{and } h(n) = \{0, 1, 2, 3, 4, 3, 2, 1\}$$

Soln. : We have,

$$y(m) = x(n) \textcircled{N} h(n)$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ y(6) \\ y(7) \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ y(6) \\ y(7) \end{bmatrix} = \begin{bmatrix} 0-1-2-3-4+3+2+1 \\ 0+1-2-3-4-3+2+1 \\ 0+1+2-3-4-3-2+1 \\ 0+1+2+3-4-3-2-1 \\ 0+1+2+3+4-3-2-1 \\ 0-1+2+3+4+3-2-1 \\ 0-1-2+3+4+3+2-1 \\ 0-1-2-3+4+3+2+1 \end{bmatrix} = \begin{bmatrix} -4 \\ -8 \\ -8 \\ -4 \\ 4 \\ 8 \\ 8 \\ 4 \end{bmatrix}$$

$$\therefore \quad y(m) = x(n) \textcircled{N} h(n) = \{-4, -8, -8, -4, 4, 8, 8, 4\} \qquad \text{...(1)}$$

$$\uparrow$$

Now we will obtain linear convolution of given sequences as shown in Fig. F-13.

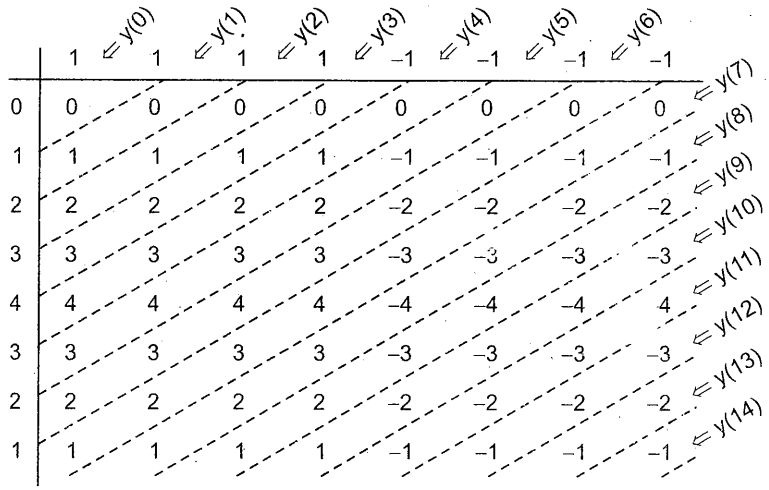


Fig. F-13 : x (n) * h (n)

$$\therefore \quad y(n) = x(n) * h(n)$$

$$\therefore \quad y(n) = \{0, 1, 3, 6, 10, 11, 9, 4, -4, -9, -11, -10, -6, -3, -1\} \qquad \text{...(2)}$$

From Equations (1) and (2) we can conclude that the results of circular convolution and linear convolution are not same.

Important note :

Why the result of circular and linear convolution is not same ?

Basically circular convolution, y (m) contains same number of samples as that of x (n) and h (n). In the previous example x (n) and h (n) have 8 samples. The circular convolution is given by Equation (1). It also contains 8 samples.

In the previous example; the linear circular is given by Equation (2). Here the number of samples are 15. We will use the following notations to indicates numbers of samples.

L = Number of samples in x (n)

M = Number of samples in h (n)

N = Number of samples in the result of linear convolution.

Thus for the linear convolution we can write the equation,

$$N = L + M - 1 \qquad ...(3)$$

Here $L = M = 8 \Rightarrow N = 15$

That's why the result of linear and circular convolution are not same.

How to obtain same result from linear and circular convolution ?

To obtain the same result from both convolutions the following steps are used :

(1) Using Equation (3), calculate the value of 'N'; that means number of samples contained in linear convolution. Let us say it is 15.

(2) By doing zero padding make the length of every sequence equal to 15. That means in this case we need to add seven zeros in x (n) as well as h (n).

(3) Perform the circular convolution. The result of circular convolution and linear convolution will be same.

Prob. 7 : Compute the circular convolution of the following sequences and compare the results with linear convolution.

$$x(n) = \{1, 0.5, 1, 0.5, 1, 0.5, 1, 0.5\}$$

$$h(n) = \{0, 1, 2, 3\}$$

Soln. :

In case of circular convolution the length of x (n) and h (n) should be same. Thus we have to add four zeros in h (n) to make its length equal to 8.

$$\therefore \quad h(n) = \{0, 1, 2, 3, 0, 0, 0, 0\}$$

Now circular convolution of x (n) and h (n) is given by,

$$y(m) = x(n) \textcircled{N} h(n)$$

$$= \begin{bmatrix} 1 & 0.5 & 1 & 0.5 & 1 & 0.5 & 1 & 0.5 \\ 0.5 & 1 & 0.5 & 1 & 0.5 & 1 & 0.5 & 1 \\ 1 & 0.5 & 1 & 0.5 & 1 & 0.5 & 1 & 0.5 \\ 0.5 & 1 & 0.5 & 1 & 0.5 & 1 & 0.5 & 1 \\ 1 & 0.5 & 1 & 0.5 & 1 & 0.5 & 1 & 0.5 \\ 0.5 & 1 & 0.5 & 1 & 0.5 & 1 & 0.5 & 1 \\ 1 & 0.5 & 1 & 0.5 & 1 & 0.5 & 1 & 0.5 \\ 0.5 & 1 & 0.5 & 1 & 0.5 & 1 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ y(6) \\ y(7) \end{bmatrix} = \begin{bmatrix} 0+0.5+2+1.5+0+0+0+0 \\ 0+1+1+3+0+0+0+0 \\ 0+0.5+2+1.5+0+0+0+0 \\ 0+1+1+3+0+0+0+0 \\ 0+0.5+2+1.5+0+0+0+0 \\ 0+1+2+3+0+0+0+0 \\ 0+0.5+2+1.5+0+0+0+0 \\ 0+1+1+3+0+0+0+0 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 4 \\ 5 \\ 4 \\ 5 \\ 4 \\ 5 \end{bmatrix}$$

$$\therefore \quad y(m) = \{4, 5, 4, 5, 4, 5, 4, 5\}$$

Comparison with linear convolution :

$x(n)$ has 8 samples and $h(n)$ has 4 samples. So linear convolution will have $8 + 4 - 1 = 11$ samples. Thus result of linear and circular convolution will not be same. We can verify this by obtaining linear convolution of $x(n)$ and $h(n)$.

Prob. 8 : Find the linear convolution of $x_1(n)$ and $x_2(n)$

if $x_1(n) = \{1, 2, 3, 4\}$ and $x_2(n) = \{1, 1, 1\}$
$\uparrow$ $\uparrow$

Obtain the same result using circular convolution.

Soln. : First we will obtain linear convolution.

Here $x_1(n) = \{1, 2, 3, 4\}$
$\uparrow$

and $x_2(n) = \{1, 1, 1\} = \{1, 1, 1, 0\}$
$\uparrow$

Using tabular method, linear convolution is obtained as shown in Fig. F-14.

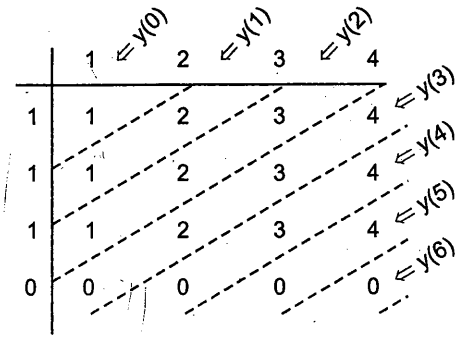


Fig. F-14 : $y(n) = x_1(n) * x_2(n)$

$$\therefore \quad y(n) = x_1(n) * x_2(n) = \{1, 3, 6, 9, 7, 4, 0\} \qquad ...(1)$$

Now we will obtain the same result using circular convolution. Here the length of linear convolution = N = 7. We have to make the number of samples in $x(n)$ and $h(n)$ equal to 7. But this length is preferred to be in terms of powers of 2. So we will make length of $x(n)$ and $h(n)$ equal to 8 by doing zero padding.

$\therefore \quad x_1(n) = \{1, 2, 3, 4, 0, 0, 0, 0\}$ and $x_2(n) = \{1, 1, 1, 0, 0, 0, 0, 0\}$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 4 & 3 & 2 \\ 2 & 1 & 0 & 0 & 0 & 0 & 4 & 3 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 & 4 \\ 4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ y(6) \\ y(7) \end{bmatrix} = \begin{bmatrix} 1+0+0+0+0+0+0+0 \\ 2+1+0+0+0+0+0+0 \\ 3+2+1+0+0+0+0+0 \\ 4+3+2+0+0+0+0+0 \\ 0+4+3+0+0+0+0+0 \\ 0+0+4+0+0+0+0+0 \\ 0+0+0+0+0+0+0+0 \\ 0+0+0+0+0+0+0+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 9 \\ 7 \\ 4 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad y(m) = \{1, 3, 6, 9, 7, 4, 0, 0\} \qquad ...(2)$$

Equations (1) and (2) are same, so the result of linear and circular convolution is same.

Prob. 9 : Obtain the linear convolution of two sequences defined as,

$$x(n) = u(n) - u(n-3)$$

$$h(n) = u(n-1) + u(n-2) - u(n-4) - u(n-5)$$

using circular convolution.

Soln. : First we will obtain the sequences x (n) and h (n).

We have, $$x(n) = u(n) - u(n-3) \qquad ...(1)$$

This sequence is obtained as shown in Fig. F-15(a).

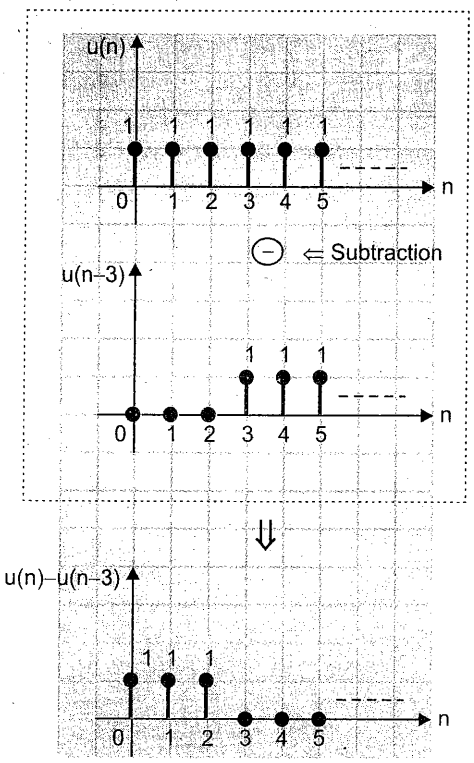
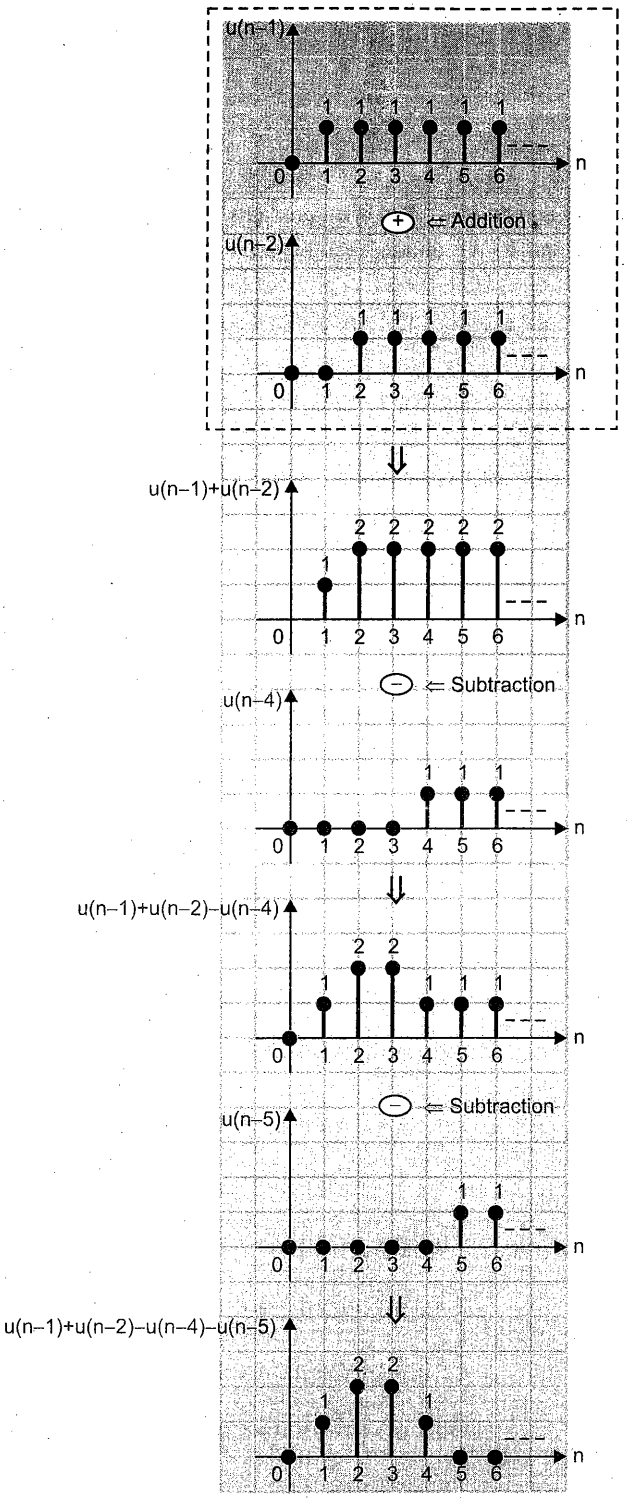


Fig. F-15(a) : x (n) = u (n) – u (n – 3)

$$\therefore \quad x(n) = \{1, 1, 1\} \qquad ...(2)$$
$$\uparrow$$

Now $$h(n) = u(n-1) + u(n-2) - u(n-4) - u(n-5) \qquad ...(3)$$

This operation is performed as shown in Fig. F-15(b).



F-15(b) : $h(n) = u(n-1) + u(n-2) - u(n-4) - u(n-5)$

$$\therefore \quad h(n) = \{0, 1, 2, 2, 1\} \qquad \text{...(4)}$$
$$\uparrow$$

Here $x(n)$ has 3 samples and $h(n)$ has 5 samples. So linear convolution will give $(3+5)-1 = 7$ samples. To obtain the same result using circular convolution we need to make length of $x(n)$ and $h(n)$ equals to 7. But we want this length (value of N) in power of 2. So we will make length of $x(n)$ and $h(n)$ equals to 8 by zero padding.

$$\therefore \quad x(n) = \{1, 1, 1, 0, 0, 0, 0, 0\} \qquad \text{...(5)}$$
$$\uparrow$$

$$\text{and} \quad h(n) = \{0, 1, 2, 2, 1, 0, 0, 0\} \qquad \text{...(6)}$$

Now we will perform circular convolution as follows.

$$y(m) = x(n) \textcircled{N} h(n)$$

$$\therefore \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ y(6) \\ y(7) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+0+0+0+0+0+0 \\ 0+1+0+0+0+0+0+0 \\ 0+1+2+0+0+0+0+0 \\ 0+1+2+2+0+0+0+0 \\ 0+0+2+2+1+0+0+0 \\ 0+0+0+2+1+0+0+0 \\ 0+0+0+0+1+0+0+0 \\ 0+0+0+0+0+0+0+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 5 \\ 5 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore \quad y(m) = x(n) * h(n) = x(n) \textcircled{N} h(n)$$

$$= \{0, 1, 3, 5, 5, 3, 1, 0\}$$
$$\uparrow$$

hence linear convolution is obtained using circular convolution.

Prob. 10 : Using circular convolution, find output of system if input $x(n)$ and impulse response $h(n)$ is given by,

$x(n) = 2u(n) - u(n-2) - u(n-4)$

$h(n) = 3\delta(n) - 2\delta(n-1) + \delta(n-2)$

Soln. :

First we will plot sequence $x(n)$ and $h(n)$ as shown in Figs. F-16(a) and F-16(b).

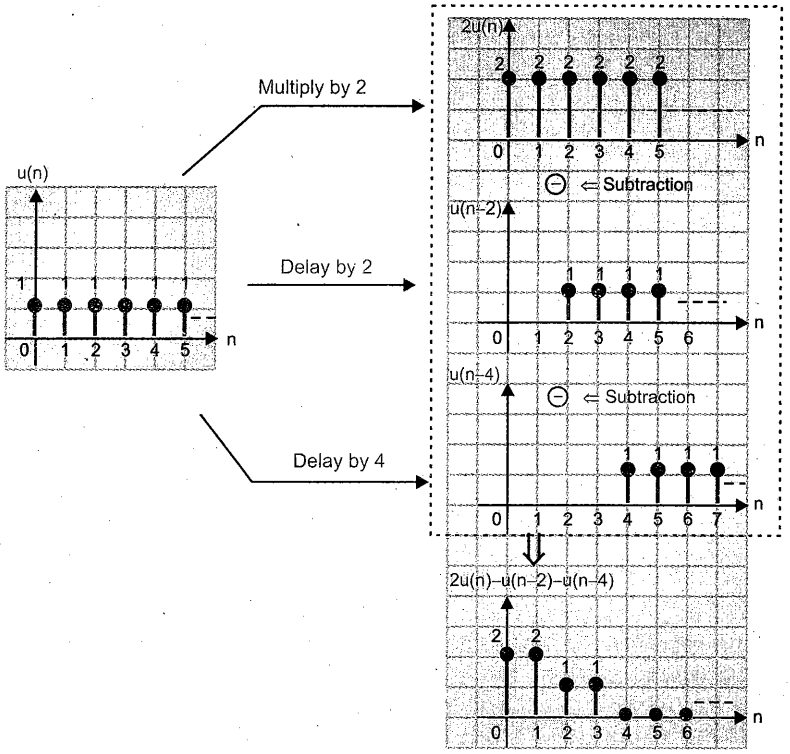


Fig. F-16(a)

$$\therefore \quad x(n) = \{2, 2, 1, 1\}$$
$$\uparrow$$

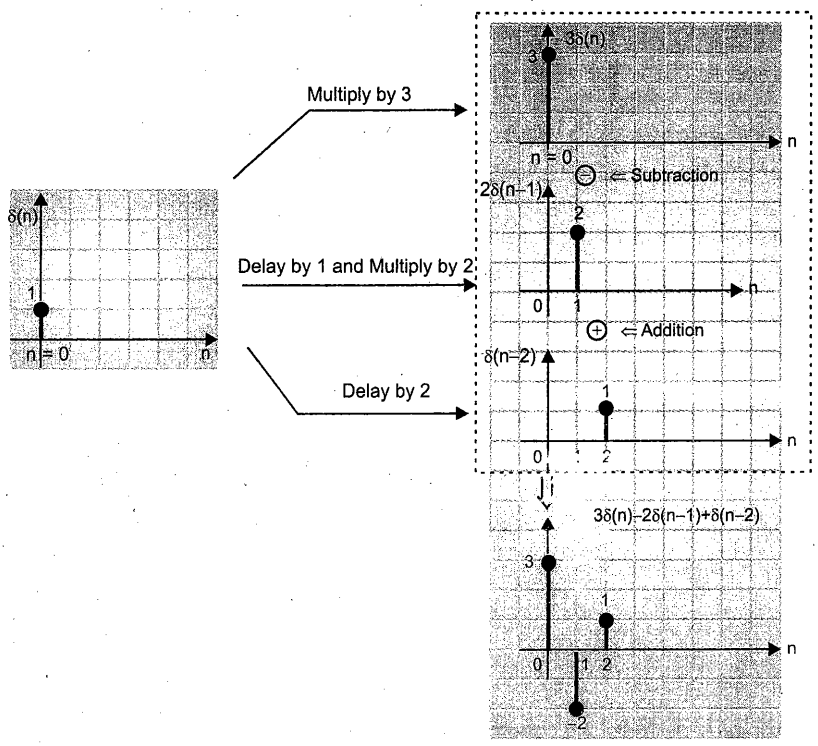


Fig. F-16(b)

$$\therefore \quad h(n) = \{3, -2, 1\}$$

Now output of system is,

$$y(n) = x(n) * h(n)$$

Thus we have to perform linear convolution using circular convolution. Here $x(n)$ has 4 samples and $h(n)$ has 3 samples. Thus result of linear convolution will contain $4 + 3 - 1 = 6$ samples. This value should be in terms of powers of 2. So to obtain same result using circular convolution we will make length of $x(n)$ and $h(n)$, equal to 8.

$$\therefore \quad x(n) = \{2, 2, 1, 1, 0, 0, 0, 0\}$$

$$\text{and} \quad h(n) = \{3, -2, 1, 0, 0, 0, 0, 0\}$$

Now we will perform circular convolution as follows :

$$y(m) = x(n) \textcircled{N} h(n)$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ y(6) \\ y(7) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 2 & 2 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 2 & 2 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6+0+0+0+0+0+0+0 \\ 6-4+0+0+0+0+0+0 \\ 3-4+2+0+0+0+0+0 \\ 3-2+2+0+0+0+0+0 \\ 0-2+1+0+0+0+0+0 \\ 0+0+1+0+0+0+0+0 \\ 0+0+0+0+0+0+0+0 \\ 0+0+0+0+0+0+0+0 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 1 \\ 3 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Thus output of system is,

$$y(m) = \{6, 2, 1, 3, -1, 1, 0, 0\}$$

1.3.9 Time Reversal of Sequence :

Statement : If $x(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X(k)$

then $x((-n))_N = x(N-n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X((-k))_N = X(N-k)$

Proof : According to the definition of DFT,

$$\text{DFT}\{x(n)\} = \sum_{n=0}^{N-1} x(n)\, e^{-j 2\pi kn/N} \quad ..(1)$$

$$\therefore \quad \text{DFT}\{x(N-n)\} = \sum_{n=0}^{N-1} x(N-n)\, e^{-j 2\pi kn/N} \quad ...(2)$$

put $l = N - n$, the limits will change as follows :

$$\text{when } n = 0 \Rightarrow l = N - 0 \quad \therefore \quad l = N$$

$$\text{and when } n = N-1 \Rightarrow l = N - N + 1 \quad \therefore \quad l = 1$$

Thus Equation (2) becomes,

$$\text{DFT}\{x(N-n)\} = \sum_{l=N}^{1} x(l)\, e^{-j2\pi k(N-l)/N} \quad \ldots(3)$$

Here $x((-n))_N$ indicates circularly folded sequence. It can also be represented as $x(N-n)$. That means the sequence $x(N-n)$ is circular in nature and we know that the DFT is periodic. As given by Equation (2); the original limits of summation are from n = 0 to N – 1. That means here summation is calculated for the period 'N'. Since the DFT is periodic in nature; if we calculate the DFT for next period then the result remains same. Now the next period is, n = 0 + N to n = N – 1 + N. That means n = N to 2N – 1. But the sequence is circular; so this period is same as n = N to n = 1. Thus even if we change the index; the limits of summation will remain same.

Basic limits of DFT are 0 to N – 1 as per Equation (2).

$$\therefore \quad \text{DFT}\{x(N-n)\} = \sum_{l=0}^{N-1} x(l)\, e^{-j2\pi k(N-l)/N}$$

$$= \sum_{l=0}^{N-1} x(l)\, e^{-j2\pi k} \cdot e^{(j2\pi kl)/N} \quad \ldots(4)$$

Now we have,

$$e^{-j2\pi k} = \cos 2\pi k - j \sin 2\pi k$$

Since k is an integer, $\cos 2\pi k = 1$ and $\sin 2\pi k = 0$

$$\therefore \quad e^{-j2\pi k} = 1 \quad \ldots(5)$$

Putting this value in Equation (4),

$$\text{DFT}\{x(N-n)\} = \sum_{l=0}^{N-1} x(l)\, e^{j2\pi kl/N} \quad \ldots(6)$$

Similar to Equation (5) we can write,

$$e^{-j2\pi l} = 1 \qquad \therefore \quad e^{\frac{-j2\pi lN}{N}} = 1 \quad \ldots(7)$$

We can multiply R.H.S. of Equation (6) by Equation (7); since its value is 1.

$$\therefore \quad \text{DFT}\{x(N-n)\} = \sum_{l=0}^{N-1} x(l)\, e^{j2\pi kl/N} \cdot e^{-j2\pi \frac{lN}{N}}$$

$$\therefore \quad \text{DFT}\{x(N-n)\} = \sum_{l=0}^{N-1} x(l)\, e^{-j2\pi l(N-k)/N} \quad \ldots(8)$$

Now according to the definition of DFT; R.H.S. of Equation (8) is $X(N-k)$

$$\therefore \quad DFT\{x(N-n)\} = X(N-k) = X((-k))_N$$

Meaning : If a sequence is circularly folded; its DFT is also circularly folded.

1.3.10 Circular Time Shift of Sequence :

Statement : If $x(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X(k)$ then,

$$x((n-l))_N \overset{DFT}{\underset{N}{\longleftrightarrow}} X(k)\, e^{-j2\pi kl/N}$$

or $$x((n-l))_N \overset{DFT}{\underset{N}{\longleftrightarrow}} X(k)\, W_N^{kl}$$

Proof : According to the definition of IDFT,

$$x(n) = IDFT\{X(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn} \qquad ...(1)$$

$$\therefore \quad IDFT\left\{X(k)\, W_N^{kl}\right\} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn} \cdot W_N^{kl}$$

$$\therefore \quad IDFT\left\{X(k)\, W_N^{kl}\right\} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-k(n-l)} \qquad ...(2)$$

Now we have, $$x(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X(k) \qquad ...(3)$$

Comparing R.H.S. of Equations (1) and (2) we can write,

$$x(n-l) \overset{DFT}{\underset{N}{\longleftrightarrow}} X(k)\, W_N^{kl}$$

The sequence is circular and DFT is periodic in nature so we can write,

$$x((n-l))_N \overset{DFT}{\underset{N}{\longleftrightarrow}} X(k)\, W_N^{kl} \qquad ..(4)$$

Hence the proof.

Meaning : Shifting the sequence in time domain by 'l' samples is equivalent to multiplying the sequence in frequency domain by W_N^{kl} or $e^{-j2\pi kl/N}$.

1.3.11 Circular Frequency Shift :

This property is also called as Quadrature Modulation Theorem.

Statement : If $x(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X(k)$ then,

$$x(n)\, e^{j2\pi ln/N} \overset{DFT}{\underset{N}{\longleftrightarrow}} X((k-l))_N = X(k+l)$$

OR $$x(n)\, e^{-j2\pi ln/N} \overset{DFT}{\underset{N}{\longleftrightarrow}} X((k+l))_N = X(k-l)$$

Meaning : Multiplication of sequence $x(n)$ by $e^{\pm j2\pi kl/N}$ is equivalent to the circular shift of DFT in time domain by 'l' samples.

1.3.12 Solved Examples using Circular Properties of DFT :

Prob. 1 : A four point sequence $x(n) = \{1, 2, 3, 4\}$ has DFT $X(k)$, $0 \le k \le 3$, without performing DFT or IDFT. Find the signal values which has DFT $X(k-1)$.

Soln. : According to the circular frequency shifting property,

$$x(n) \cdot e^{-j2\pi ln/N} \overset{DFT}{\underset{N}{\longleftrightarrow}} X((k+l))_N = X(k-l)$$

Here $l = 1$. Let the signal whose DFT is $X(k-1)$ be denoted by $x_1(n)$.

$$\therefore \quad x_1(n) = x(n)\, e^{-j2\pi \cdot 1 \cdot n/4}$$

Since $N = 4$ in this case.

The given sequence is $x(n) = \{1, 2, 3, 4\}$

$$\therefore \quad x(0) = 1, \quad x(1) = 2, \quad x(2) = 3 \quad \text{and} \quad x(3) = 4$$

We will find the sequence $x_1(n)$ as follows :

For $n = 0 \Rightarrow x_1(0) = x(0) \cdot e^0 = 1$

For $n = 1 \Rightarrow x_1(1) = x(1)\, e^{-\frac{j2\pi}{4}} = 2\, e^{-\frac{j\pi}{2}} = 2\left[\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right]$

$$\therefore \quad x_1(1) = -2j$$

For $n = 2 \Rightarrow x_1(2) = x(2)\, e^{-\frac{j4\pi}{4}} = 3\, e^{-j\pi} = 3\,[\cos\pi - j\sin\pi]$

$$\therefore \quad x_1(2) = -3$$

For $n = 3 \Rightarrow x_1(3) = x(3)\, e^{-\frac{j6\pi}{4}} = 4\, e^{-\frac{j3\pi}{2}} = 4\left[\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right]$

$$\therefore \quad x_1(3) = 4j$$

$$\therefore \quad x_1(n) = \{1, -2j, -3, 4j\}$$

Prob. 2 : Consider a real finite length sequence,

$$x(n) = \{4, 3, 2, 1, 0, 0, 1, 1\}$$

(i) $y(n)$ is a sequence related to $x(n)$ such that,

$Y(k) = W_8^{4k} X(k)$ where $X(k)$ is 8 point DFT of $x(n)$. Obtain $y(n)$.

(ii) Also obtain finite length sequence $q(n)$ related to $x(n)$ such that its 8 point DFT is $Q(k) = R_e\{X(k)\}$.

Soln. :

(i) Given $Y(k) = W_8^{4k} X(k)$ ...(1)

According to circular time shifting property,

$$x((n-l))_N \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X(k)\, W_N^{kl} \qquad \ldots(2)$$

Here $N = 8$ and $l = 4$

Thus comparing Equations (1) and (2).

$$y(n) = x((n-4))_8$$

That means $y(n)$ represents circular delay of sequence $x(n)$ by 4 samples. It is represented in Fig. F-17(a).

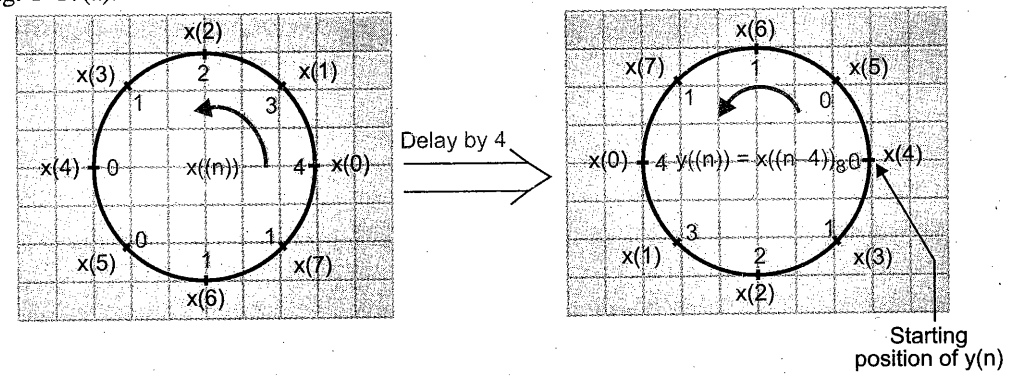


Fig. F-17(a)

$$\therefore \quad y(n) = \{0, 0, 1, 1, 4, 3, 2, 1\}$$

(ii) Given, $Q(k) = R_e\{X(k)\}$ ...(3)

Let $X(k) = M(k) + jN(k)$ ...(4)

Thus M (k) represents real part of X (k).

$$\text{Now} \quad X^*(k) = M(k) - jN(k) \qquad \ldots(5)$$

Adding Equations (4) and (5),

$$X(k) + X^*(k) = 2M(k)$$

$$\therefore \quad M(k) = \frac{X(k) + X^*(k)}{2} \qquad \ldots(6)$$

Taking IDFT of both sides,

$$m(n) = \frac{x(n) + x^*(-n)}{2} \qquad \ldots(7)$$

$$\text{As} \quad x^*(-n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X^*(k)$$

$$\text{We have,} \quad x(n) = \{4, 3, 2, 1, 0, 0, 1, 1\} \qquad \ldots(8)$$

$$\therefore \quad x^*(n) = \{4, 3, 2, 1, 0, 0, 1, 1\}$$

$X^*(-n)$ represents circular folding of $x^*(n)$. It is shown in Fig. F-17(b).

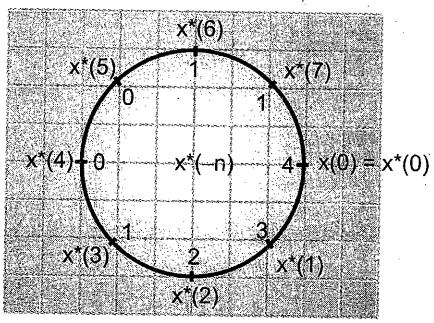


Fig. F-17(b)

$$\therefore \quad x^*(-n) = \{4, 1, 1, 0, 0, 1, 2, 3\} \qquad \ldots(9)$$

Putting Equations (8) and (9) in Equation (7) we can find sequence m (n) as follows :

$$\text{For} \quad n = 0 \Rightarrow m(0) = \frac{x(0) + x^*(0)}{2} = \frac{4+4}{2} = 4$$

$$\text{For} \quad n = 1 \Rightarrow m(1) = \frac{x(1) + x^*(1)}{2} = \frac{3+1}{2} = 2$$

$$\text{For} \quad n = 2 \Rightarrow m(2) = \frac{x(2) + x^*(2)}{2} = \frac{2+1}{2} = \frac{3}{2}$$

For $n = 3 \Rightarrow m(3) = \frac{x(3)+x^*(3)}{2} = \frac{1+0}{2} = \frac{1}{2}$

For $n = 4 \Rightarrow m(4) = \frac{x(4)+x^*(4)}{2} = \frac{0+0}{2} = 0$

For $n = 5 \Rightarrow m(5) = \frac{x(5)+x^*(5)}{2} = \frac{0+1}{2} = \frac{1}{2}$

For $n = 6 \Rightarrow m(6) = \frac{x(6)+x^*(6)}{2} = \frac{1+2}{2} = \frac{3}{2}$

For $n = 7 \Rightarrow m(7) = \frac{x(7)+x^*(7)}{2} = \frac{1+3}{2} = 2$

$$\therefore m(n) = q(n) = \left\{4, 2, \frac{3}{2}, \frac{1}{2}, 0, \frac{1}{2}, \frac{3}{2}, 2\right\}$$

1.3.13 Multiplication of Two Sequences :

Statement : If $x_1(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X_1(k)$ and $x_2(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X_2(k)$ then,

$$x_1(n) \cdot x_2(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} \frac{1}{N}\left[X_1(k) \textcircled{N} X_2(k)\right]$$

Meaning : The multiplication of two sequences in time domain is equivalent to its circular convolution in the frequency domain.

1.3.14 Circular Correlation :

Statement : If $x(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X(k)$ and $y(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} Y(k)$ then,

$$r_{xy}(l) \overset{DFT}{\underset{N}{\longleftrightarrow}} R_{xy}(k) = X(k)\,Y^*(k)$$

Meaning : The circular crosscorrelation of two sequences in time domain is equivalent to the multiplication of DFT of one sequence with the complex conjugate DFT of other sequence.

1.3.15 Complex Conjugate Property :

Statement : If $x(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X(k)$ then

$$x^*(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X^*((-k))_N = X^*(N-k)$$

Meaning : The DFT of complex conjugate of the sequence is equal to the complex conjugate of DFT of that sequence; with the sequence delayed by k samples in the frequency domain.

1.3.16 Parseval's Theorem :

It states that, If $x(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X(k)$ and $y(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} Y(k)$

then $$\sum_{n=0}^{N-1} x(n)\,y^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\,Y^*(k)$$

1.3.17 Solved Problems on Circular Correlation :

Prob. 1 : DFT of a sequence $x(n)$ is given by

$X(k) = \{4, 1+2j, j, 1-3j\}$

Using DFT property only find DFT of $x^*(n)$ if $x^*(n)$ is complex conjugate of $x(n)$.

Soln. : According to complex conjugate property,

$$\text{If} \quad x(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X(k) \text{ then}$$

$$x^*(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X^*(-k)_N$$

We have, $X(k) = \{4, 1+2j, j, 1-3j\}$

$\therefore \quad X^*(k) = \{4, 1-2j, -j, 1+3j\}$

But DFT of $x^*(n)$ is $X^*(-k)_N$

Here $X^*(-k)_N$ indicates circular folding of $X^*(k)$. That means sequence $X^*(-k)_N$ is obtained by plotting the samples of $X^*(k)$ in clockwise direction.

$$\therefore X^*(-k)_N = \{4, 1+3j, -j, 1-2j\}$$

This is the DFT of $x^*(n)$.

Prob. 2 : DFT of a sequence $x(n)$ is given by,

$X(k) = \{6, 0, -2, 0\}$

(i) Determine $x(n)$

(ii) Plot $x_1(n)$ if $X_1(k)$ is $X(k) \cdot e^{-j 2\pi k/2}$

(iii) Determine circular autocorrelation of $x(n)$ using DFT and IDFT only.

Soln. :

(i) According to the definition of IDFT

$$x(n) = \frac{1}{N}[W_N^*] \cdot X_N$$

Here $$[W_N^*] = [W_4^*] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

$$\therefore \quad x(n) = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix}$$

$$\therefore \quad x(n) = \frac{1}{4}\begin{bmatrix} 6+0-2+0 \\ 6+0+2+0 \\ 6+0-2+0 \\ 6+0+2+0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ 8 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

$$\therefore x(n) = \{1, 2, 1, 2\}$$

(ii) Given $X_1(k) = X(k) e^{-j 2\pi k/2}$

According to circular time shift property,

$$x((n-l))_N \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X(k) e^{-j 2\pi kl/N} \qquad ...(1)$$

Given term can be expressed as,

$$X(k) e^{-j 2\pi k/2} = X(k) e^{-\frac{j 2\pi k \cdot 2}{4}} \qquad ...(2)$$

Comparing Equations (1) and (2),

$$x((n-2))_N \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_1(k) = X(k) e^{-\frac{j 2\pi k \cdot 2}{4}}$$

That means $x_1(n) = x((n-2))_N$

Here $x((n-2))_N$ indicates circular delay of $x(n)$ by 2 samples. It is plotted as shown in Fig. F-18.

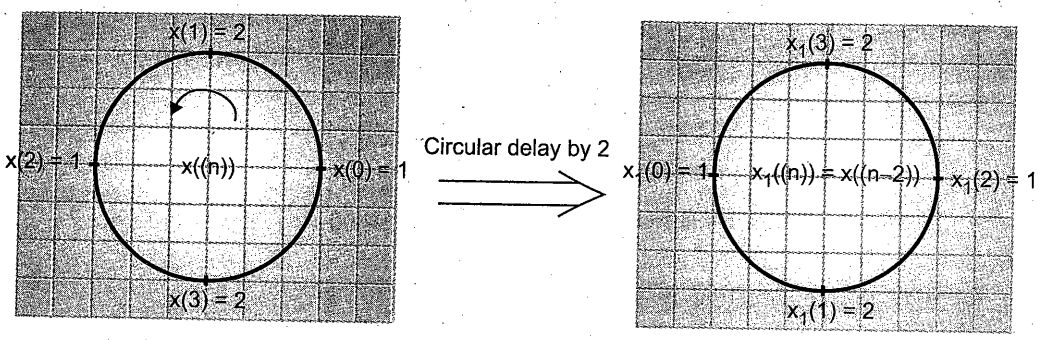


Fig. F-18

(iii) According to circular correlation property we have

$$r_{xx}(l) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} = R_{xx}(k) = X(k) \cdot X^*(k)$$

We have $X(k) = \{6, 0, -2, 0\}$

$X^*(k)$ is complex conjugate of $X(k)$

$$\therefore \quad X^*(k) = \{6, 0, -2, 0\}$$

$$\therefore \quad X(k) \cdot X^*(k) = \{36, 0, 4, 0\}$$

Now $r_{xx}(l)$ is obtained by taking IDFT of $X(k) \cdot X^*(k)$.

$$\therefore \quad r_{xx}(l) = \frac{1}{4}[W_4^*] \cdot X(k) \cdot X^*(k)$$

$$\therefore \quad r_{xx}(l) = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 36 \\ 0 \\ 4 \\ 0 \end{bmatrix}$$

$$\therefore \quad r_{xx}(l) = \frac{1}{4}\begin{bmatrix} 36+0+4+0 \\ 36+0-4+0 \\ 36+0+4+0 \\ 36+0-4+0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 40 \\ 32 \\ 40 \\ 32 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 10 \\ 8 \end{bmatrix}$$

$$\therefore r_{xx}(l) = \{10, 8, 10, 8\}$$

Summary of DFT properties :

Table F-4 shows the summary of DFT properties.

Table F-4

Sr. No.	Property	Time domain	Frequency domain
(1)	Periodicity	$x[n] = x[n+N]$	$X[k] = X[k+n]$
(2)	Linearity	$a_1 x_1[n] + a_2 x_2[n]$	$a_1 X_1[k] + a_2 X_2[k]$
(3)	Time Reversal	$x[N-n]$	$X[N-k]$
(4)	Circular time shift	$x[(n-l)]_N$	$X[k] e^{-j2\pi kl/N}$
(5)	Circular frequency shift	$x[n] e^{j\pi 2ln/N}$	
(6)	Circular convolution	$x_1[n] \textcircled{N} x_2[n]$	$X_1[k] X_2[k]$
(7)	Circular correlation	$x[n] \textcircled{N} y^*[-n]$	$X[k] Y^*[k]$
(8)	Multiplication of two sequences	$x_1[n] x_2[n]$	$\frac{1}{N} X_1[k] \textcircled{N} X_2[k]$
(9)	Complex conjugate	$x^*[n]$	$X^*[N-k]$
(10)	Parseval's theorem	$\sum_{n=0}^{N-1} x[n] y^*[n]$	$\frac{1}{N} \sum_{k=0}^{N-1} X[k] Y^*[k]$

1.4 Linear Filtering Methods based on DFT :

Linear filtering is same as linear convolution. In this section we will discuss how the linear convolution is obtained using DFT. We know that the linear convolution of $x(n)$ and $h(n)$ is given by,

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

$$= \sum_{k=-\infty}^{\infty} h(k) x(n-k) \quad ..(1)$$

If we obtain the fourier transform of $x(n)$ and $h(n)$ then we will get $X(\omega)$ and $H(\omega)$. We know that convolution is equivalent to multiplication in the frequency domain. So by multiplying $X(\omega)$ and $H(\omega)$ we will get $Y(\omega)$.

$$\therefore \quad Y(\omega) = X(\omega) \cdot H(\omega) \quad ...(2)$$

If we take inverse fourier of Equation (2) then we will get the sequence $y(n)$. This sequence will be same as the linear convolution of $x(n)$ and $h(n)$. But we cannot use fourier transform to obtain linear convolution because of the following reasons :

(1) In fourier transform, 'ω' is continuous function of frequency. So the computation cannot be done on digital computers. Because for the digital signal processors we want discrete signals and not the continuous signal.

(2) If we use DFT then the computation will be more efficient because of the availability of fast fourier transform (FFT) algorithms.

So we must use DFT to obtain the linear filtering operation.

In case of DFT we have studied that the multiplication of two DFTs in frequency domain is equivalent to the circular convolution. That means,

$$X(k) \cdot H(k) = x(n) \textcircled{N} h(n) \quad ...(3)$$

But in this case, we want linear convolution (linear filtering) and not the circular convolution. Earlier we have studied that if we adjust the length of two sequences, $x(n)$ and $h(n)$ then, the same result can be obtained using linear convolution and circular convolution.

Consider FIR filter having impulse response $h(n)$ as shown in Fig. F-19.

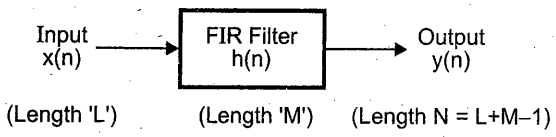


Fig. F-19 : FIR filter

Here, $x(n)$ = Input sequence having length 'L'

$\therefore$ $x(n) = \{0, 1, 2, ..., L-1\}$

$h(n)$ = Impulse response of filter having length 'M'.

$\therefore$ $h(n) = \{0, 1, 2, ..., M-1\}$

Here the linear convolution of $x(n)$ and $h(n)$ produces the output sequence $y(n)$ and the length of $y(n)$ is,

$$N = L + M - 1 \quad ...(4)$$

In this case, both the sequences $x(n)$ and $h(n)$ are finite; so linear convolution will be finite. Thus Equation (1) becomes,

$$y(n) = \sum_{k=0}^{M-1} h(k)\, x(n-k) \quad ...(5)$$

Now if we adjust the length of $x(n)$ and $h(n)$ equal to 'N' and if we perform the circular convolution of $x(n)$ and $h(n)$ then, the result will be same as linear convolution. The length of $x(n)$ and $h(n)$ can be made equal to 'N' by adding required number of zeros in $x(n)$ and $h(n)$. This is called as zero padding. That means we have to increase the length of $x(n)$ by M points and length of $h(n)$ by L points to make the total length $N = L + M - 1$. Then, we can obtain the DFT of $x(n)$ and $h(n)$ that is $X(k)$ and $H(k)$. The multiplication of these two DFTs gives sequence $Y(k)$.

$$\therefore \quad Y(k) = X(k) \cdot H(k) \quad ...(6)$$

Now by taking IDFT of $Y(k)$ the output sequence $y(n)$ can be obtained. Thus the linear filtering can be obtained using DFTs.

Steps to calculate linear filtering using DFT :

Step I : Calculate the value of N using $N = L + M - 1$. Here 'L' represents number of samples in $x(n)$ and M represents number of samples in $h(n)$.

Step II : By adding zeros, make the length of $x(n)$ and $h(n)$ equal to N.

Step III : Calculate DFT of $x(n)$ that means $X(k)$.

Step IV : Calculate DFT of $h(n)$ that means $H(k)$.

Step V : Multiply $X(k)$ and $H(k)$ to get $Y(k) = X(k) \cdot H(k)$.

Step VI : Obtain IDFT of $Y(k)$ that means $y(n)$.

Solved Problem :

Prob. 1 : Determine the response of FIR filter using DFT if :

$$x(n) = \underset{\uparrow}{\{1, 2\}} \text{ and } h(n) = \underset{\uparrow}{\{2, 2\}}$$

Soln. :

Step I : Here length of $x(n) = L = 2$

and length of $h(n) = M = 2$

$$\therefore \quad N = L + M - 1 = 2 + 2 - 1 = 3$$

That means we have to calculate 3-point DFT. We will compute standard 4-point DFT. That means take $N = 4$.

Step II : We will make length of $x(n)$ and $h(n)$ equal to 4 by adding zeros at the end.

$$\therefore \quad x(n) = \underset{\uparrow}{\{1, 2, 0, 0\}} \qquad ...(1)$$

$$\text{and} \quad h(n) = \underset{\uparrow}{\{2, 2, 0, 0\}} \qquad ...(2)$$

Step III : Calculation of $X(k)$:

We will calculate DFT of $x(n)$, $X(k)$ using matrix method.

$$\text{We have} \quad X[k] = W_N \cdot x_N \qquad ..(3)$$

Earlier we have obtained the matrix for twiddle factor W_4. It is,

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \qquad ..(4)$$

We have input matrix

$$x_N = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \qquad ...(5)$$

Putting Equations (4) and (5) in Equation (3),

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2+0+0 \\ 1-2j+0+0 \\ 1-2+0+0 \\ 1+2j+0+0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1-2j \\ -1 \\ 1+2j \end{bmatrix}$$

$$\therefore \quad X(k) = \{3, 1-2j, -1, 1+2j\} \qquad \ldots(6)$$

Step IV : Calculation of $H(k)$:

$$H[k] = W_N \cdot h_N$$

$$\therefore \quad \begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2+2+0+0 \\ 2-2j+0+0 \\ 2-2+0+0 \\ 2+2j+0+0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2-2j \\ 0 \\ 2+2j \end{bmatrix}$$

$$\therefore \quad H(k) = \underset{\uparrow}{\{4, 2-2j, 0, 2+2j\}} \qquad \ldots(7)$$

Step V : Now, $Y(k) = X(k) \cdot H(k)$

$$\therefore \quad Y(k) = \underset{\uparrow}{\{3, 1-2j, -1, 1+2j\}} \cdot \underset{\uparrow}{\{4, 2-2j, 0, 2+2j\}}$$

$$\therefore \quad Y(0) = X(0) \cdot H(0) = 3 \times 4 = 12$$

$$Y(1) = X(1) \cdot H(1) = (1-2j)(2-2j) = 2-2j-4j+4j^2 = 2-6j-4 = -2-6j$$

$$Y(2) = X(2) \cdot H(2) = (-1)(0) = 0$$

$$Y(3) = X(3) \cdot H(3) = (1+2j)(2+2j) = 2+2j+4j+4j^2 = 2+6j-4 = -2+6j$$

Thus sequence $Y(k)$ is,

$$Y(k) = \{Y(0), Y(1), Y(2), Y(3)\}$$

$$\therefore \quad Y(k) = \{12, (-2-6j), 0, (-2+6j)\}$$

Step VI : Now we will obtain $y(n)$ by taking IDFT of $Y(k)$.

We have, $$y(n) = \frac{1}{N} W_N^* Y(k)$$

Here W_N^* is complex conjugate of W_N; which is obtained by changing the sign of 'j' term.

$$\therefore \quad W_N^* = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

$$\therefore \quad y(n) = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 12 \\ -2-6j \\ 0 \\ -2+6j \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 12-2-6j+0-2+6j \\ 12-2j+6+0+2j+6 \\ 12+2+6j+0+2-6j \\ 12+2j-6+0-2j-6 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 8 \\ 24 \\ 16 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 4 \\ 0 \end{bmatrix}$$

$$\therefore \quad y(n) = \{2, 6, 4, 0\}$$
$$\uparrow$$

This is the required response of the filter.

Verification of result :

We will verify the result by calculating linear convolution of x (n) and h (n). It is shown in Fig. F-20.

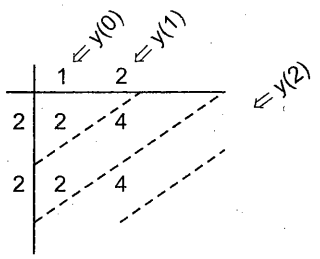


Fig. F-20 : x (n) * h (n)

$$\therefore \quad y(n) = \{2, 6, 4\}$$
$$\uparrow$$

Hence two results are same.

1.4.1 Linear Filtering of Long Data Sequences :

In real time applications; the input data sequence is very long. It is not possible to do the calculations for such long data sequences. This is because the digital computer has limited memory. So long input data sequence is broken into small length sequences (blocks). The computation of each block is done separately. Then all processed blocks are fitted one after other to get the final output.

There are two methods to do such computation. These are as follows :

(1) Overlap save method (2) Overlap add method

1. Overlap save method :

In this method, the filtering of each block is done separately. Let 'M' be the impulse response of filter. Here input data sequence is very long. This data sequence is divided into blocks. Let 'N' be the length of each block.

How to form different blocks ?

In this method, each block of length 'N' is formed by taking 'L' samples of the current segment and 'M – 1' samples from the previous segment. Thus N = L + M – 1.

For the first block, $x_1(n)$ there are no previous segments. So this block is padded with 'M – 1' zeros. And the first 'L' samples are taken from the input sequence.

$$\therefore x_1(n) = \{\underbrace{0, 0, 0, ...0,}_{(M-1)\text{ zeros}} \quad \underbrace{x(0), x(1).....x(L-1)}_{\text{First L samples from input sequence x(n)}}\} \qquad \ldots (1)$$

Let the long input sequence is,

$$x(n) = \{1, 2, 3, 4, 5, -1, 2, -3, 4, 5, 1, 2, 3, 4, 5, ...\}$$

Let $M = 4$ and $L = 5$

$\therefore \quad M - 1 = 3$ and $N = M + L - 1 = 4 + 5 - 1 = 8$

Thus we can write the first sequence $x_1(n)$ as,

$$\therefore x_1(n) = \{\underbrace{0, 0, 0,}_{\text{Three zeros}} \underbrace{1, 2, 3, 4, 5}_{\text{First five samples of x(n)}}\}$$

Now $x_2(n)$ is formed by taking last 'M – 1' samples from $x_1(n)$ and next L samples from input sequence x(n). Here M – 1 = 3. So last 3 samples of $x_1(n)$ are 3, 4, 5.

$$\therefore x_2(n) = \{\underbrace{3, 4, 5,}_{\substack{(M-1)\text{ samples} \\ \text{of } x_1(n)}} \underbrace{-1, 2, -3, 4, 5}_{\text{Next L samples of x(n)}}\}$$

Thus general equation of $x_2(n)$ becomes,

$$\therefore x_2(n) = \{\underbrace{x(L-M+1).....x(L-1),}_{(M-1)\text{ samples from } x_1(n)} \quad \underbrace{x(L), x(L+1).....x(2L-1)}_{\text{Next L samples of x(n)}}\} \qquad \ldots(2)$$

Similarly $x_3(n)$ is formed by taking last M – 1 samples from $x_2(n)$ and next L samples from x(n).

$$\therefore x_3(n) = \{\underbrace{x(2L-M+1).....x(2L-1),}_{(M-1)\text{ samples from } x_2(n)} \quad \underbrace{x(2L), x(2L+1).....x(3L-1)}_{\text{Next L samples of x(n)}}\} \qquad \ldots(3)$$

Likewise the different blocks of length 'N' are formed. Now we have the length of each block equals to 'N' and N = L + M – 1. We have assumed that the length of impulse response is M. This length should be same as the length of each block (N). This is done by adding 'L – 1' zeros in the sequence h (n).

$$\therefore h(n) = \{\underbrace{h(0), h(1).....h(M-1),}_{M \text{ samples of } h(n)} \quad \underbrace{0, 0, 0.....(L-1 \text{ zeros})}_{(L-1) \text{ zeros}}\} \qquad ...(4)$$

The 'N' point DFT of h (n) is H (k). Similarly 'N' point DFT of each block is computed separately and it is stored in the memory. Let the DFT of m^{th} input block be $X_m (k)$. Now the corresponding output is obtained by taking multiplication of H (k) and $X_m (k)$. Let this output be $\hat{Y}_m (k)$.

$$\therefore \quad \hat{Y}_m (k) = H (k) \cdot X_m (k) \qquad ...(5)$$

Now the corresponding time domain sequence is obtained by taking IDFT of $\hat{Y}_m (k)$. Let this sequence be $y_m (n)$.

$$\therefore \quad y_m (n) = IDFT \left\{ \hat{Y}_m (k) \right\} \qquad ..(6)$$

Similarly the time domain sequence corresponding to the DFT of each input block is obtained.

How to avoid loss of data due to aliasing ?

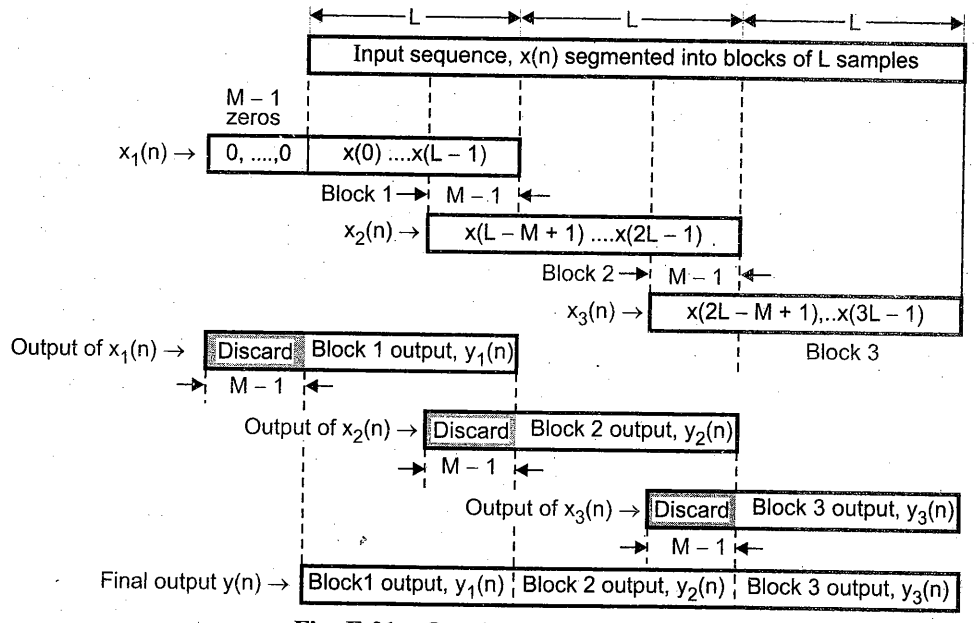


Fig. F-21 : Overlap save method

Observe each input block $x_1 (n)$, $x_2 (n)$ ... . Here each input block consists of initial 'M – 1' samples taken from the previous block. That means there is overlap of sequences. While the last 'L' samples of each block are the actual input samples.

Now because of the circular shift of DFT and because of the overlapping of input data blocks; there is aliasing in the initial 'M – 1' samples in the corresponding output block. To avoid this; first 'M – 1' samples of $y_m(n)$ are discarded. That means after computing time domain sequences; initial 'M – 1' samples are discarded. Then each block is fitted one after other to obtain the final output. The overlap save method is described as shown in Fig. F-21.

2. Overlap add method :

In the overlap add method; the size of each input data block is 'N'. Each input data block is formed by taking 'L' samples from the input sequence and adding 'M – 1' zeros at the end of each input sequence.

$$\therefore x_1(n) = \{\underbrace{x(0), ...x(L-1)}_{\text{First L samples of input sequence x(n)}}, \underbrace{0, 0.....0}_{(M-1)\text{ zeros}}\} \quad ...(1)$$

$x_2(n)$ is formed by taking next 'L' samples from long input sequence $x(n)$ and padding 'M – 1' zeros at the end.

$$\therefore x_2(n) = \{\underbrace{x(L), x(L+1) .....x(2L-1)}_{\text{Next L samples of input sequence x(n)}}, \underbrace{0, 0.....0}_{(M-1)\text{ zeros}}\} \quad ...(2)$$

Similarly,

$$\therefore x(3) = \{\underbrace{x(2L), x(2L+1).....x(3L-1)}_{\text{Next L samples of input sequence x(n)}}, \underbrace{0, 0.....0}_{(M-1)\text{ zeros}}\} \quad ...(3)$$

Impulse response of the filter is $h(n)$ and its length is M. This length should be again made equal to N. So L – 1 zeros are added to form the sequence $h(n)$.

$$\therefore h(n) = \{\underbrace{h(0), h(1).....h(M-1)}_{\text{M samples of impulse sequence}}, \underbrace{0, 0.....0}_{\substack{(L-1)\text{ zeros} \\ \text{to make total length = N}}}\} \quad ...(4)$$

Now the DFT of each input data block is computed separately. Similarly the DFT of $h(n)$ is computed that is, $H(k)$. Now the output of m^{th} block is obtained by multiplying DFT of m^{th} input block by $\hat{H}(k)$.

$$\therefore \quad Y_m(k) = H(k) \cdot X_m(k) \quad ...(5)$$

The time domain sequence $Y_m(n)$ is obtained by taking IDFT of $Y_m(k)$. Here we are performing 'N' point DFT; so the length of each output block is 'N'. The different output sequences are as follows :

$$y_1(n) = \{y_1(0), y_1(1) .... y_1(L-1), y_1(L), y_1(L+1) .... y_1(L+M-1)\}$$

$$y_2(n) = \{y_2(0), y_2(1) .... y_2(L-1), y_2(L), y_2(L+1) .... y_2(L+M-1)\}$$

Similarly all output sequences are obtained.

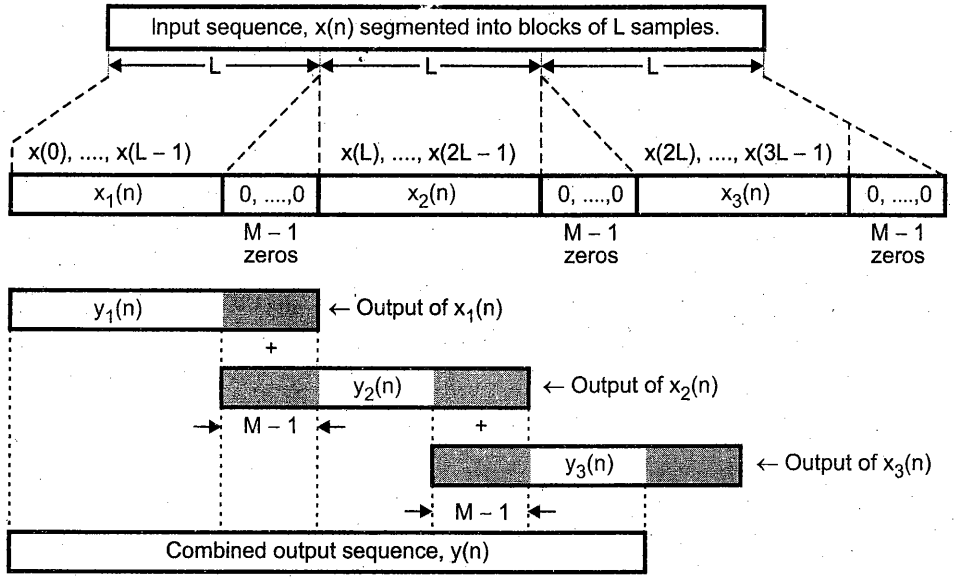


Fig. F-22 : Overlap add method

In this method, each input sequence is padded with 'M – 1' zeros at the end to make the length N = L + M – 1. So last 'M – 1' samples of each output block must be overlapped and added to the first M – 1 samples of succeeding block as shown in Fig. F-22. That is why this method is called as overlap add method. Here 'M – 1' samples are not discarded because there is no aliasing effect.

1.5 Frequency Analysis of Signals using DFT :

The frequency analysis is also called as spectrum analysis. To do the frequency analysis; time domain signal should be first converted into the frequency domain.

If input signal is analog then first it is passed through the antialiasing filter. Then this signal is sampled at the rate F_s; where F_s is the sampling frequency. To avoid the aliasing, F_s should be greater than or equals to 2W. Here W is the maximum frequency of input signal.

In practical cases; the time interval of signal is maintained at T_0 and $T_0 = LT$. Here L = number of samples and T is the sampling interval.

Suppose we want to analyze the long input sequence x (n). Now to limit the time interval of sequence means this input sequence is multiplied by a rectangular window. A rectangular window is

denoted by $W_R(n)$ and it is defined as,

$$W_R(n) = \begin{cases} 1 & , 0 \le n \le L-1 \\ 0 & \text{otherwise} \end{cases} \quad ...(1)$$

This rectangular window is shown in Fig. F-23.

Since output of $W_R(n)$ is zero after the interval $L-1$; multiplying $x(n)$ by $W_R(n)$ produces signal $\hat{x}(n)$ only in the range 0 to $L-1$.

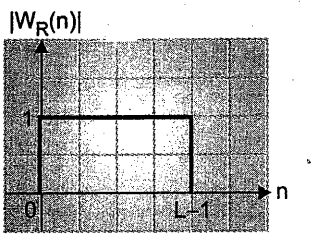


Fig. F-23 : Rectangular window

$$\therefore \quad \hat{x}(n) = x(n) W_R(n) \quad ...(2)$$

Then by taking DFT of Equation (2) we get,

$$\hat{X}(k) = \sum_{n=0}^{L-1} \left[x(n) W_R(n) \right] e^{-j 2\pi kn/N} \quad ...(3)$$

Here we are considering only 'L' samples of input signal and not the complete input signal. The spectrum will be more accurate if the value of 'L' is large.

Spectral Leakage :

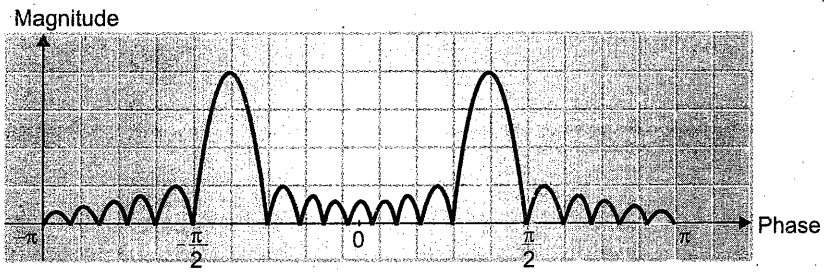


Fig. F-24 : Magnitude spectrum

As shown in Fig. F-24; the magnitude spectrum is not localized to a single frequency, but it is spreadout over the entire frequency range. That means the power of a signal is spreadout in entire frequency range. This leakage of the power is called as spectral leakage which is taking place because of 'windowing' of input sequence.

Advantages and limitations of spectrum analysis using DFT :

Advantages :

(1) Fast processing of DFT can be done using FFT algorithms.
(2) Estimation of power spectrum can be done.
(3) Calculation of harmonics can also be done.
(4) The resolution can be improved by increasing number of samples in the calculation of DFT.

Limitations :

(1) The frequency spectrum of entire input signal is not obtained because of windowing.

(2) The leakage of power (spectral leakage) takes place.

(3) If we increase number of samples to obtain the better accuracy then, the processing time is increased.

1.6 Relationship between DFT and Z-Transform :

The Z-transform of sequence x(n) is,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \quad ...(1)$$

We know that at $Z = e^{j\omega}$,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \quad ...(2)$$

It means that X(Z) is evaluated on unit circle.

Now suppose X(Z) is sampled at 'N' equally spaced points on the unit circle. Then we have

$$\omega = \frac{2\pi K}{N} \quad ...(3)$$

Now if X(Z) is evaluated at $Z = e^{j\omega K/N}$ then by putting Equation (3) in Equation (2) we get,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi Kn/N} \quad \text{at } Z = e^{j2\pi K/N} \quad ...(4)$$

In Equation (4), if x(n) is causal sequence and has 'N' number of samples then we can write Equation (4) as,

$$X(Z) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \quad \text{at } Z = e^{j2\pi K/N} \quad ...(5)$$

But R.H.S. of Equation (5) is DFT of x(n).

$$\therefore \quad X(K) = X(Z) \text{ at } Z = e^{j2\pi K/N}$$

This means if Z-transform is evaluated on the unit circle at evenly spaced points only; then it becomes DFT.

Theory Questions

Q. 1 Write note on : Overlap-save and Overlap-add algorithms.

Q. 2 What is relation between Z-transform and DFT

Q. 3 State DFT and IDFT and describe any four important properties of the same.

Q. 4 Explain in brief method to reduce complexity in computation of FFT. Give practical

considerations in reducing - (i) memory size for storage of coefficients and (ii) computation time.

Q. 5 Explain clearly : Circular convolution.

Q. 6 Prove that linear convolution is obtained using circular convolution property of DFT.

Q. 7 With the help of neat flow graph explain DIF FFT algorithm consider N = 8.

Q. 8 Explain the difference between FT, DTFT, DFT.

Q. 9 How linear convolution of two sequences can be obtained using DFT ?

Q. 10 Explain circular shifting, circular reflection and circular convolution.

Q. 11 Explain the relationship between DTFT and DFT.

Q. 12 For DFT computation why it is necessary to have - $N \geq L$ where,

N : N-point DFT.

L : Length of the signal $x(n)$.

Ans. :

Definition of DFT :

It is a finite duration discrete frequency sequence which is obtained by sampling one period of fourier transform. Sampling is done at 'N' equally spaced points over the period extending from $\omega = 0$ to $\omega = 2\pi$.

N is very important parameter in the definition of DFT and IDFT. 'N' decides the limits of 'k' and 'n'. 'N' also decides the amount of processing time required to calculate N point DFT. The values of 'N' is decided according to the number of elements present in the given sequence x(n). That means the length of sequence

$$x(n) = \{ x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7) \}$$

Here the last value $x(7)$ is denoted by $x_h(n)$. Normally value of N is takes as $N = x_h(n) + 1$ or $N \geq L$. If it is less than 'L' then it is not possible to compute DFT or IDFT.

Now in case of fast fourier transformation (FFT) 'N' is required to be a power of 2. Accordingly the length of sequence $x(n)$ is adjusted by padding zeros at the end.

Q. 13 How N-point DFT is obtained from FT ?

Q. 14 Explain how N-point circular convolution can be obtained using DFT and IDFT.

Q. 15 What is the relationship between ZT and DFT.

Unsolved Numericals

Q. 1 Using definition of DFT, compute DFT of $x(n) = \{1, 0, 0, 1\}$
$\uparrow$

Ans. : $X(k) = \{2, 1 + j, 0, 1 - j\}$

Q. 2 Using definition of DFT, compute DFT of sequence,

$$x(n) = \begin{cases} \frac{1}{3}, & 0 \le n \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Ans. :

$$X(k) = e^{-j2\pi k/N}\left[\frac{1+2\cos(2\pi k/N)}{3}\right]$$

Q. 3 Using matrix method, determine 8 point DFT of sequence x (n) = {0, 0, 1,1, 1, 0, 0, 0}

Ans. :

$$X(k) = \{3, -1.707 + j\,1.707, j, -0.293 + j\,0.293, 1, -0.293 - j\,0.293, -j, -1.707 + j\,1.707\}$$

Q. 4 Plot the magnitude and phase spectrum of sampled data sequence {2, 0, 0, 1} which is obtained using sampling frequency of 20 kHz. Select N = 4.

Hint : $X(k) = \{3, 2 + j, 1, 2 - j\} = \{3\angle 0^\circ, 2.236\angle 26.57^\circ, 1\angle 0^\circ, 2.236\angle -26.57^\circ\}$

To plot the spectrum we have to select frequency on X axis.

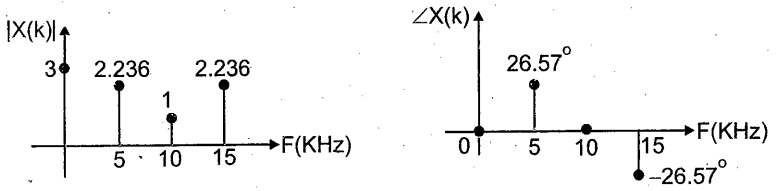
$$\therefore \quad F = \frac{1}{NT_s} = \frac{F_s}{N} = \frac{20\text{ kHz}}{4} = 5\text{ kHz}$$

$\therefore$ X (0) represents DC component

X (1) represents 5 kHz component

X (2) represents 10 kHz component

X (3) represents 15 kHz component



(a) Magnitude spectrum **(b) Phase spectrum**

Fig. F-25

Q. 5 Determine IDFT of $X(k) = \{2, 1 + j, 0, 1 - j\}$

Ans. :

$$x(n) = \{1, 0, 0, 1\}$$

Q. 6 A 4 point DFT of sampled data sequence {2, 0, 0, 1} is $\{3, 2 + j, 1, 2 - j\}$ verify

(i) X (7) = X (3) (ii) X (12) = X (0)

Hint : (i) Put N = 4 and n = 7 in the equation of IDFT and expand the summation.

(ii) Put N = 4 and k = 12 in the equation of IDFT and expand the summation.

Q. 7 First five points of 8 point DFT of a real valued sequence are given by X (0) = 0, X (1) = 2 + j 2, X (2) = – j4 , X (3) = 2 – j2, X (4) = 0. Determine remaining points and obtain time domain sequence x (n).

Ans. :

$$X(5) = 2 + j2,\ X(6) = 0 + j\,4,\ X(7) = 2 - j2$$

and $x(n) = \{1, 1, -1, -1, -1, 1, 1, -1\}$

Q. 8 Calculate DFT of a sequence whose values for one period are given by, $x(n) = \{1, 1, -2, -2\}$

Ans. :

$$X(k) = \{2, 3-j3, 0, 3+j3\}$$

Q. 9 Obtain circular convolution of,

$x_1(n) = \{1, 3, 5, 3\}$, $x_2(n) = \{2, 3, 1, 1\}$

Ans. :

$$y(m) = \{19, 17, 23, 25\}$$

Q. 10 Using DFT and IDFT obtain circular convolution of,
$x(n) = \{2, 0, 0, 1\}$ $h(n) = \{4, 3, 2, 1\}$

Ans. :

$$Y(m) = \{11, 8, 5, 6\}$$

Q. 11 Find out sequence $Y(n)$ if $h(n) = \{1, 1, 1\}$, $x(n) = \{1, 2, 3, 1\}$ using circular convolution.
Hint : Obtain linear convolution using circular convolution

Ans. :

$$Y(n) = \{1, 3, 6, 6, 4, 1, 0, 0\}$$

Q. 12 Obtain circular convolution of,
$x_1(n) = \{1, 2, 3, 4\}$ $x_2(n) = \{4, 3, 2, 1\}$

Ans. :

$$Y(m) = \{24, 22, 24, 30\}$$

Q. 13 Determine response $Y(n)$ of a filter if $x(n) = \{1, 2, 2, 1\}$ and $h(n) = \{1, 2, 3\}$

Ans. :

$$Y(n) = \{1, 4, 9, 11, 8, 3\}$$

❑❑❑

FFT Algorithms

Section No.	Name of the Topic
1.1	Introduction
1.2	Radix-2 FFT Algorithm
1.3	Radix-2 Decimation In Time (DIT) Algorithm (DIT FFT)
1.4	Radix-2 Decimation In Frequency (DIF) FFT Algorithm
1.5	Computation of Inverse DFT (IDFT) using FFT Algorithms
1.6	Computation of Linear Filtering and Correlation using FFT
1.7	Computation of DFT by Linear Filtering Approach
1.8	Practical Considerations in FFT Implementation
1.9	Chirp-Z Algorithm

1.1 Introduction :

We have studied how to obtain DFT of a sequence by using direct computation. Basically, the direct computation of DFT requires large number of computations. So more processing time is required.

For the computation of N-point DFT, N^2 complex multiplications and $N^2 - N$ complex additions are required. If the value of N is large then the number of computations will go into lakhs. This proves inefficiency of direct DFT computation.

In 1965, Cooley and Tukey developed very efficient algorithm to implement the DFT. This algorithm is called as Fast Fourier Transform (FFT). These FFT algorithms are very efficient in terms of computations. By using these algorithms, number of arithmetic operations involved in the computation of DFT are greatly reduced.

Different FFT algorithms are available : out of which Radix-2 FFT algorithm is most important FFT algorithm.

1.2 Radix-2 FFT Algorithm :

While calculating DFT; we have discussed that we always calculate 'N'-point DFT. The number N can be factored as,

$$N = r_1 , r_2 , r_3 \ldots r_v \quad \ldots(1)$$

Here every 'r' is a prime.

Now if $r_1 = r_2 = r_3 = ... = r_v = r$

then we can write,

$$N = r^v \quad \ldots(2)$$

Here 'r' is called as radix (base) of FFT algorithm and

'v' indicates number of stages in FFT algorithm.

Now radix means base and if its value is '2' then it is called as radix-2 FFT algorithm. Thus when r = 2; Equation (2) becomes,

$$N = 2^v \quad \ldots(3)$$

Thus if we are computing 8 point DFT then N = 8

$$\therefore \quad 8 = 2^v$$

$$\therefore \quad v = 3 \quad \ldots(4)$$

So for 8 point DFT, there are three stages of FFT algorithm. While computing FFT, divide number of input samples by 2, till you reach minimum two samples. Based on this division there are two algorithms as follows :

(1) Radix-2 Decimation in Time (DIT) algorithm.

(2) Radix-2 Decimation in Frequency (DIF) algorithm.

Before studying these algorithms we will derive some important properties of twiddle factor W_N. The twiddle factor W_N is given by,

$$W_N = e^{-\frac{j2\pi}{N}} \qquad ...(5)$$

1. $\mathbf{W_N^k = W_N^{k+N}}$

Using Equation (5) we can write,

$$W_N^{k+N} = \left[e^{-\frac{j2\pi}{N}}\right]^{k+N} = e^{-\frac{j2\pi k}{N}} \cdot e^{-j2\pi}$$

But $e^{-j2\pi} = \cos 2\pi - j \sin 2\pi = 1 - j0 = 1$

$$\therefore \quad W_N^{k+N} = e^{-j\frac{2\pi}{N}k}$$

$$\therefore \quad W_N^{k+N} = \left(e^{-j\frac{2\pi}{N}}\right)^k \qquad ...(6)$$

In Equation (6), the bracket term is W_N

$$\therefore \quad W_N^{k+N} = W_N^k \qquad ...(7)$$

Equation (7) indicates that twiddle factor is periodic.

2. $\mathbf{W_N^{k+\frac{N}{2}} = -W_N^k}$

Using Equation (6) we can write,

$$W_N^{k+\frac{N}{2}} = \left[e^{-\frac{j2\pi}{N}}\right]^{\left(k+\frac{N}{2}\right)} = e^{-\frac{j2\pi k}{N}} \cdot e^{-\frac{j2\pi}{N}\cdot\frac{N}{2}}$$

$$\therefore \quad W_N^{k+\frac{N}{2}} = e^{-\frac{j2\pi k}{N}} \cdot e^{-j\pi} \qquad ...(8)$$

But $e^{-j\pi} = \cos \pi - j \sin \pi = -1 - 0 = -1$

$$\therefore \quad W_N^{k+\frac{N}{2}} = -e^{-j\frac{2\pi k}{N}}$$

$$\therefore \quad W_N^{k+\frac{N}{2}} = -\left(e^{-\frac{j2\pi}{N}}\right)^k \qquad ...(9)$$

In Equation (9), the bracket term is, W_N

$$\therefore \quad W_N^{k+\frac{N}{2}} = -W_N^k \qquad ...(10)$$

Equation (10) indicates that twiddle factor is symmetric.

3. $W_N^2 = W_{N/2}$

From Equation (5) we can write,

$$W_{N/2} = e^{-\frac{j2\pi}{N/2}} = e^{-\frac{j2\pi}{N}\cdot 2}$$

$$\therefore \quad W_{N/2} = \left[e^{-\frac{j2\pi}{N}}\right]^2$$

$$\therefore W_{N/2} = W_N^2$$

1.3 Radix-2 Decimation In Time (DIT) Algorithm (DIT FFT) :

To decimate means to break into parts. Thus DIT indicates dividing (splitting) the sequence in time domain. The different stages of decimation are as follows :

First stage of decimation :

Let $x(n)$ be the given input sequence containing 'N' samples. Now for decimation in time we will divide $x(n)$ into even and odd sequences.

$$\therefore \quad x(n) = f_1(m) + f_2(m) \qquad ...(1)$$

Here $f_1(m)$ is even sequence and $f_2(m)$ is odd sequence

$$\therefore \quad f_1(m) = x(2n), \qquad m = 0, 1, \ldots \frac{N}{2} - 1 \qquad ...(2)$$

$$\text{and} \quad f_2(m) = x(2n+1), \qquad m = 0, 1, \ldots, \frac{N}{2} - 1 \qquad ..(3)$$

Input sequence $x(n)$ has 'N' samples. So after decimation; $f_1(m)$ and $f_2(m)$ will contain $\frac{N}{2}$ samples.

Now according to the definition of DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad ...(4)$$

Since we have divided $x(n)$ into two parts; we can write separate summation for even and odd sequences as follows :

$$X(k) = \sum_{n \text{ even}} x(n) W_N^{k} + \sum_{n \text{ odd}} x(n) W_N^{kn} \qquad ...(5)$$

The first summation represents even sequence. So we will put n = 2m in first summation. While the second summation represents odd sequence, so we will put n = (2m + 1) in second summation. Since even and odd sequences contain $\frac{N}{2}$ samples each; the limits of summation will be from m = 0 to $\frac{N}{2} - 1$.

$$\therefore \quad X(k) = \sum_{m=0}^{\frac{N}{2}-1} x(2m)\, W_N^{2km} + \sum_{m=0}^{\frac{N}{2}-1} x(2m+1)\, W_N^{k(2m+1)} \qquad ...(6)$$

But x (2m) is even sequence, so it is $f_1(m)$ and x (2m + 1) is odd sequence, so it is $f_2(m)$.

$$\therefore \quad X(k) = \sum_{m=0}^{\frac{N}{2}-1} f_1(m)\, W_N^{2km} + \sum_{m=0}^{\frac{N}{2}-1} f_2(m)\, W_N^{2km} \cdot W_N^{k}$$

$$\therefore \quad X(k) = \sum_{m=0}^{\frac{N}{2}-1} f_1(m) \left(W_N^2 \right)^{km} + W_N^k \sum_{m=0}^{\frac{N}{2}-1} f_2(m) \left(W_N^2 \right)^{km} \qquad ...(7)$$

Now we have $W_N^2 = W_{N/2}$

$$\therefore \quad X(k) = \sum_{m=0}^{\frac{N}{2}-1} f_1(m)\, W_{N/2}^{km} + W_N^k \sum_{m=0}^{\frac{N}{2}-1} f_2(m)\, W_{N/2}^{km} \qquad ...(8)$$

Comparing each summation with the definition of DFT,

$$X(k) = F_1(k) + W_N^k \; F_2(k), \quad K = 0, 1, ..., N-1 \qquad ...(9)$$

We will consider an example of 8 point DFT. That means N = 8.

Here $F_1(k)$ is $\frac{N}{2}$ point DFT of $f_1(m)$ and $F_2(k)$ is $\frac{N}{2}$ point DFT of $f_2(m)$. That means $F_1(k)$ and $F_2(k)$ are 4-point DFTs.

Equation (9) indicates that $F_2(k)$ is multiplied by W_N^k and it is added with $F_1(k)$, to obtain (4 + 4) i.e. 8-point DFT. Graphically Equation (9) represented as shown in Fig. G-1.

Remember that in Equation (9), K varies from 0 to N – 1 (i.e. 0 to 7 for 8 point DFT).



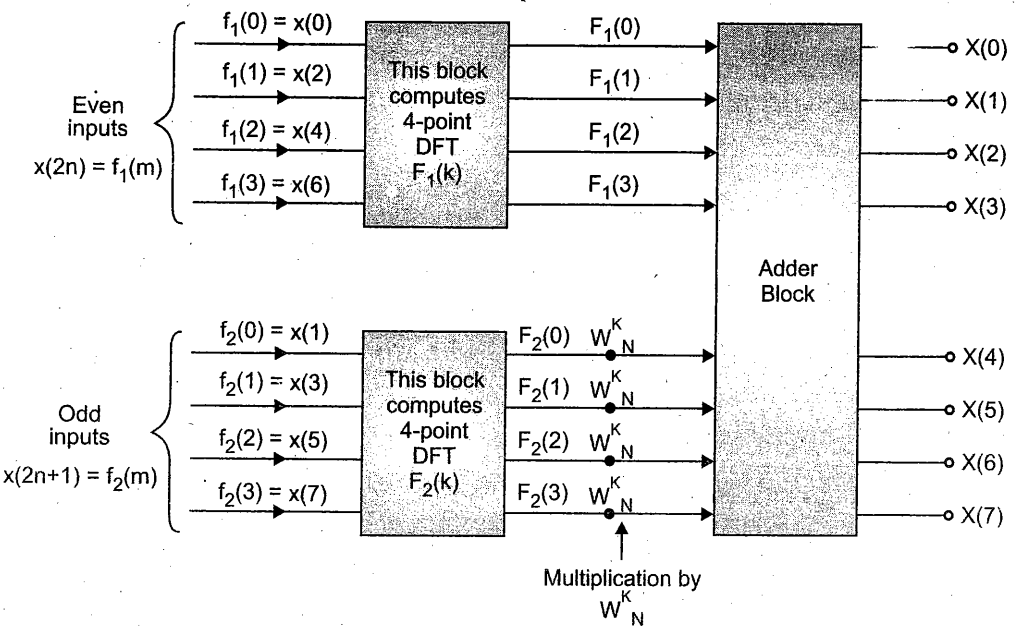


Fig. G-1 : Graphical representation of $X(k) = F_1(k) + W_N^k F_2(k)$

Now $F_1(k)$ and $F_2(k)$ are 4-point $\left(\frac{N}{2}\right)$ DFTs. They are periodic with period $\frac{N}{2}$. Using periodicity property of DFT we can write,

$$F_1\left(k+\frac{N}{2}\right) = F_1(k) \qquad ...(10)$$

and $$F_2\left(k+\frac{N}{2}\right) = F_2(k) \qquad ...(11)$$

Replacing k by $k+\frac{N}{2}$ in Equation (9) we get,

$$X\left(k+\frac{N}{2}\right) = F_1\left(k+\frac{N}{2}\right) + W_N^{k+\frac{N}{2}} F_2\left(k+\frac{N}{2}\right) \qquad ...(12)$$

Now we have, $$W_N^{k+\frac{N}{2}} = -W_N^k$$

$$\therefore \quad X\left(k+\frac{N}{2}\right) = F_1\left(k+\frac{N}{2}\right) - W_N^k \; F_2\left(k+\frac{N}{2}\right) \qquad ...(13)$$

Using Equations (10) and (11) we get,

$$X\left(k+\frac{N}{2}\right) = F_1(k) - W_N^k F_2(k) \qquad ....(14)$$

Here $X(k)$ is 'N' point DFT. We can take k = 0 to $\frac{N}{2}-1$ then, by using Equations (9) and (14) we can obtain combined N-point DFT.

$$\therefore \quad X(k) = F_1(k) + W_N^k F_2(k), \quad k = 0, 1 \ldots \frac{N}{2} - 1 \qquad \ldots(15)$$

$$\text{and} \quad X\left(k + \frac{N}{2}\right) = F_1(k) - W_N^k F_2(k), \quad k = 0, 1, \ldots \frac{N}{2} - 1 \qquad \ldots(16)$$

We are considering an example of 8 point DFT (N = 8). So in Equations (15) and (16), k varies from 0 to 3. Now putting k = 0 to 3 in Equations (15) and (16) we get,

$$\left.\begin{aligned} X(0) &= F_1(0) + W_N^0 F_2(0) \\ X(1) &= F_1(1) + W_N^1 F_2(1) \\ X(2) &= F_1(2) + W_N^2 F_2(2) \\ X(3) &= F_1(3) + W_N^3 F_2(3) \end{aligned}\right\} \qquad \ldots(17)$$

and

$$\left.\begin{aligned} X(0+4) = X(4) &= F_1(0) - W_N^0 F_2(0) \\ X(1+4) = X(5) &= F_1(1) - W_N^1 F_2(1) \\ X(2+4) = X(6) &= F_1(2) - W_N^2 F_2(2) \\ X(3+4) = X(7) &= F_1(3) - W_N^3 F_2(3) \end{aligned}\right\} \qquad \ldots(18)$$

The graphical representation of first stage of decimation for 8 point DFT is as shown in Fig. G-2.

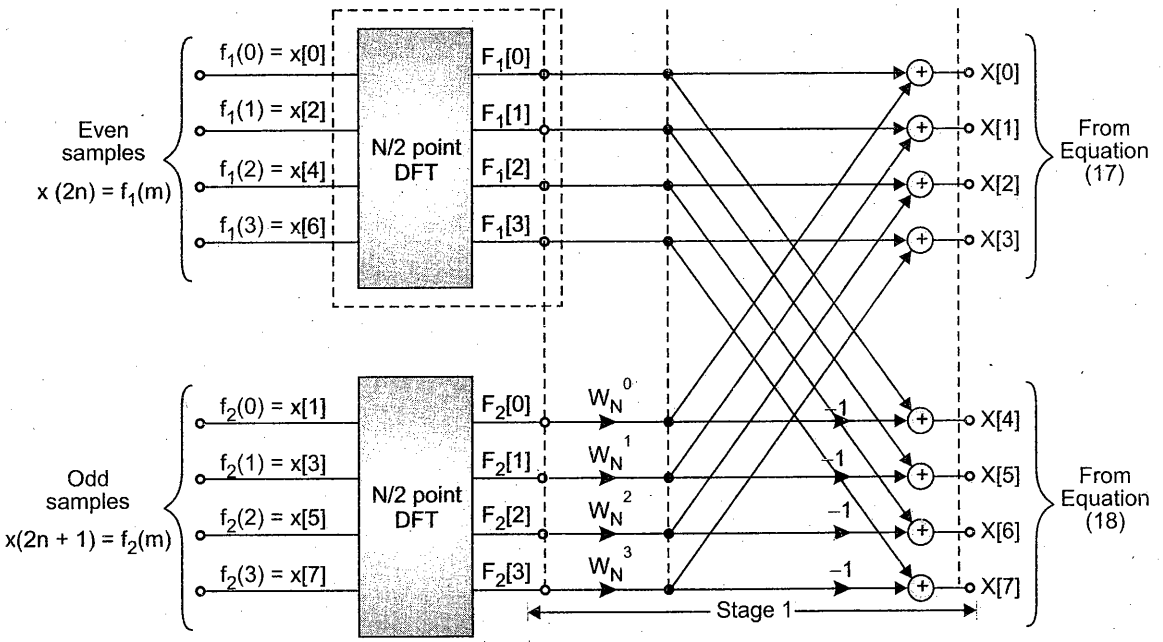


Fig. G-2 : First stage of decimation

In Fig. G-2, input sequences are

$$f_1(m) = x(2n) = \{x(0), x(2), x(4), x(6)\} \quad ...(19)$$

and $$f_2(m) = x(2n+1) = \{x(1), x(3), x(5), x(7)\} \quad ...(20)$$

That means each sequence contains $\frac{N}{2}$ samples.

Second stage of decimation :

In the first stage of decimation; we obtained the sequences of length $\frac{N}{2}$. That means for 8-point DFT (N = 8); the length of each sequence is '4' as given by Equations (19) and (20). We discussed that we have to continue this process till we get '2' point sequence.

We can further decimate $f_1(m)$ into even and odd samples. Let $g_{11}(n) = f_1(2m)$, which contains even samples and let $g_{12}(n) = f_1(2m+1)$, which contains odd samples of $f_1(m)$.

Note that here range of 'n' and 'm' is from 0 to $\frac{N}{4} - 1$.

Now recall Equations (15) and (16). We obtained sequences $X(k)$ and $X\left(k+\frac{N}{2}\right)$ from $F_1(k)$ and $F_2(k)$. The length of each sequence was $\frac{N}{2}$. Here in the second stage of decimation; we are further dividing the sequences into even and odd parts. So similar to Equations (15) and (16) we can write; For $F_1(k)$,

$$F_1(k) = G_{11}(k) + W_{N/2}^{k} G_{12}(k) \qquad k = 0, 1, \ldots \frac{N}{4} - 1 \quad ...(21)$$

and $$F_1\left(k+\frac{N}{4}\right) = G_{11}(k) - W_{N/2}^{k} G_{12}(k) \qquad k = 0, 1, \ldots \frac{N}{4} - 1 \quad ...(22)$$

Thus, for N = 8 we have the range of K, from K = 0 to K = 1. Here $G_{11}(k)$ is DFT of $g_{11}(n)$ and $G_{12}(k)$ is DFT of $g_{12}(n)$. Now putting the values of 'K' in Equation (21) we get,

$$\left.\begin{aligned} F_1(0) &= G_{11}(0) + W_{N/2}^{0} G_{12}(0) \\ F_1(1) &= G_{11}(1) + W_{N/2}^{1} G_{12}(1) \end{aligned}\right\} \quad ...(23)$$

Similarly from Equation (22) we get,

$$\left.\begin{aligned} F_1\left(0+\frac{8}{4}\right) &= F_1(2) = G_{11}(0) - W_{N/2}^{0} G_{12}(0) \\ F_1\left(1+\frac{8}{4}\right) &= F_1(3) = G_{11}(1) - W_{N/2}^{1} G_{12}(1) \end{aligned}\right\} \quad ...(24)$$

Here the values of K are '0' and 1. That means it is 2-point DFT. Thus Equations (23) and (24) shows that we can obtain 4-point DFT by combining two 2-point DFTs. The graphical representation is shown in Fig. G-3.

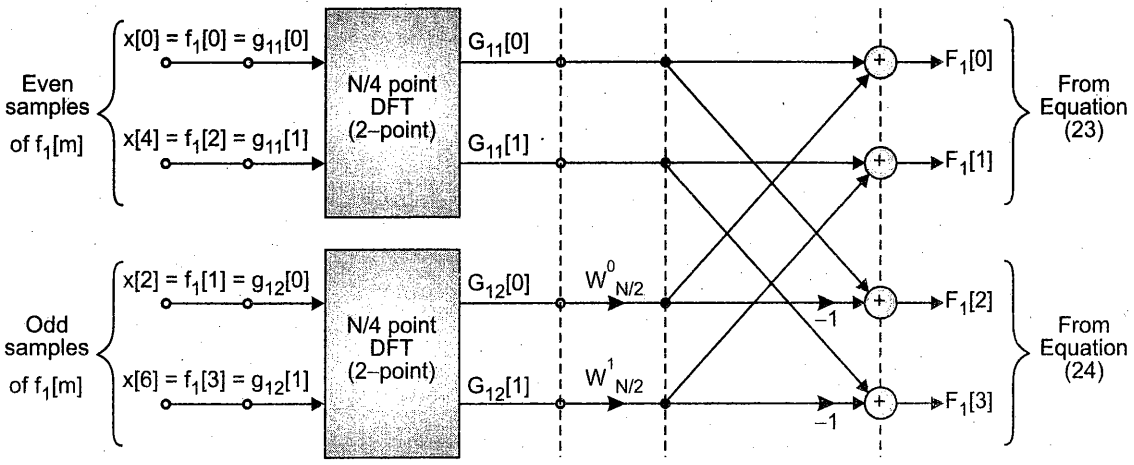


Fig. G-3 : $F_1(k)$, $\frac{N}{2}$ point DFT

Note that here,

$$\left.\begin{aligned} g_{11}(n) &= f_1(2m) = x(4n) = \{x(0), x(4)\} \\ \text{and} \quad g_{12}(n) &= f_1(2m+1) = x(4n+2) = \{x(2), x(6)\} \end{aligned}\right\} \quad ...(25)$$

Now similar to Equations (21) and (22) we can write equations for $F_2(k)$ as follows :

$$F_2(k) = G_{21}(k) + W^k_{N/2}\, G_{22}(k), \qquad k = 0, 1, \ldots \frac{N}{4} - 1 \quad ...(26)$$

$$\text{and} \quad F_2\left(k + \frac{N}{4}\right) = G_{21}(k) - W^k_{N/2}\, G_{22}(k), \qquad k = 0, 1, \ldots, \frac{N}{4} - 1 \quad ...(27)$$

Here $G_{21}(k)$ is DFT of $g_{21}(n)$ and $G_{22}(k)$ is DFT of $g_{22}(n)$. The values of K are 0 and 1. Putting these values in Equation (26),

$$\left.\begin{aligned} F_2(0) &= G_{21}(0) + W^0_{N/2}\, G_{22}(0) \\ F_2(1) &= G_{21}(1) + W^1_{N/2}\, G_{22}(1) \end{aligned}\right\} \quad ...(28)$$

Similarly from Equation (27) we get,

$$\left.\begin{aligned} F_2\left(0+\frac{8}{4}\right) &= F_2(2) = G_{21}(0) - W_{N/2}^{0}\, G_{22}(0) \\ F_2\left(1+\frac{8}{4}\right) &= F_2(3) = G_{21}(1) - W_{N/2}^{1}\, G_{22}(1) \end{aligned}\right\} \quad \ldots(29)$$

The graphical representation of Equations (28) and (29) is shown in Fig. G-4.

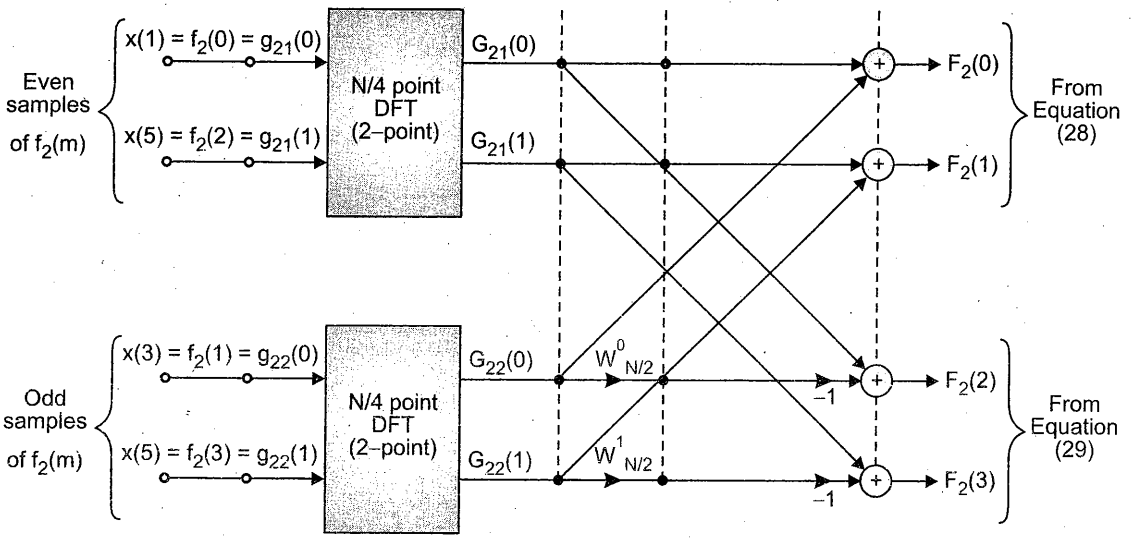


Fig. G-4 : $F_2(k)$, $\frac{N}{2}$ point DFT

Note that here,

$$\left.\begin{aligned} g_{21}(n) &= f_2(2n) = x(4n+1) = \{x(1), x(5)\} \\ g_{22}(n) &= f_2(2n+1) = x(4n+3) = \{x(3), x(7)\} \end{aligned}\right\} \quad \ldots(30)$$

Combination of first and second stage of decimation :

Combining Fig. G-3 and Fig. G-4 in Fig. G-2 we get the combination of first and second stage of decimation. It is shown in Fig. G-5.

At this stage we have $\frac{N}{4}$ that means 2 point sequences. So further decimation is not possible. As shown in Fig. G-5; we have to compute 2-point DFT.

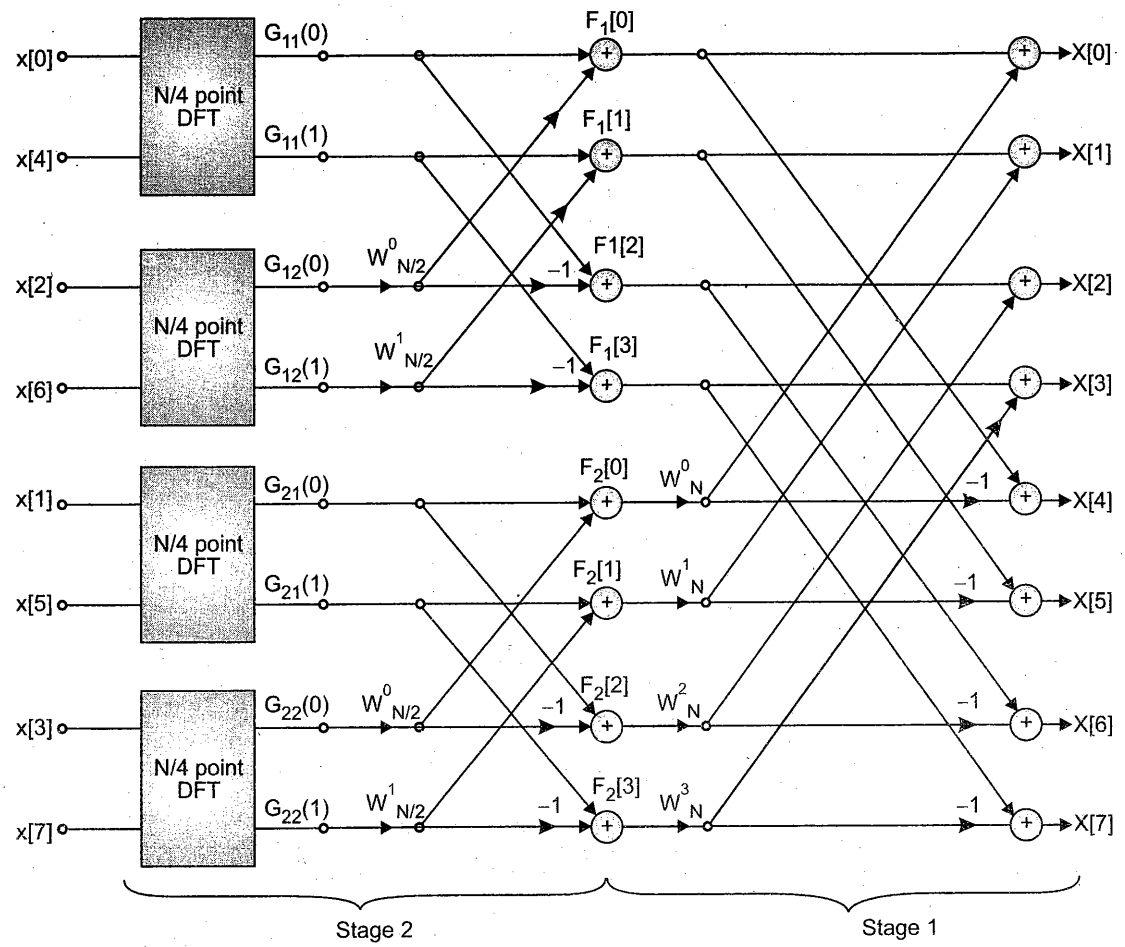


Fig. G-5 : Combination of first and second stage of decimation

Computation of 2-point DFT : According to the basic definition of DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad k = 0, 1, ..., N-1 \qquad ...(31)$$

We will use Equation (31) to compute 2-point DFT. From Fig. G-6, consider the first block of 2-point DFT. It is separately drawn as shown in Fig. G-6.

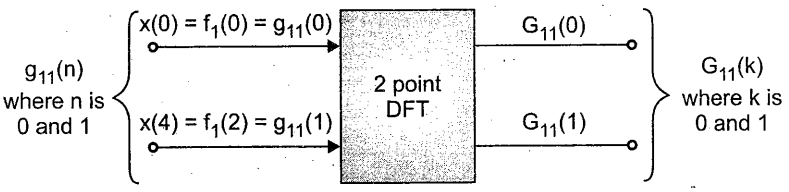


Fig. G-6 : Block of 2-point DFT

Here input sequences are $g_{11}(0)$ and $g_{11}(1)$. We can denote it by $g_{11}(n)$; where n varies from 0 to 1. Now the output sequences are $G_{11}(0)$ and $G_{11}(1)$. We can denote it by $G_{11}(k)$; where 'k' varies from 0 to 1. Here $G_{11}(k)$ is DFT of $g_{11}(n)$.

Thus for $G_{11}(k)$ we can write Equation (31) as,

$$G_{11}(k) = \sum_{n=0}^{1} g_{11}(n) W_2^{kn}, \quad k = 0, 1 \qquad ...(32)$$

Note that this is 2 point DFT, so we have put N = 2.

Now putting values of k in Equation (32) we get,

$$\text{For } k = 0 \Rightarrow G_{11}(0) = \sum_{n=0}^{1} g_{11}(n) W_2^{0}$$

$$\text{But} \quad W_2^0 = 1$$

$$\therefore \quad G_{11}(0) = \sum_{n=0}^{1} g_{11}(n)$$

Expanding the summation we get,

$$\boxed{G_{11}(0) = g_{11}(0) + g_{11}(1)} \qquad ...(33)$$

$$\text{For } k = 1 \Rightarrow G_{11}(1) = \sum_{n=0}^{1} g_{11}(n) W_2^{n}$$

Expanding the summation we get,

$$G_{11}(1) = g_{11}(0) W_2^0 + g_{11}(1) W_2^1 \qquad ...(34)$$

We have $W_N = e^{-\frac{j2\pi}{N}}$

$$\therefore \quad W_2^1 = \left(e^{-\frac{j2\pi}{2}}\right)^1 = e^{-j\pi} = \cos\pi - j\sin\pi = -1 - j0$$

$$\therefore \quad W_2^1 = -1 \qquad \text{And} \quad W_2^0 = 1$$

Putting these values in Equation (34) we get,

$$\boxed{G_{11}(1) = g_{11}(0) - g_{11}(1)} \qquad ...(35)$$

Using Equations (33) and (35), we can represent the computation of 2-point DFT as shown in Fig. G-7. This structure looks like a butterfly. So it is called as FFT butterfly structure.

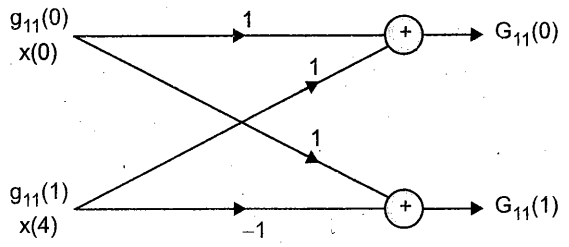


Fig. G-7 : FFT butterfly structure

Now we know that $W_2^0 = 1$. Thus we can modify Equation (33) and (35) as follows :

$$G_{11}(0) = g_{11}(0) + W_2^0 g_{11}(1) \quad ...(36)$$

and $$G_{11}(1) = g_{11}(0) - W_2^0 g_{11}(1) \quad ...(37)$$

This modified butterfly structure is shown in Fig. G-8.

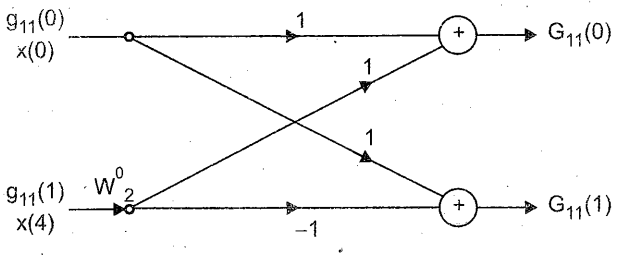


Fig. G-8 : Modified butterfly structure

Similarly for other 2-point DFTs we can draw the butterfly structure.

Total signal flow-graph for 8-point DIT FFT :

The total signal flow graph is obtained by interconnecting all stages of decimation. In this case, it is obtained by interconnecting first and second stage of decimation. But the starting block is the block used to compute 2-point DFT (butterfly structure). The total signal flow graph is shown in Fig. G-9.

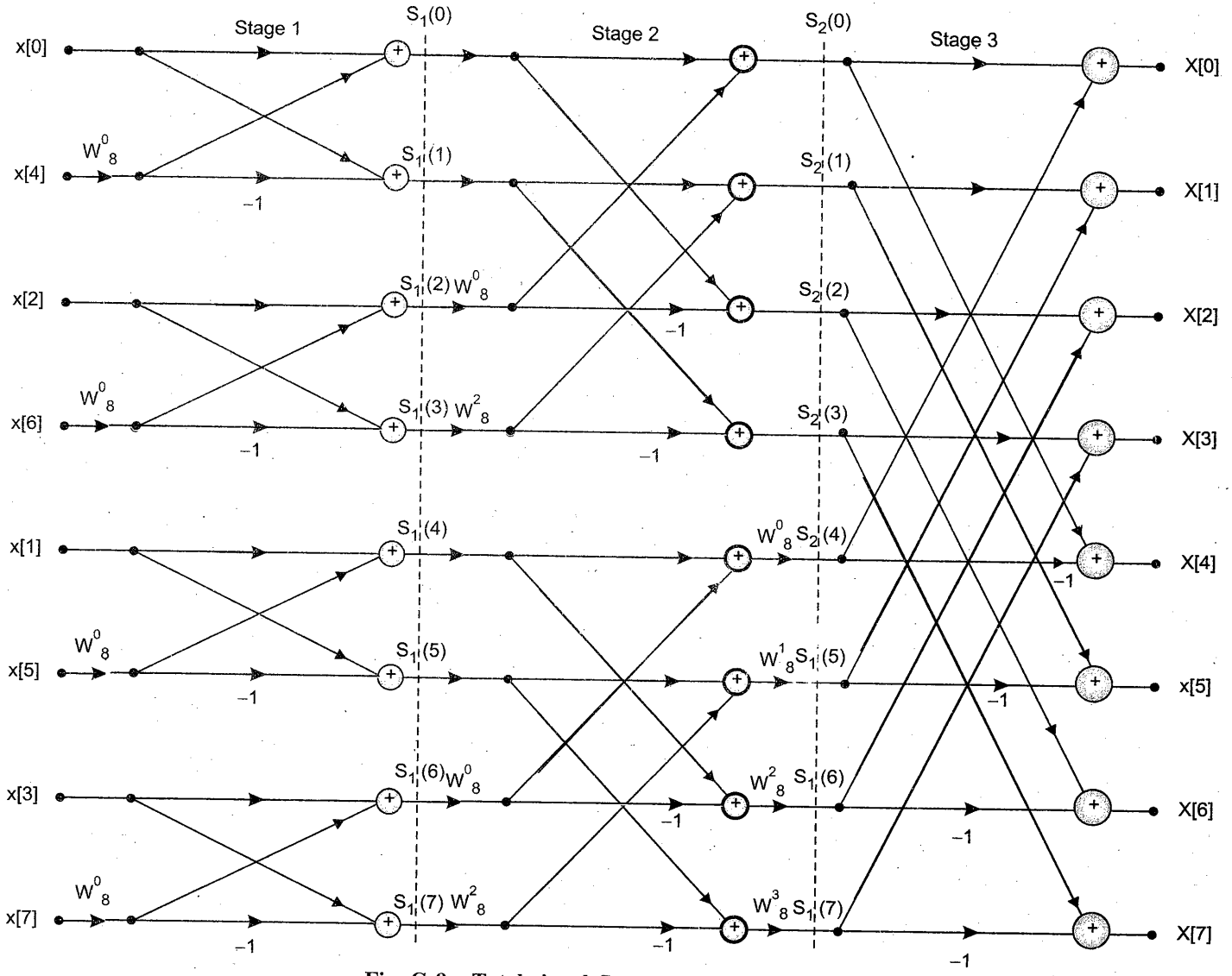


Fig. G-9 : Total signal flow graph for 8 point DIT FFT

Prob. 1 : Compute the eight-point DFT of a sequence.

$x(n) = \left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0\right\}$ Using in-place radix-2 decimation in time FFT algorithm.

Soln. : This flow graph is shown in Fig. G-10.

Here $s_1(n)$ represents output of stage - 1 and $s_2(n)$ represents output of stage - 2. The different values of twiddle factor are

$$W_8^0 = e^0 = 1$$

$$W_8^1 = e^{-j\frac{\pi}{4}} = 0.707 - j\,0.707$$

$$W_8^2 = e^{-j\frac{\pi}{2}} = -j$$

$$W_8^3 = -0.707 - j\,0.707$$

Output of stage - 1 :

$$s_1(0) = x(0) + W_8^0 x(4) = \frac{1}{2} + 1(0) = \frac{1}{2}$$

$$s_1(1) = x(0) - W_8^0 x(4) = \frac{1}{2} - 1(0) = \frac{1}{2}$$

$$s_1(2) = x(2) + W_8^0 x(6) = \frac{1}{2} + 1(0) = \frac{1}{2}$$

$$s_1(3) = x(2) - W_8^0 x(6) = \frac{1}{2} - 1(0) = \frac{1}{2}$$

$$s_1(4) = x(1) + W_8^0 x(5) = \frac{1}{2} + 1\cdot(0) = \frac{1}{2}$$

$$s_1(5) = x(1) - W_8^0 x(5) = \frac{1}{2} - 1\cdot(0) = \frac{1}{2}$$

$$s_1(6) = x(3) + W_8^0 x(7) = \frac{1}{2} + 1\cdot(0) = \frac{1}{2}$$

$$s_1(7) = x(3) - W_8^0 x(7) = \frac{1}{2} - 1\cdot(0) = \frac{1}{2}$$

Output of stage - 2 :

$$s_2(0) = s_1(0) + W_8^0 s_1(2) = \frac{1}{2} + 1\cdot\left(\frac{1}{2}\right) = 1$$

$$s_2(1) = s_1(1) + W_8^2 s_1(3) = \frac{1}{2} - j\frac{1}{2}$$

$$s_2(2) = s_1(0) - W_8^0 s_1(2) = \frac{1}{2} - \frac{1}{2} = 0$$

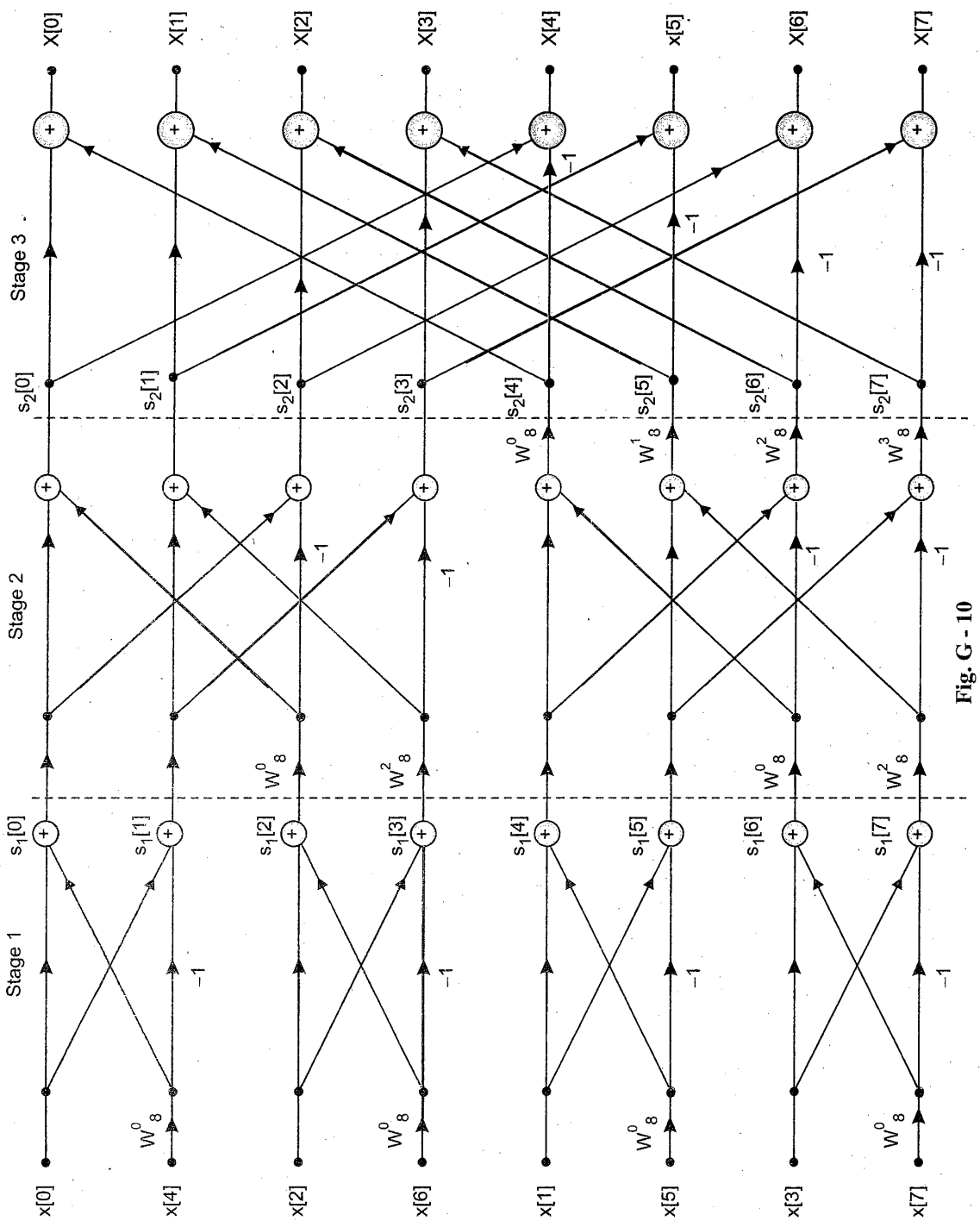

Stage 1
Stage 2
Stage 3
x[0]
x[4]
x[2]
x[6]
x[1]
x[5]
x[3]
x[7]
$s_1[0]$
$s_1[1]$
$s_1[2]$
$s_1[3]$
$s_1[4]$
$s_1[5]$
$s_1[6]$
$s_1[7]$
$s_2[0]$
$s_2[1]$
$s_2[2]$
$s_2[3]$
$s_2[4]$
$s_2[5]$
$s_2[6]$
$s_2[7]$
X[0]
X[1]
X[2]
X[3]
X[4]
x[5]
X[6]
X[7]
W_8^0
W_8^1
W_8^2
W_8^3
−1

Fig. G - 10

$$s_2(3) = s_1(1) - W_8^2 s_1(3) = \frac{1}{2} + j\frac{1}{2}$$

$$s_2(4) = s_1(4) + W_8^0 s_1(6) = \frac{1}{2} + \frac{1}{2} = 1$$

$$s_2(5) = s_1(5) + W_8^2 s_1(7) = \frac{1}{2} - j\frac{1}{2}$$

$$s_2(6) = s_1(4) - W_8^0 s_1(6) = \frac{1}{2} - \frac{1}{2} = 0$$

$$s_2(7) = s_1(5) - W_8^2 s_1(7) = \frac{1}{2} + j\frac{1}{2}$$

Final output :

$$X(0) = s_2(0) + W_8^0 s_2(4) = 1 + 1 = 2$$

$$X(1) = s_2(1) + W_8^1 s_2(5) = \left(\frac{1}{2} - j\frac{1}{2}\right) + (0.707 - j\,0.707)\left(\frac{1}{2} - j\frac{1}{2}\right)$$

$$= 0.5 - j\,1.207$$

$$X(2) = s_2(2) + W_8^2 s_2(6) = 0 + (-j)(0) = 0$$

$$X(3) = s_2(3) + W_8^3 s_2(7) = \left(\frac{1}{2} + j\frac{1}{2}\right) + (-0.707 - j\,0.707)\left(\frac{1}{2} + j\frac{1}{2}\right)$$

$$= \left(\frac{1}{2} + j\frac{1}{2}\right) + (0 - j\,0.707) = 0.5 - j\,0.207$$

$$X(4) = s_2(0) - W_8^0 s_2(4) = 1 - 1.1 = 0$$

$$X(5) = s_2(1) - W_8^1 s_2(5)$$

$$= \left(\frac{1}{2} - j\frac{1}{2}\right) - (0.707 - j\,0.707)\left(\frac{1}{2} - j\frac{1}{2}\right)$$

$$= \left(\frac{1}{2} - j\frac{1}{2}\right) - (-0.707\,j)$$

$$\therefore \quad X(5) = 0.5 + j\,0.207$$

$$X(6) = s_2(2) - W_8^2 s_2(6) = 0 + j \cdot (0) = 0$$

$$X(7) = s_2(3) - W_8^3 s_2(7)$$

$$= \left(\frac{1}{2} + j\frac{1}{2}\right) - (-0.707 - j\,0.707)\left(\frac{1}{2} + j\frac{1}{2}\right)$$

$$= \left(\frac{1}{2} + j\frac{1}{2}\right) + 0.707\,j = 0.5 + j\,1.21$$

Thus,

$$X(k) = \{X(0), X(1), X(2), X(3), X(4), X(5), X(6), X(7)\}$$

$$\therefore\ X(k) = \{2,\ 0.5 - j\,1.207,\ 0,\ 0.5 - j\,0.207,\ 0,\ 0.5 + j\,0.207,\ 0,\ 0.5 + j\,1.21\}$$

Prob. 2 : Let $x(n)$ be a finite duration sequence of length 8 such that $x(n) = \{-1, 0, 2, 0, -4, 0, 2, 0\}$

(a) Find $X(k)$ using DITFFT flow graph.

(b) Using the result in (a) and not otherwise find DFT of sequence $x_1(n) = \{-1, 2, -4, +2\}$. Justify your answer.

(c) Using result in (b) find DFT of sequence $x_2(n) = \{-4, +2, -1, +2\}$

Soln. :

(a) This flow graph is as shown in Fig. G-11(a)

Here $s_1(n)$ represents output of stage - 1 and $s_2(n)$ represents output of stage - 2. The different values of twiddle factor are as follows :

$$W_8^0 = e^0 = 1$$

$$W_8^1 = e^{-j\frac{\pi}{4}} = 0.707 - j\,0.707$$

$$W_8^2 = e^{-\frac{j\pi}{2}} = -j$$

$$W_8^3 = e^{-\frac{j3\pi}{4}} = -0.707 - j\,0.707$$

Output of stage - 1 :

$$s_1(0) = x(0) + W_8^0 x(4) = -1 + 1\cdot(-4) = -5$$

$$s_1(1) = x(0) - W_8^0 x(4) = -1 - 1(-4) = 3$$

$$s_1(2) = x(2) + W_8^0 x(6) = 2 + 1\cdot(2) = 4$$

$$s_1(3) = x(2) - W_8^0 x(6) = +2 - 1\cdot(2) = 0$$

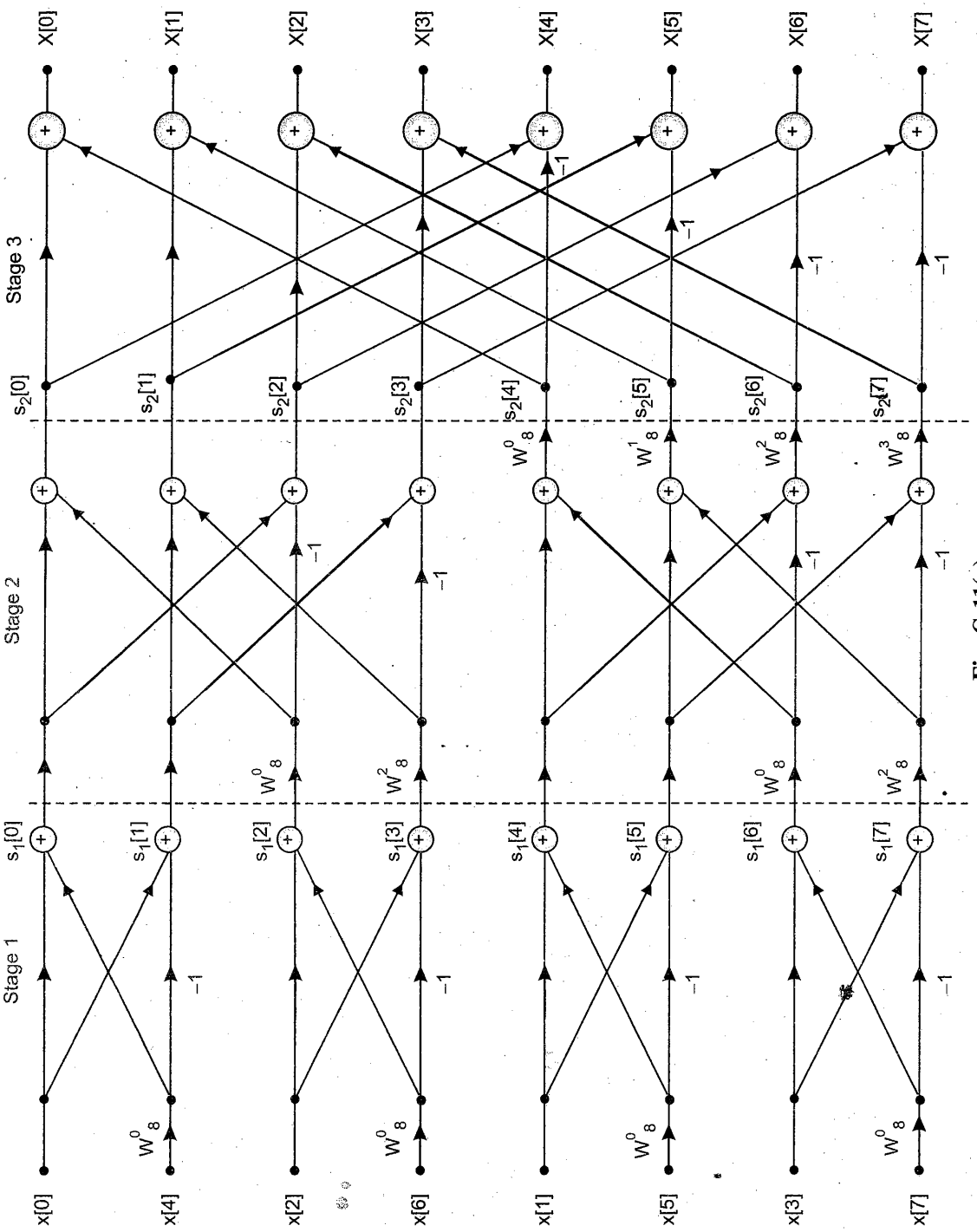

Stage 1
Stage 2
Stage 3
x[0]
x[4]
x[2]
x[6]
x[1]
x[5]
x[3]
x[7]
W_8^0
W_8^2
W_8^1
W_8^3
−1
$s_1[0]$
$s_1[1]$
$s_1[2]$
$s_1[3]$
$s_1[4]$
$s_1[5]$
$s_1[6]$
$s_1[7]$
$s_2[0]$
$s_2[1]$
$s_2[2]$
$s_2[3]$
$s_2[4]$
$s_2[5]$
$s_2[6]$
$s_2[7]$
X[0]
X[1]
X[2]
X[3]
X[4]
X[5]
X[6]
X[7]

Fig. G-11(a)

$$s_1(4) = x(1) + W_8^0 x(5) = 0 + 1 \cdot (0) = 0$$

$$s_1(5) = x(1) - W_8^0 x(5) = 0 - 1(0) = 0$$

$$s_1(6) = x(3) + W_8^0 x(7) = 0$$

$$s_1(7) = x(3) - W_8^0 x(7) = 0 - 1 \cdot (0) = 0$$

Output of stage - 2 :

$$s_2(0) = s_1(0) + W_8^0 s_1(2) = -5 + 1 \cdot (4) = -1$$

$$s_2(1) = s_1(1) + W_8^2 s_1(3) = 3 - j(0) = 3$$

$$s_2(2) = s_1(0) - W_8^0 s_1(2) = -5 - 1 \cdot (4) = -9$$

$$s_2(3) = s_1(1) - W_8^2 s_1(3) = 3 + j(0) = 3$$

$$s_2(4) = s_1(4) + W_8^0 s_1(6) = 0 + 1 \cdot (0) = 0$$

$$s_2(5) = s_1(5) + W_8^2 s_1(7) = 0 - j(0) = 0$$

$$s_2(6) = s_1(4) - W_8^0 s_1(6) = 0 - 1 \cdot (0) = 0$$

$$s_2(7) = s_1(5) - W_8^2 s_1(7) = 0 + j(0) = 0$$

Final output :

$$X(0) = s_2(0) + W_8^0 s_2(4) = -1 + 1 \cdot (0) = -1$$

$$X(1) = s_2(1) + W_8^1 s_2(5) = 3 + (0.707 - j\,0.707) \cdot 0 = 3$$

$$X(2) = s_2(2) + W_8^2 s_2(6) = -9 - j(0) = -9$$

$$X(3) = s_2(3) + W_8^3 s_2(7) = 3 + (-0.707 - j\,0.707) \cdot 0 = 3$$

$$X(4) = s_2(0) - W_8^0 s_2(4) = -1 - 1 \cdot (0) = -1$$

$$X(5) = s_2(1) - W_8^1 s_2(5) = 3 - 1 \cdot (0) = 3$$

$$X(6) = s_2(2) - W_8^2 s_2(6) = -9 + j(0) = -9$$

$$X(7) = s_2(3) - W_8^3 s_2(7) = 3 - (-0.707 - j\,0.707) \cdot 0 = 3$$

Thus,

$$X(k) = \{X(0), X(1), X(2), X(3), X(4), X(5), X(6), X(7)\}$$

$$\therefore X(k) = \{-1, 3, -9, 3, -1, 3, -9, 3\}$$

(b) Let $x(n) = \{a, b, c, d\}$ and let its DFT be the sequence $X(k) = \{A, B, C, D\}$. If we add one zero after each sample in $x(n)$ then we will get the sequence.

$$x_1(n) = \{a, 0, b, 0, c, 0, d, 0\}$$

This process is called as upsampling process. Since in this sequence one zero is added after each sample; the entire DFT repeats one time. If we will add two zeros after each sample then entire DFT will repeat two times.

$$\therefore \quad \text{DFT}\{x_1(n)\} = X_1(k) = \{A, B, C, D, A, B, C, D\}$$

In part (a) for the sequence $x(n)$.

$$x(n) = \{-1, 0, 2, 0, -4, 0, 2, 0\}$$

We have obtained the DFT,

$$X(k) = \{-1, 3, -9, 3, -1, 3, -9, 3\}$$

Observe that first four DFT samples are repeated only once. This is because in $x(n)$, zero is added after each sample.

The given sequence is,

$$x_1(n) = \{-1, 2, -4, 2\}$$

So its DFT is,

$$X_1(k) = \{-1, 3, -9, 3\}$$

Justification of answer :

We will prove the property of DFT used in this example. Let $x(n) = \{a, b, c, d\}$ and $X(k) = \{A, B, C, D\}$.

Consider the sequence,

$$x_1(n) = \{a, 0, b, 0, c, 0, d, 0\}$$

According to the definition of DFT we can write,

$$X_1(k) = \sum_{n=0}^{7} x_1(n) \cdot W_8^{kn} \qquad \ldots(1)$$

We will divide the sequence $x_1(n)$ into odd part and even part. Let $x_1(2n)$ represent even part and $x_1(2n+1)$ represent odd part.

$$\therefore \quad X_1(k) = \sum_{n=0}^{3} x_1(2n)\, W_8^{2kn} + \sum_{n=0}^{3} x_1(2n+1)\, W_8^{(2n+1)\cdot k} \qquad ...(2)$$

Observe that in the first summation 'n' is replaced by 2n and in the second summation 'n' is replaced by $(2n + 1)$. But in the second summation $x_1(2n+1)$ represents odd samples of sequence $x_1(n)$ and all these samples are zero.

$$\therefore \quad X_1(k) = \sum_{n=0}^{3} x_1(2n)\, W_8^{2kn} \qquad ..(3)$$

Now we have the property of twiddle factor.

$$W_N^{2kn} = W_{N/2}^{kn}$$

$$\therefore \quad W_8^{2kn} = W_4^{kn}$$

$$\therefore \quad X_1(k) = \sum_{n=0}^{3} x_1(2n) \cdot W_4^{kn} \qquad ...(4)$$

But $x_1(2n)$ represents even samples of $x_1(n)$. That means $x_1(2n) = x(n)$.

$$\therefore \quad X_1(k) = \sum_{n=0}^{3} x(n)\, W_4^{kn} = X(k)$$

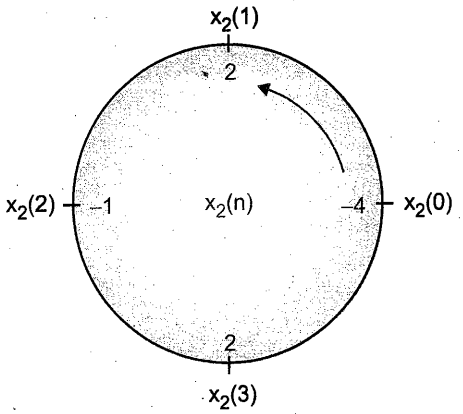
But $x_1(n)$ is eight point sequence.

$$\therefore \quad X_1(k) = \{A, B, C, D, A, B, C, D\}$$

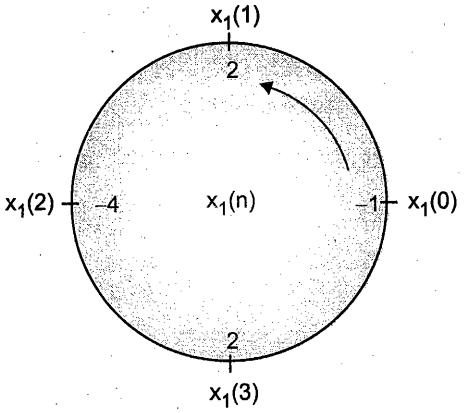
(c) Here $x_2(n) = \{-4, 2, -1, 2\}$

We have $x_1(n) = \{-1, 2, -4, 2\}$

Let us plot the sequences $x_1(n)$ and $x_2(n)$ as shown in Fig. G-11(b) and G-11(c).



(b)



(c)

Fig. G-11

From these diagrams we can conclude that $x_2(n)$ is obtained by circularly rotating $x_1(n)$ by 2 positions in anticlockwise direction. That means $x_2(n)$ is obtained by delaying $x_1(n)$ by 2 positions.

$$\therefore \quad x_2(n) = x_1((n-2))$$

Now according to circular time shifting property,

$$x((n-l))_N \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X(k)\, W_N^{kl}$$

Thus in this case we can write,

$$X_2(k) = X_1(k) \cdot W_4^{2k} = X_1(k) \cdot e^{-\frac{j2\pi}{4} \cdot k}$$

$$\therefore \quad X_2(k) = e^{-\frac{j\pi}{2} \cdot k} \cdot X_1(k)$$

We have, $\quad X_1(k) = \{-1, 3, -9, 3\}$

We will calculate sequence $X_2(k)$ for different values of k as follows :

For $k = 0 \Rightarrow X_2(0) = e^0 \cdot X_1(0) = -1$

For $k = 1 \Rightarrow X_2(1) = e^{-j\pi} X_1(1) = (\cos\pi - j\sin\pi) \cdot 3 = -3$

For $k = 2 \Rightarrow X_2(2) = e^{-j\pi 2} X_1(2) = (\cos 2\pi - j\sin 2\pi) \cdot (-9) = -9$

For $k = 3 \Rightarrow X_2(3) = e^{-j\pi 3} X_1(3) = (\cos 3\pi - j\sin 3\pi) \cdot 3 = -3$

$$\therefore \quad X_2(k) = \{X_2(0), X_2(1), X_2(2), X_2(3)\}$$

$$\therefore \quad X_2(k) = \{-1, -3, -9, -3\}$$

Prob. 3 : Derive DIT FFT flow graph for N = 4 hence find DFT of x (n) = {1, 2, 3, 4}

Soln. :

First stage of decimation :

We have the equations for first stage of decimation.

$$X(k) = F_1(k) + W_N^k F_2(k), \qquad k = 0, 1 \ldots \frac{N}{2} - 1 \qquad \ldots(1)$$

and $$X\left(k + \frac{N}{2}\right) = F_1(k) - W_N^k F_2(k), \qquad k = 0, 1, \ldots \frac{N}{2} - 1 \qquad \ldots(2)$$

Here N = 4

$$\therefore \quad X(k) = F_1(k) + W_4^k F_2(k), \qquad k = 0, 1 \qquad \ldots(3)$$

and $$X(k+2) = F_1(k) - W_4^k F_2(k), \qquad k = 0, 1 \qquad \ldots(4)$$

Putting values of k in Equation (3) we get,

$$\left.\begin{aligned} X(0) &= F_1(0) + W_4^0 F_2(0) \\ \text{and} \quad X(1) &= F_1(1) + W_4^1 F_2(1) \end{aligned}\right\} \qquad \ldots(5)$$

Similarly putting values of k in Equation (4) we get,

$$\left.\begin{aligned} X(2) &= F_1(0) - W_4^0 F_2(0) \\ \text{and} \quad X(3) &= F_1(1) - W_4^1 F_2(1) \end{aligned}\right\} \qquad \ldots(6)$$

This signal flow graph is shown in Fig. G-12(a)

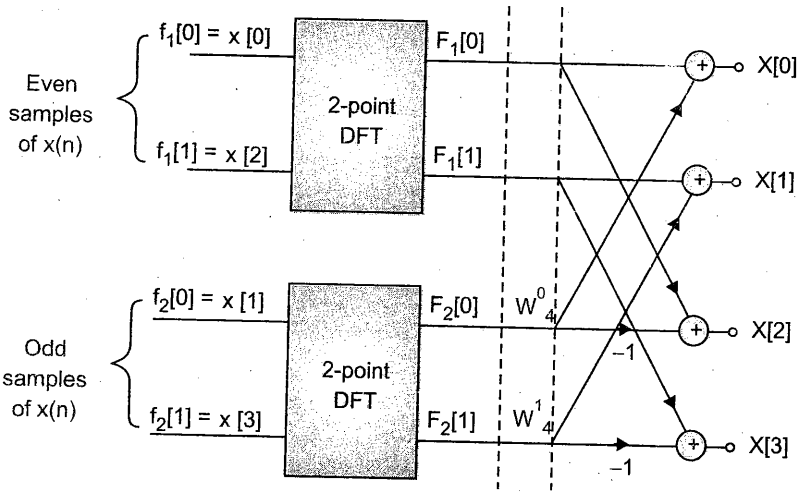


Fig. G-12(a)

Now we will replace each 2-point DFT by butterfly structure as shown in Fig. G-12(b).

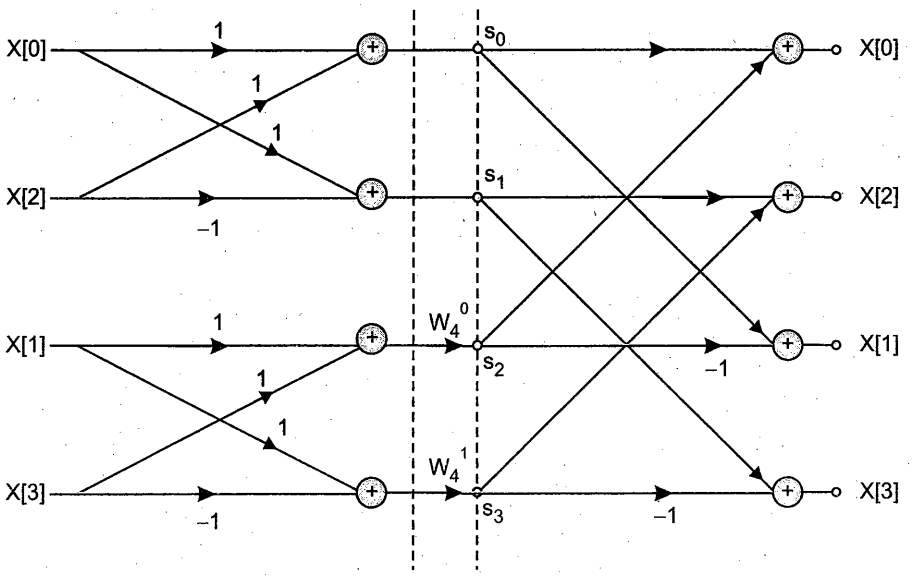


Fig. G-12(b)

The given sequence is,

$$x(n) = \{1, 2, 3, 4\}$$

The different values of twiddle factor are as follows :

$$W_4^0 = 1$$

$$W_4^1 = e^{-\frac{j2\pi}{4} \cdot 1} = e^{-\frac{j\pi}{2}}$$

$$= \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

The output s (n) is,

$$s_0 = x(0) + x(2) = 1 + 3 = 4$$

$$s_1 = x(0) - x(2) = 1 - 3 = -2$$

$$s_2 = [x(1) + x(3)]\, W_4^0 = 2 + 4 = 6$$

$$s_3 = [x(1) - x(3)]\, W_4^1 = (2 - 4) \cdot (-j) = 2j$$

The final output is,

$$X(0) = s_0 + s_2 = 4 + 6 = 10$$

$$X(1) = s_1 + s_3 = -2 + j2$$

$$X(2) = s_0 - s_2 = 4 - 6 = -2$$

$$X(3) = s_1 - s_3 = -2 - j2$$

Thus,

$$X(k) = \{X(0), X(1), X(2), X(3)\}$$

$$\therefore X(k) = \{10, -2 + j2, -2, -2 - j2\}$$

Prob. 4 : Draw flow diagram of DITFFT for N = 16.

Soln. :

(1) Here N = 16, means it is 16 point DFT.

(2) Total number of stages = 4

(3) The first stage of decimation using two 8-point DFT is shown in Fig. G-13(a).

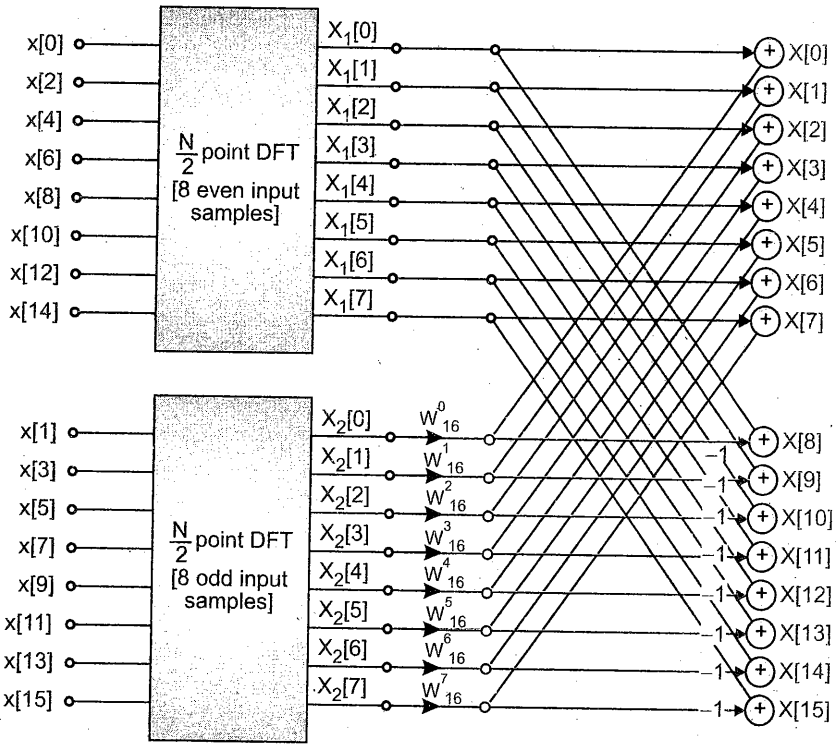


Fig. G-13(a)

(4) In the second stage each 8 point DFT is divided into 2 four point DFTs as shown in Fig. G-13(b).

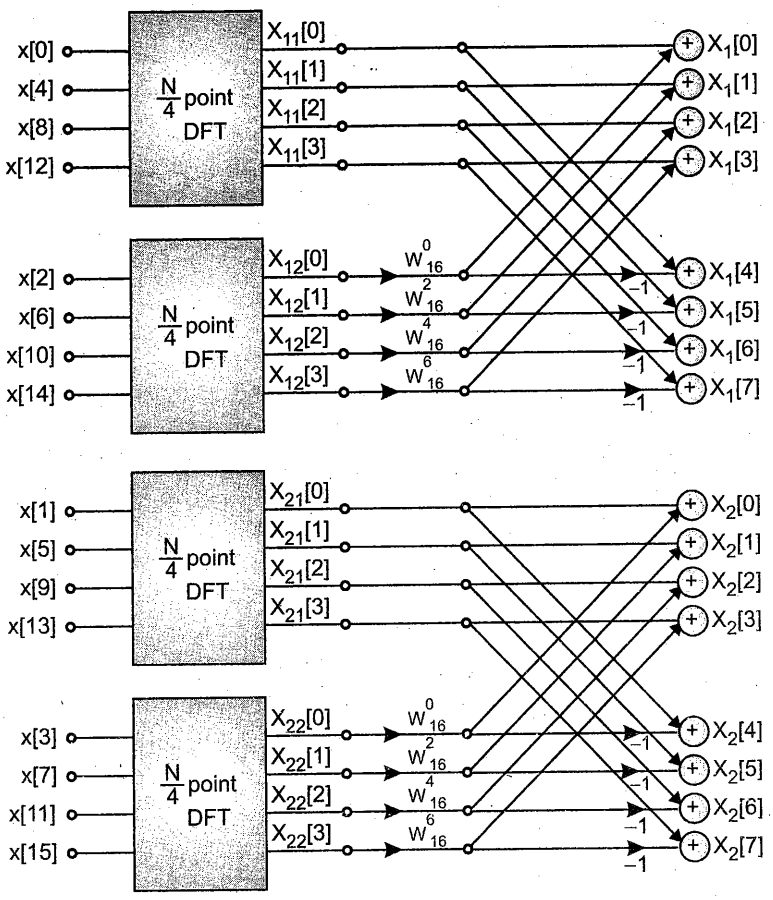


Fig. G-13(b)

(5) The total flow graph is shown in Fig. G-13(c)

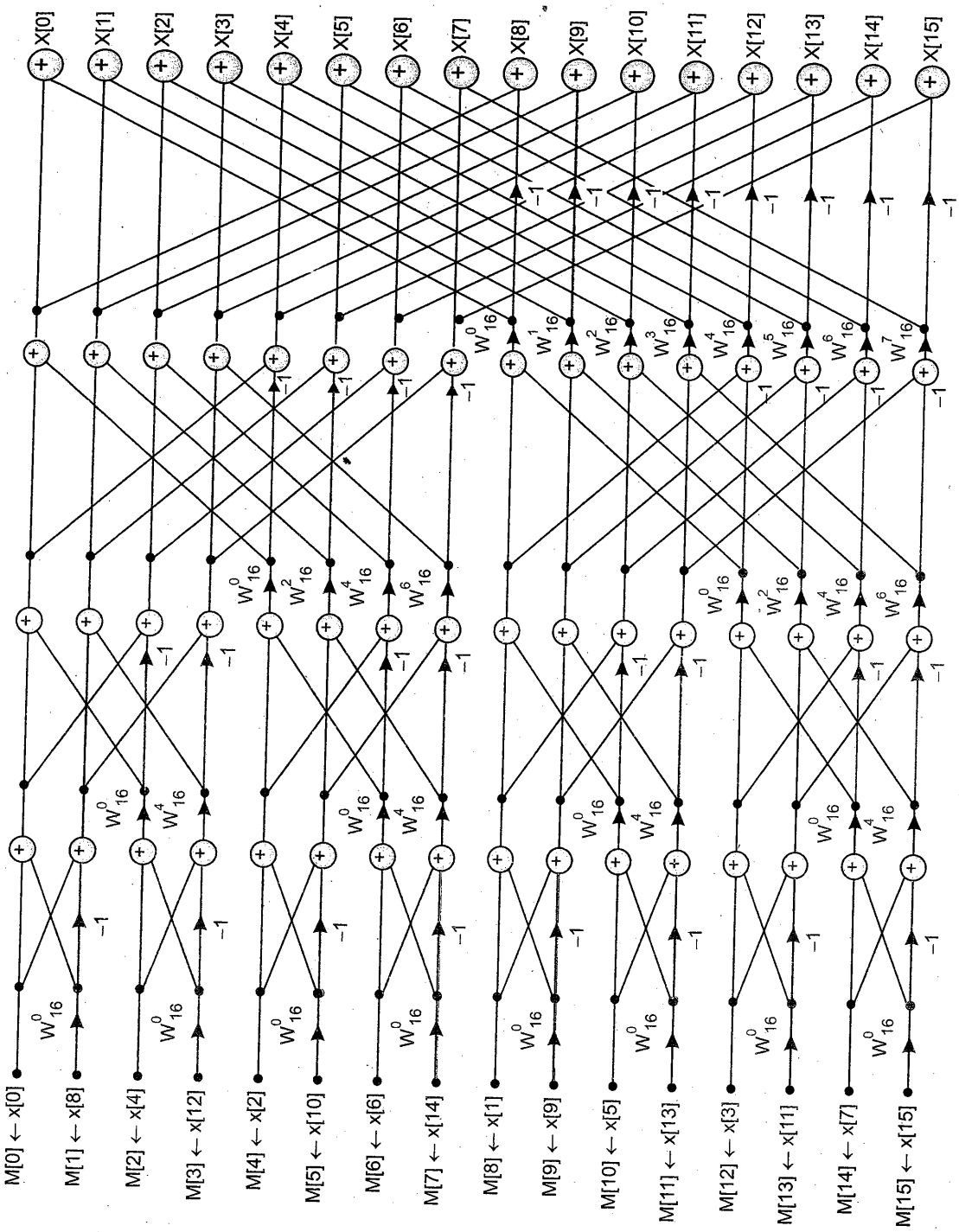

M[0] ← x[0]
M[1] ← x[8]
M[2] ← x[4]
M[3] ← x[12]
M[4] ← x[2]
M[5] ← x[10]
M[6] ← x[6]
M[7] ← x[14]
M[8] ← x[1]
M[9] ← x[9]
M[10] ← x[5]
M[11] ← x[13]
M[12] ← x[3]
M[13] ← x[11]
M[14] ← x[7]
M[15] ← x[15]
X[0]
X[1]
X[2]
X[3]
X[4]
X[5]
X[6]
X[7]
X[8]
X[9]
X[10]
X[11]
X[12]
X[13]
X[14]
X[15]

Fig. G-13(c)

1.3.1 Comparison of Computational Complexity with Direct Computation :

First we will calculate the computational complexity for direct DFT calculation.

A] For direct computation :

According to the definition of DFT we have,

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad k = 0, 1, 2, ... N-1 \qquad ...(1)$$

Equation (1) indicates that we have to take multiplication of x (n) and twiddle factor. Then we have to add all the terms. Since twiddle factor is complex; we need to perform complex multiplications and complex additions.

Complex multiplications :

As given by Equation (1), for one value of 'k' multiplication should be performed for all values of 'n'. The range of 'n' is from 0 to N – 1. So for one value of 'k'; N complex multiplications are required. Now the range of k is also from k = 0 to k = N – 1. The total complex multiplications are,

$$\text{Complex multiplications} = N \times N = N^2 \qquad ..(2)$$

Complex additions :

According to Equation (1), for each value of K we need to add the product terms of $x(n) W_N^{kn}$. For example, let us say N = 4.

$$\text{For } k = 0 \Rightarrow X(0) = \sum_{n=0}^{3} x(n) W_4^{0 \times n} = \sum_{n=0}^{3} x(n) W_4^0$$

$$\therefore \quad X(0) = x(0) W_4^0 + x(1) W_4^0 + x(2) W_4^0 + x(3) W_4^0 \qquad ...(3)$$

In Equation (3); four complex multiplications are required and three complex additions are required. Here we have considered N = 4. Thus for each value of 'k'; N complex multiplications are required and 'N – 1' complex additions are required. Now the total values of k and 'N'.

$$\therefore \text{ Complex Additions} = N(N-1) = N^2 - N$$

B] Computational complexity using FFT algorithm :

Firstly we will calculate the computation complexity required for one butterfly. Consider the general structure of butterfly as shown in Fig. G-14.

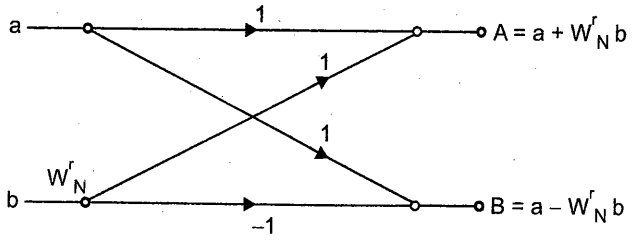


Fig. G-14 : General structure of butterfly

Here 'a' and 'b' are inputs and A and B are outputs of butterfly. The outputs are given by,

$$A = a + W_N^r b \quad ...(4)$$

and $$B = a - W_N^r b \quad ...(5)$$

(1) To calculate any output (A or B), we need to multiple input 'b' by twiddle factor W_N^r. So one complex multiplication is required for one butterfly.

(2) To calculate output 'A', one complex addition is required, while to calculate output 'B' one complex subtraction is required as given by Equation (5). But the computational complexity of addition and subtraction is same. So we can say that for one butterfly two complex additions are required.

(3) As shown in Fig. G-10, for 8 point DFT; 4 butterflies are there at each stage. So for 'N' point DFT, at each stage, $\frac{\text{'N'}}{2}$ butterflies are required.

(4) As shown in Fig. G-10; three stages are required to compute 8-point DFT. In general, for 'N' point DFT, $\log_2 N$ stages are required.

Complex multiplications :

At each stage there are $\frac{N}{2}$ butterflies. Total number of stages are $\log_2 N$. And for each butterfly, one complex multiplication is required.

$$\therefore \text{ Total complex multiplications} = \frac{N}{2}\log_2 N \quad ...(6)$$

Complex additions :

Total number of stages are $\log_2 N$. At each stage, $\frac{N}{2}$ butterflies are required. And for each butterfly, '2' complex additions are required.

$$\therefore \text{ Total complex additions} = 2 \times \frac{N}{2}\log_2 N$$

$$\therefore \text{ Total complex additions} = N \log_2 N$$

Table G-1 shows comparison of direct DFT computation and computation using FFT algorithms.

Table G-1

Number of points 'N'	Direct Computation		Using FFT	
	Complex Multiplications (N^2)	Complex Additions ($N^2 - N$)	Complex Multiplication $\left(\frac{N}{2} \log_2 N\right)$	Complex Addition ($N \log_2 N$)
4	16	12	4	8
8	64	56	12	24
16	256	240	32	64
32	1024	992	40	80
64	4096	4032	96	192
128	16,384	16,256	224	448

The Table G-1 shows that, by the use of FFT algorithms the number of complex multiplications and complex additions are reduced. So there is tremendous improvement in the speed.

1.3.2 In-place Computation to Reduce Memory Size :

Firstly, we will discuss the memory requirement of each butterfly. As shown in Fig. G-11(a); a butterfly calculates the values of 'A' and 'B' for the inputs 'a' and 'b'. Remember that 'a' and 'b' are complex inputs. So two memory locations are required to store any one of the inputs; 'a' or 'b'. One memory location is required to store real part and other memory location is required to store imaginary part. Now, to store both inputs 'a' and 'b'; 2 + 2 = 4 memory locations are required.

Now the outputs are computed as follows :

$$A = a + W_N^r b \quad ...(1)$$

$$\text{and} \quad B = a - W_N^r b \quad ...(2)$$

Thus outputs 'A' and 'B' are calculated by using the values of 'a' and 'b' stored in the memory. Now 'A' and 'B' are also complex numbers; so 2 + 2 = 4 memory locations are required to store both the outputs A and B.

Once the computation of 'A' and 'B' is done then, values of 'a' and 'b' are not required. So instead of storing 'A' and 'B' at other memory locations; these values are stored at the same place where 'a' and 'b' were stored. That means 'A' and 'B' are stored in the place of 'a' and 'b'. This is called as in-place computation. In place computation reduces the memory size.

Memory requirement :

We discuss that 'four' memory locations are required for every butterfly to store input and output values. Now, there are $\frac{N}{2}$ butterflies per stage. Thus for each stage,

Memory locations required to store inputs and outputs $= 4 \times \frac{N}{2} = 2N$

Now as shown in Fig. G-14, one value of twiddle factor is required to compute A and B. To store one value of twiddle factor, one memory location is required for each butterfly. Now there are $\frac{N}{2}$ butterflies at each stage. Thus for each stage,

$$\therefore \text{ Memory locations required to store twiddle factor} = \frac{N}{2}$$

Thus combined memory required per stage is '$2N + \frac{N}{2}$'. These many number of memory locations are required to store input values, output values and the twiddle factor per stage.

Now the actual computation of N-point FFT is done stage wise. That means computation of one stage is done at a time. So these memory locations can be used for other stages also. This will again reduce the memory size.

$$\therefore \text{ Total memory locations} = 2N + \frac{N}{2}$$

1.4 Radix-2 Decimation In Frequency (DIF) FFT Algorithm :

Decimation in frequency stands for splitting the sequences in terms of frequency. That means we have to split output sequences into smaller subsequences. This decimation is done as follows :

First stage of decimation :

According to the definition of DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad \ldots(1)$$

We can divide the summation into two parts as follows :

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{n=\frac{N}{2}}^{N-1} x(n) W_N^{kn} \qquad \ldots(2)$$

Now consider the second summation that means, $\sum_{n=\frac{N}{2}}^{N-1} x(n) W_N^{kn}$.

Put $n = n + \frac{N}{2}$; the limits will change as follows :

$$\text{when } n = \frac{N}{2} \Rightarrow \frac{N}{2} = n + \frac{N}{2} \quad \therefore n = 0$$

$$\text{and when } n = N-1 \Rightarrow N-1 = n + \frac{N}{2} \quad \therefore n = N - 1 - \frac{N}{2} = \frac{N}{2} - 1$$

$$\therefore \sum_{n=\frac{N}{2}}^{N-1} x(n) W_N^{kn} = \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) W_N^{k\left(n+\frac{N}{2}\right)}$$

Putting this value in Equation (2) we get,

$$X(K) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) \cdot W_N^{k\left(n+\frac{N}{2}\right)}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) \cdot W_N^{kn} \cdot W_N^{kN/2}$$

$$\therefore X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + W_N^{kN/2} \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) W_N^{kn} \quad ...(3)$$

Now we have, $W_N = e^{-\frac{j2\pi}{N}}$

$$\therefore W_N^{\frac{kN}{2}} = e^{-\frac{j2\pi}{N} \times \frac{kN}{2}} = e^{-j\pi K} = \left(e^{-j\pi}\right)^k$$

$$\therefore W_N^{\frac{kN}{2}} = (\cos\pi - j\sin\pi)^k = (-1 - j0)^k$$

$$\therefore W_N^{\frac{kN}{2}} = (-1)^K$$

Putting this value in Equation (3) we get,

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + (-1)^k \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) W_N^{kn}$$

$\therefore$ Taking the summation common we get,

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^k x\left(n+\frac{N}{2}\right) \right] W_N^{kn} \quad ...(4)$$

Here we have to split the sequence in terms of frequency. So we will split X (k) in terms of even numbered and odd numbered DFT coefficients. Let X (2r) represents even numbered DFT and X (2r + 1) represent odd numbered DFT.

Thus putting k = 2r in Equation (4), we will get even numbered sequence.

$$\therefore \quad X(2r) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^{2r} x\left(n+\frac{N}{2}\right) \right] W_N^{2rn} \qquad ...(5)$$

By putting k = 2r + 1, in Equation (4), we will get odd numbered sequence.

$$\therefore \quad X(2r+1) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^{2r+1} x\left(n+\frac{N}{2}\right) \right] W_N^{(2r+1)n} \qquad ...(6)$$

Here 'r' is an integer similar to k and it varies from 0 to $\frac{N}{2} - 1$.

$$\therefore \quad (-1)^{2r} = 1 \qquad ...(7)$$

$$\text{and} \quad (-1)^{2r+1} = (-1)^{2r}(-1)^1 = -1 \qquad ...(8)$$

Putting these values in Equations (5) and (6) we get,

$$X(2r) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + x\left(n+\frac{N}{2}\right) \right] W_N^{2rn} \qquad ...(9)$$

$$\text{and} \quad X(2r+1) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) - x\left(n+\frac{N}{2}\right) \right] W_N^{(2r+1)n} \qquad ...(10)$$

Now consider the term W_N^{2rn}

$$W_N^{2rn} = \left(W_N^2\right)^{rn}$$

But we have $W_N^2 = W_{N/2}$

$$\therefore \quad W_N^{2rn} = \left(W_{N/2}\right)^{rn} = W_{N/2}^{rn} \qquad ...(11)$$

Now we can write,

$$W_N^{(2r+1)n} = W_N^{2rn} \cdot W_N^n = W_{N/2}^{rn} \cdot W_N^n \qquad ...(12)$$

Putting these values in Equations (9) and (10) we get,

$$X(2r) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + x\left(n+\frac{N}{2}\right) \right] W_{N/2}^{rn} \qquad ...(13)$$

$$\text{and} \quad X(2r+1) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) - x\left(n+\frac{N}{2}\right) \right] W_{N/2}^{rn} \cdot W_N^n \qquad ...(14)$$

$$\text{Now let,} \quad g(n) = x(n) + x\left(n+\frac{N}{2}\right) \qquad ...(15)$$

$$\text{and} \quad h(n) = \left[x(n) - x\left(n+\frac{N}{2}\right) \right] W_N^n \qquad ...(16)$$

Putting these values in Equations (13) and (14) we get,

$$X(2r) = \sum_{n=0}^{\frac{N}{2}-1} g(n) W_{N/2}^{rn} \qquad ...(17)$$

$$\text{and} \quad X(2r+1) = \sum_{n=0}^{\frac{N}{2}-1} h(n) W_{N/2}^{rn} \qquad ...(18)$$

Note that at this stage we have decimated the sequence of 'N' point DFT into two $\frac{N}{2}$ point DFTs given by Equations (17) and (18). Let us consider an example of 8-point DFT. That means N = 8. So combining Equations (17) and (18) (that means $\frac{N}{2} = 4$) we can obtain N (8 point) point DFT. This is first stage of decimation. Note that Equation (17) indicates $4\left(\frac{N}{2}\right)$ point DFT of $g(n)$ and Equation (18) indicates $4\left(\frac{N}{2}\right)$ point DFT of $h(n)$. For 8-point DFT Equation (15) becomes,

$$g(n) = x(n) + x(n+4) \qquad ...(19)$$

Here we are computing '4' point DFT. So range of 'n' is n = 0 to n = 3. Putting these values in Equation (19) we get,

$$\left.\begin{aligned} &\text{For } n = 0 \Rightarrow g(0) = x(0) + x(4) \\ &\text{For } n = 1 \Rightarrow g(1) = x(1) + x(5) \\ &\text{For } n = 2 \Rightarrow g(2) = x(2) + x(6) \\ &\text{For } n = 3 \Rightarrow g(3) = x(3) + x(7) \end{aligned}\right\} \qquad ...(20)$$

Similarly for 8 point DFT Equation (16) becomes,

$$h(n) = [x(n) - x(n+4)] W_8^n \qquad ...(21)$$

$$\left.\begin{aligned}
&\text{For } n = 0 \Rightarrow h(0) = [x(0) - x(4)]\, W_8^0 \\
&\text{For } n = 1 \Rightarrow h(1) = [x(1) - x(5)]\, W_8^1 \\
&\text{For } n = 2 \Rightarrow h(2) = [x(2) - x(6)]\, W_8^2 \\
&\text{For } n = 3 \Rightarrow h(3) = [x(3) - x(7)]\, W_8^3
\end{aligned}\right\} \quad \ldots(22)$$

Using Equations (20) and (22), and Equations (17) and (18) we can draw the flow graph of first stage of decimation as shown in Fig. G-15.

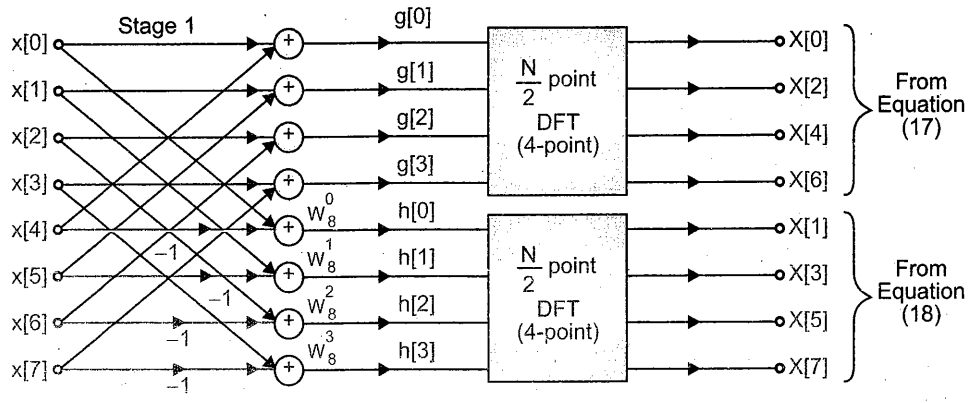


Fig. G-15 : First stage of decimation

Second stage of decimation :

In the first stage of decimation we have used 4-point DFT. We can further decimate the sequence by using 2-point DFT. The second stage of decimation is shown in Fig. G-16.

This is similar to DIT-FFT.

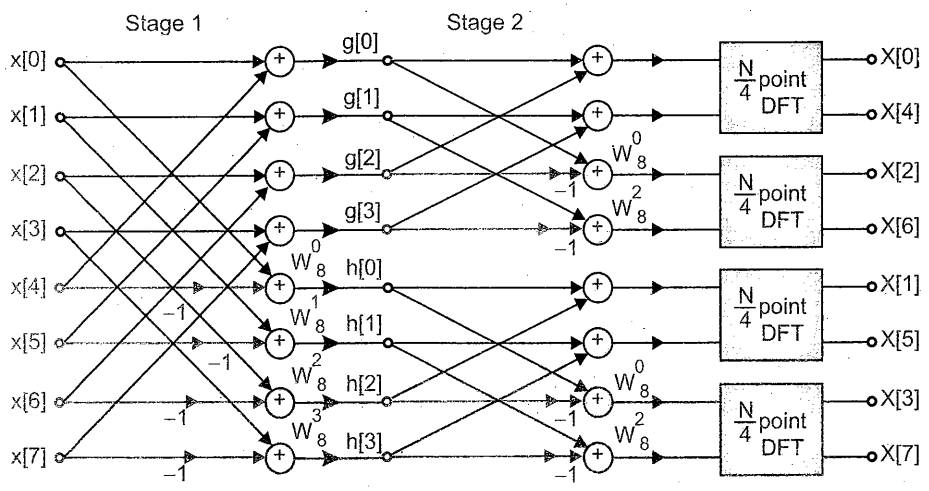


Fig. G-16 : Second stage of decimation

Third stage of decimation :

In the second stage of decimation we have used 2-point DFT. So further decimation is not possible. Now we will use a butterfly structure to obtain 2-point DFT. This butterfly is same as shown in Fig. G-14. Thus the total flow graph for 8 point DIF-FFT is shown in Fig. G-17.

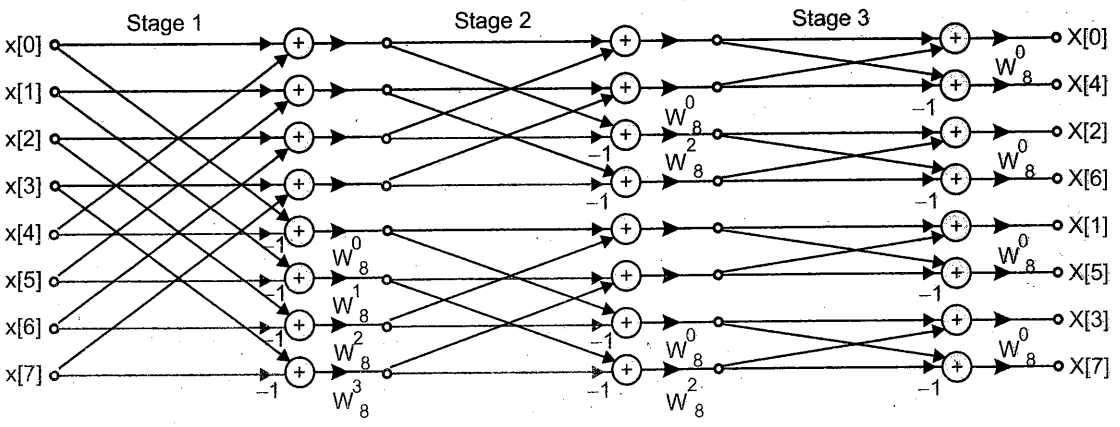


Fig. G-17 : Total flow graph for 8-point DIF-FFT

This flow graph is similar to the flow graph of DIF-FFT but it is in the opposite direction. Note that here input x (n) is in sequence but output is shuffled. Similar to DIT-FFT there are $\log_2 N = \log_2 8 = 3$ stages.

The computational complexity and the memory requirement is same as that of DIT-FFT.

Prob. 1 : Obtain DFT of a sequence

$$x(n) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0\right)$$

Using decimation in frequency FFT algorithm.

Soln. : The total flow graph is shown in Fig. G-18.

Here g (n) is output of first stage.

h (n) is output of second stage.

The values of twiddle factor are as follows :

$$W_8^0 = e^0 = 1$$

$$W_8^1 = e^{-\frac{j\pi}{4}} = 0.707 - j\,0.707$$

$$W_8^2 = e^{\frac{-j\pi}{2}} = -j$$

$$W_8^3 = -0.707 - j\,0.707$$

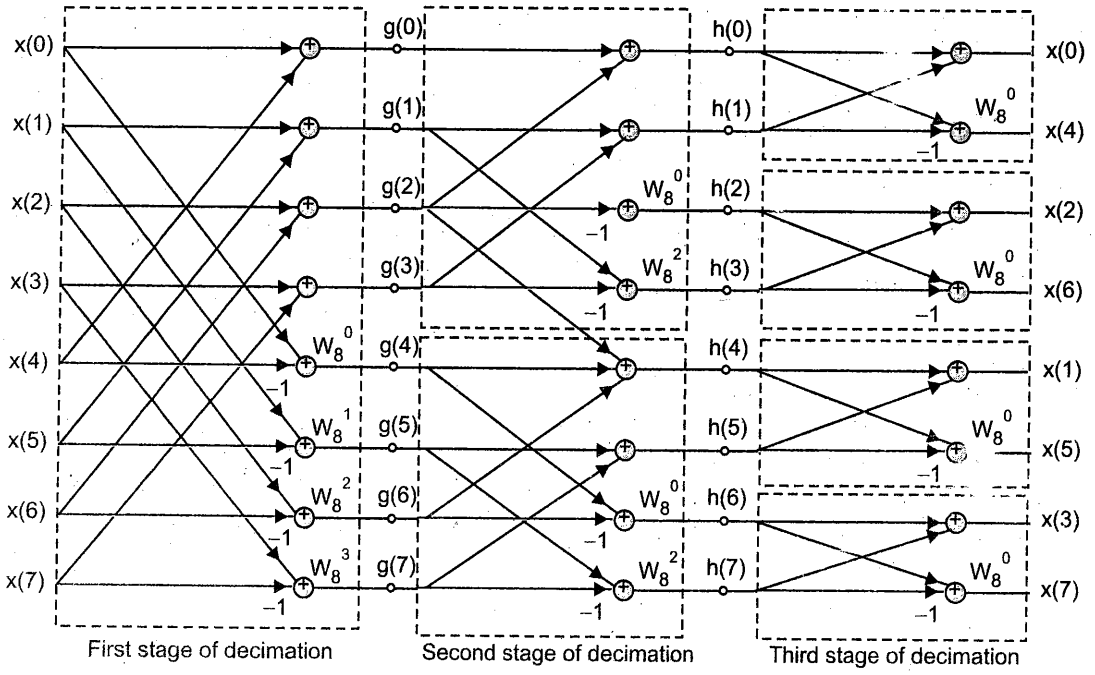


Fig. G-18

Output of stage - 1 :

$$g(0) = x(0)+x(4) = \frac{1}{2}+0 = 0.5$$

$$g(1) = x(1)+x(5) = \frac{1}{2}+0 = 0.5$$

$$g(2) = x(2)+x(6) = \frac{1}{2}+0 = 0.5$$

$$g(3) = x(3)+x(7) = \frac{1}{2}+0 = 0.5$$

$$g(4) = [x(0)-x(4)]W_8^0 = \left[\frac{1}{2}-0\right]1 = 0.5$$

$$g(5) = [x(1)-x(5)]W_8^1 = \left[\frac{1}{2}-0\right](0.707-j\,0.707)$$

$$= 0.3535-j\,0.3535$$

$$g(6) = [x(2)-x(6)]W_8^2 = \left[\frac{1}{2}-0\right](-j) = -j\,0.5$$

$$g(7) = [x(3)-x(7)]W_8^3 = \left[\frac{1}{2}-0\right](-0.707-j\,0.707)$$

$$= -0.3535-j\,0.3535$$

Output of stage - 2 :

$$h(0) = g(0)+g(2) = 0.5 + 0.5 = 1$$

$$h(1) = [g(1)+g(3)] = (0.5+0.5) = 1$$

$$h(2) = [g(0)-g(2)]\,W_8^0 = (0.5-0.5)(+1) = 0$$

$$h(3) = [g(1)-g(3)]\,W_8^2 = (0.5-0.5)(-j) = 0$$

$$h(4) = g(4)+g(6) = 0.5 - j\,0.5$$

$$h(5) = g(5)+g(7) = 0.3535 - j\,0.3535 - 0.3535 - j\,0.3535$$

$$\therefore \quad h(5) = -j\,0.707$$

$$h(6) = [g(4)-g(6)]\,W_8^0 = [0.5 + j\,0.5]\,1 = 0.5 + j\,0.5$$

$$h(7) = [g(5)-g(7)]\,W_8^2 = [0.3535 - j\,0.3535 + 0.3535 + j\,0.3535](-j)$$

$$\therefore \quad h(7) = -j\,0.707$$

Final output :

$$X(0) = h(0)+h(1) = 1 + 1 = 2$$

$$X(1) = h(4)+h(5) = 0.5 - j\,0.5 - j\,0.707 = 0.5 - j\,1.207$$

$$X(2) = h(2)+h(3) = 0 + 0 = 0$$

$$X(3) = [h(6)+h(7)]\,W_8^0 = [0.5 + j\,0.5 - j\,0.707]\cdot 1 = 0.5 - j\,0.207$$

$$X(4) = [h(0)-h(1)]\,W_8^0 = [1-1]\cdot 1 = 0$$

$$X(5) = [h(4)-h(5)]\,W_8^0 = [(0.5 - j\,0.5) + j\,0.707]\cdot 1$$

$$= 0.5 + j\,0.207$$

$$X(6) = [h(2)-h(3)]\,W_8^0 = 0$$

$$X(7) = [h(6)-h(7)]\,W_8^0 = [0.5 + j\,0.5 + j\,0.707]$$

$$= 0.5 + j\,1.21$$

$$\therefore \quad X(k) = \{X(0), X(1), X(2), X(3), X(4), X(5), X(6), X(7)\}$$

$$\therefore \quad X(k) = \{2,\ 0.5 - j\,1.207,\ 0,\ 0.5 - j\,0.207,\ 0,\ 0.5 + j\,0.207,\ 0,\ 0.5 + j\,1.21\}$$

Prob. 2 : Using DIFFFT Find

(i) DFT $X_1(k)$ of following sequence $x_1(n) = \{1, 2, -1, 2, 4, 2, -1, 2\}$

(ii) If $x_2(n) = x_1(-n)$ without performing FFT find $X_2(k)$.

Soln. : The total flow graph is shown in Fig. G-19(a).

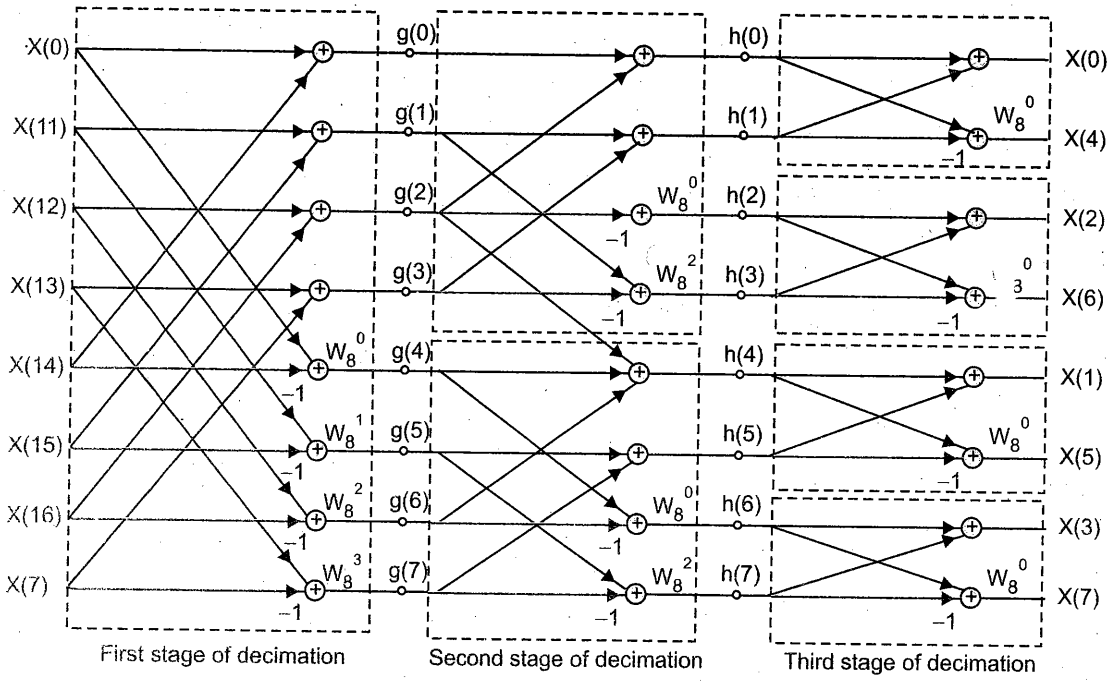


Fig. G-19(a)

The values of twiddle factor are as follows :

$$W_8^0 = 1$$

$$W_8^1 = 0.707 - j\,0.707$$

$$W_8^2 = -j$$

$$W_8^3 = -0.707 - j\,0.707$$

Here, $x(0) = 1$, $x(1) = 2$, $x(2) = -1$, $x(3) = 2$, $x(4) = 4$, $x(5) = 2$, $x(6) = -1$, $x(7) = 2$

Output of stage - 1 :

$$g(0) = x(0) + x(4) = 1 + 4 = 5$$

$$g(1) = x(1) + x(5) = 2 + 2 = 4$$

$$g(2) = x(2) + x(6) = -1 - 1 = -2$$

$$g(3) = x(3)+x(7) = 2+2 = 4$$
$$g(4) = [x(0)-x(4)]W_8^1 = [1-4]\times 1 = -3$$
$$g(5) = [x(1)-x(5)]W_8^1 = (2-2)W_8^1 = 0$$
$$g(6) = [x(2)-x(6)]W_8^2 = (-1+1)W_8^2 = 0$$
$$g(7) = [x(3)-x(7)]W_8^3 = (2-2)W_8^3 = 0$$

Output of stage - 2 :

$$h(0) = g(0)+g(2) = 5-2 = 3$$
$$h(1) = g(1)+g(3) = 4+4 = 8$$
$$h(2) = [g(0)-g(2)]W_8^0 = 5+2 = 7$$
$$h(3) = [g(1)-g(3)]W_8^2 = (4-4)(-j) = 0$$
$$h(4) = g(4)+g(6) = -3$$
$$h(5) = g(5)+g(7) = 0+0 = 0$$
$$h(6) = [g(4)-g(6)]W_8^0 = -3$$
$$h(7) = [g(5)-g(7)]W_8^2 = 0$$

Final output :

$$X(0) = h(0)+h(1) = 3+8 = 11$$
$$X(1) = h(4)+h(5) = -3+0 = -3$$
$$X(2) = h(2)+h(3) = 7+0 = 7$$
$$X(3) = [h(6)+h(7)]W_8^0 = -3$$
$$X(4) = [h(0)-h(1)]W_8^0 = 3-8 = -5$$
$$X(5) = [h(4)-h(5)]W_8^0 = -3$$
$$X(6) = [h(2)-h(3)]W_8^0 = 7$$
$$X(7) = [h(6)-h(7)]W_8^0 = -3$$
$$\therefore\ X(k) = \{X(0), X(1), X(2), X(3), X(4), X(5), X(6), X(7)\}$$
$$\therefore\ X(k) = \{11, -3, 7, -3, -5, -3, 7, -3\}$$

(ii) Given $x_2(n) = x_1(-n)$

According to time reversal property,

$$x((-n))_N \overset{DFT}{\underset{N}{\longleftrightarrow}} X((-k))_N$$

Here $x((-n))_N$ represents circular folding of $x(n)$ and $X((-k))_N$ represents circular folding of $X(k)$. Thus $X_2(k)$ is obtained by circularly folding $X_1(k)$ as shown in Fig. G-19(b).

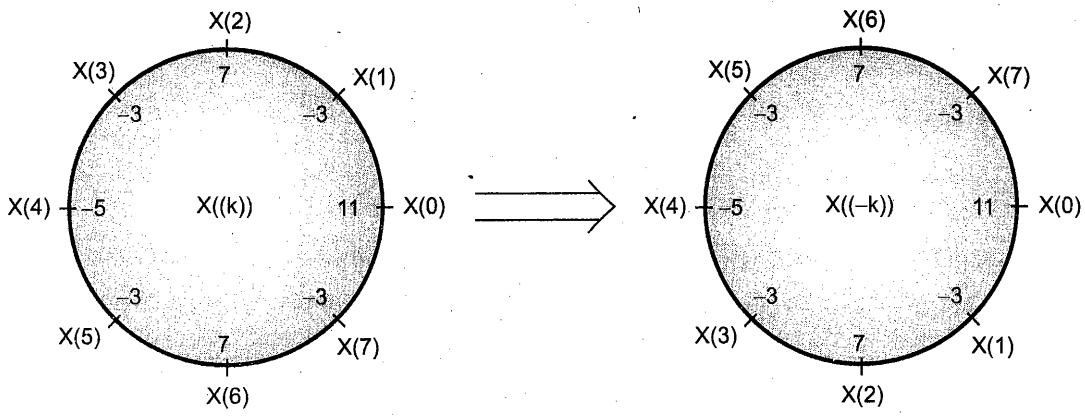


Fig. G-19(b)

Thus in this case $X_2(k)$ is same as $X_1(k)$.

1.5 Computation of Inverse DFT (IDFT) using FFT Algorithms :

We have studied Radix-2 DIT and DIF FFT algorithms to compute the DFT, $X(k)$. The same algorithm can be used to obtain input sequence $x(n)$ from its DFT. That means to compute IDFT. Recall the definition of IDFT,

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}, \quad n = 0, 1, \ldots N-1 \qquad ..(1)$$

Just for comparison we will write the definition of DFT.

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{+kn}, \quad k = 0, 1, \ldots N-1 \qquad ...(2)$$

Thus IDFT differs from DFT by,

(a) Multiplication by $\frac{1}{N}$ factor.

(b) Negative sign of imaginary part osf (W_N)

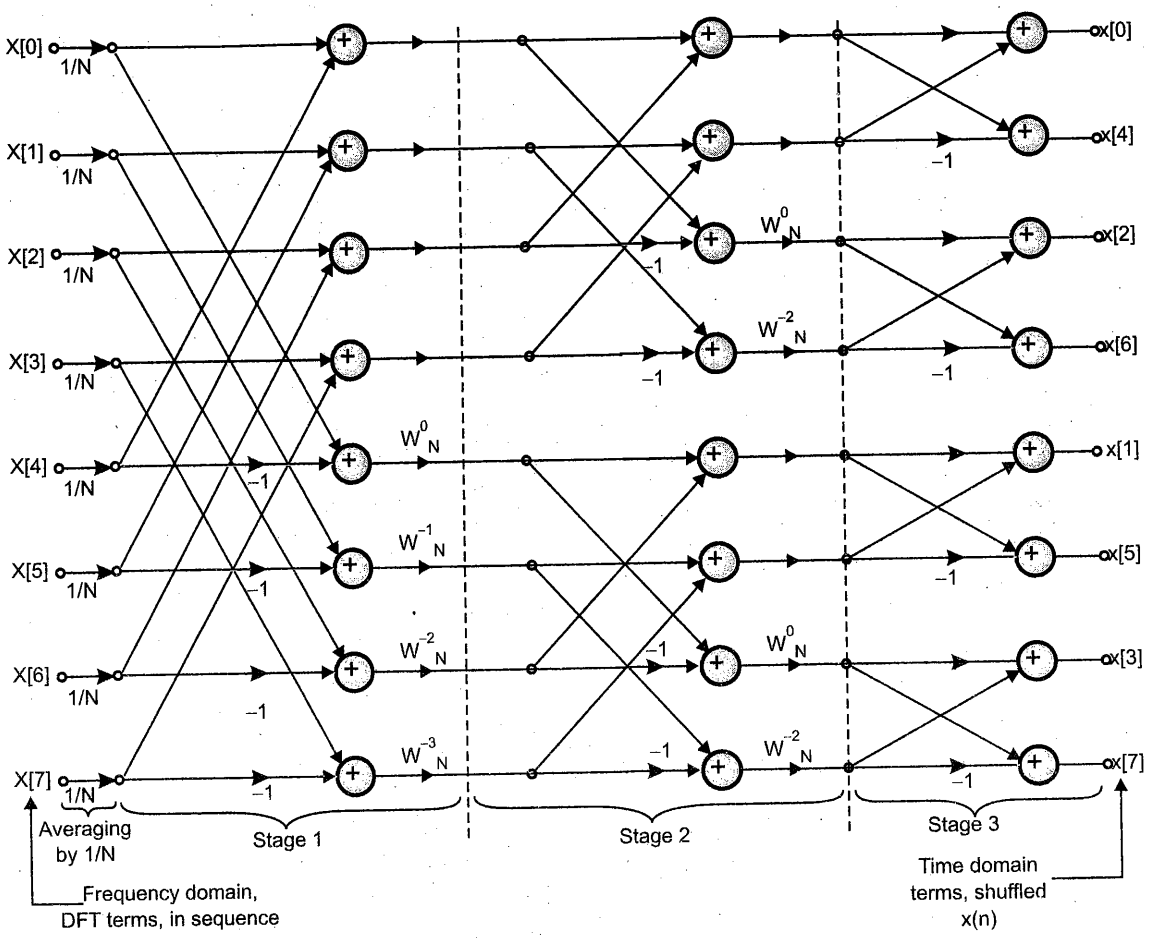


Fig. G-20 : Computation of IDFT using FFT

Thus we can use the same algorithm to compute IDFT; but we have to change the sign of twiddle factor and for DIF FFT algorithm we have to multiply input sequence $X(k)$ by $\frac{1}{N}$. The total flow graph is shown in Fig. G-20.

Prob. 1 : Using FFT and IFFT, find the output of system if input $x(n)$ and impulse response $h(n)$ are given by,

$x(n) = \{2, 2, 4\}$

$h(n) = \{1, 1\}$

Soln. : If the given sequences are very long then we have to divide such sequences into smaller segments. In this case it is not necessary to divide the sequences.

Given $x(n) = \{2, 2, 4\}$ and $h(n) = \{1, 1\}$

Here L = Number of samples in $x(n) = 3$

M = Number of samples in $h(n) = 2$

$$\therefore \quad L+M-1 = 3+2-1 = 4$$

That means we have to make length of x (n) and h (n) equal to 4.

$$\therefore \quad x(n) = \{2, 2, 4, 0\} \text{ and } h(n) = \{1, 1, 0, 0\}$$

First we will obtain DFT of x (n) using DITFFT algorithms. We can also use DIFFFT algorithm.

The calculations are shown in Fig. G-21(a)

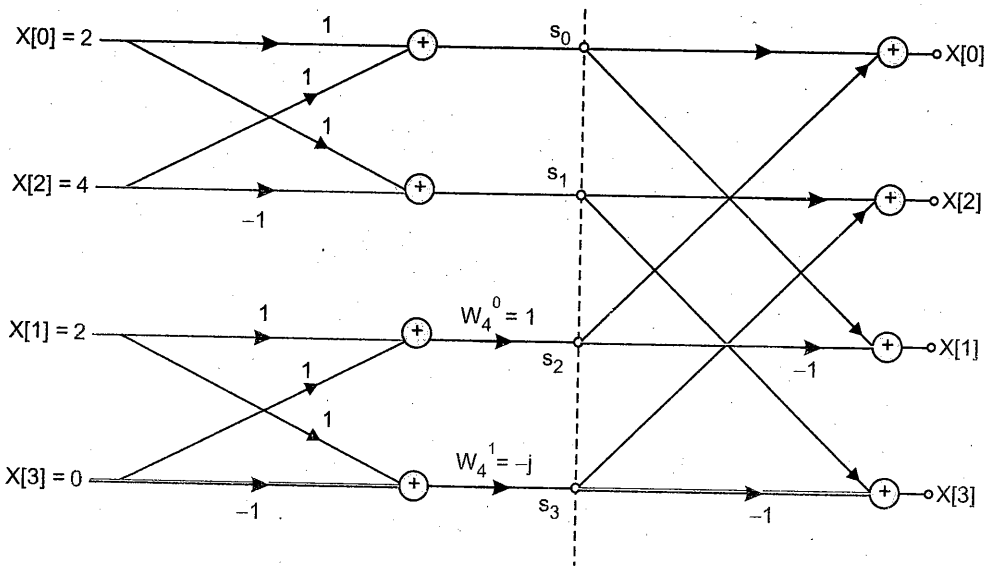


Fig. G-21(a)

Here $s_0 = x(0)+x(2) = 2 + 4 = 6$

$s_1 = x(0)-x(2) = 2-4 = -2$

$s_2 = [x(1)+x(3)]\,W_4^0 = 2 + 0 = 2$

$s_3 = [x(1)-x(3)]\,W_4^1 = (2-0)(-j) = -j2$

The final output is,

$X(0) = s_0+s_2 = 6 + 2 = 8$

$X(1) = s_1+s_3 = -2-j2$

$X(2) = s_0-s_2 = 6-2 = 4$

$X(3) = s_1-s_3 = -2+j2$

$\therefore \quad X(k) = \{8, -2-j2, 4, -2+j2\}$

Now we will obtain DFT of h (n) using DITFFT as shown in Fig. G-21(b).

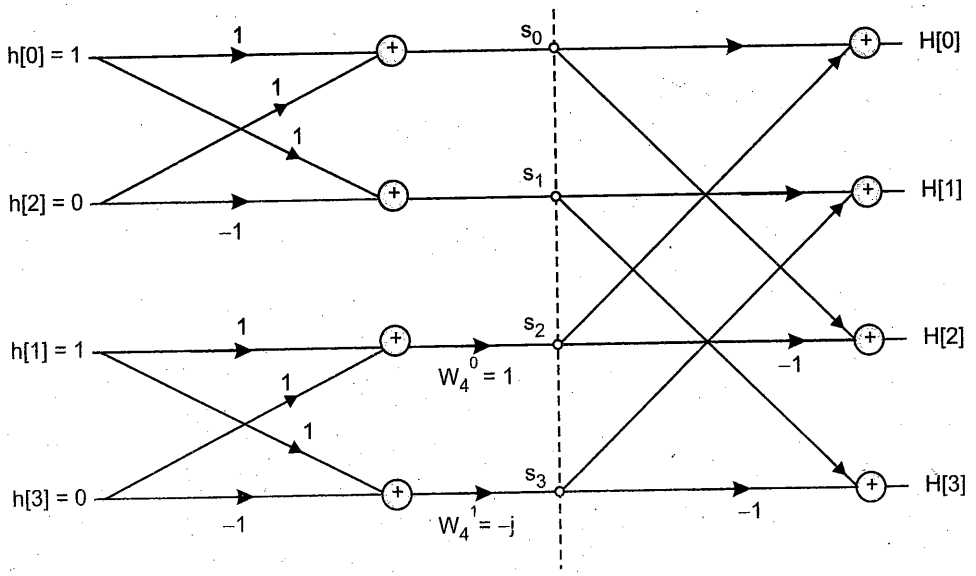


Fig. G-21(b)

$$s_0 = h(0)+h(2) = 1+0 = 1$$

$$s_1 = h(0)-h(2) = 1-0 = 1$$

$$s_2 = [h(1)+h(3)]\,W_4^0 = 1+0 = 1$$

$$s_3 = [h(1)-h(3)]\,W_4^1 = (1-0)-j = -j$$

The final output is,

$$H(0) = s_0+s_2 = 1+1 = 2$$

$$H(1) = s_1+s_3 = 1-j$$

$$H(2) = s_0-s_2 = 1-1 = 0$$

$$H(3) = s_1-s_3 = 1+j$$

$$\therefore \quad H(k) = \{2,\ 1-j,\ 0,\ 1+j\}$$

Now we will multiply H (k) and X (k).

$$\text{Let} \quad Y(k) = X(k)\cdot H(k)$$

$$\therefore \quad Y(k) = \{8,\ -2-j2,\ 4,\ -2+j2\}\cdot\{2,\ 1-j,\ 0,\ 1+j\}$$

$$\therefore \quad Y(k) = \{16,\ -4,\ 0,\ -4\}$$

Now we will perform IFFT to obtain sequence y (n). For this we have to multiply each input by $\frac{1}{N}$ that means $\frac{1}{4}$ and we have to change the sign of imaginary part of twiddle factor. This computation is shown in Fig. G-21(c)

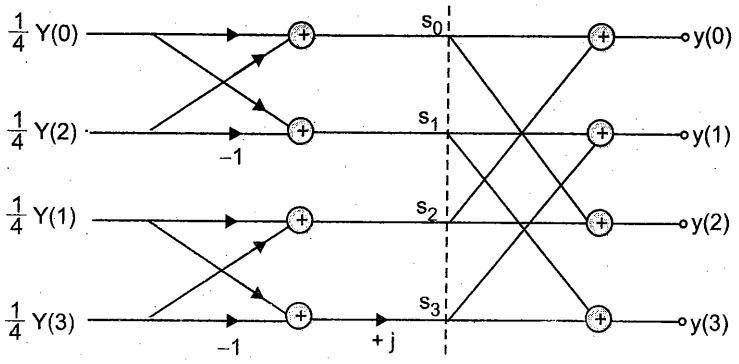


Fig. G-21(c)

$$s_0 = \frac{1}{4}Y(0)+\frac{1}{4}Y(2) = \frac{1}{4}(16)+\frac{1}{4}(0) = 4$$

$$s_1 = \frac{1}{4}Y(0)-\frac{1}{4}Y(2) = \frac{1}{4}(16)-\frac{1}{4}(0) = 4$$

$$s_2 = \left[\frac{1}{4}Y(1)+\frac{1}{4}Y(3)\right]\cdot 1 = \frac{1}{4}(-4)+\frac{1}{4}(-4) = -2$$

$$s_3 = \left[\frac{1}{4}Y(1)-\frac{1}{4}Y(3)\right](-j) = \left[\frac{1}{4}(-4)-\frac{1}{4}(-4)\right](-j) = 0$$

The final output is,

$$y(0) = s_0+s_2 = 4-2 = 2$$

$$y(1) = s_1+s_3 = 4+0 = 4$$

$$y(2) = s_0-s_2 = 4+2 = 6$$

$$y(3) = s_1-s_3 = 4-0 = 4$$

$$\therefore\ y(n) = \{2, 4, 6, 4\}$$

Prob. 2 : Let x (n) be 8 point sequence. Its corresponding DFT X (k) is

$X(k) = \{(0.5), (2+j), (3+j2), (j), (3), (-j), (3-j2), (2-j)\}$

Find x (n) by performing IFFT.

Soln. : We will use inverse DIF FFT algorithm. The flow graph is shown in Fig. G-22.

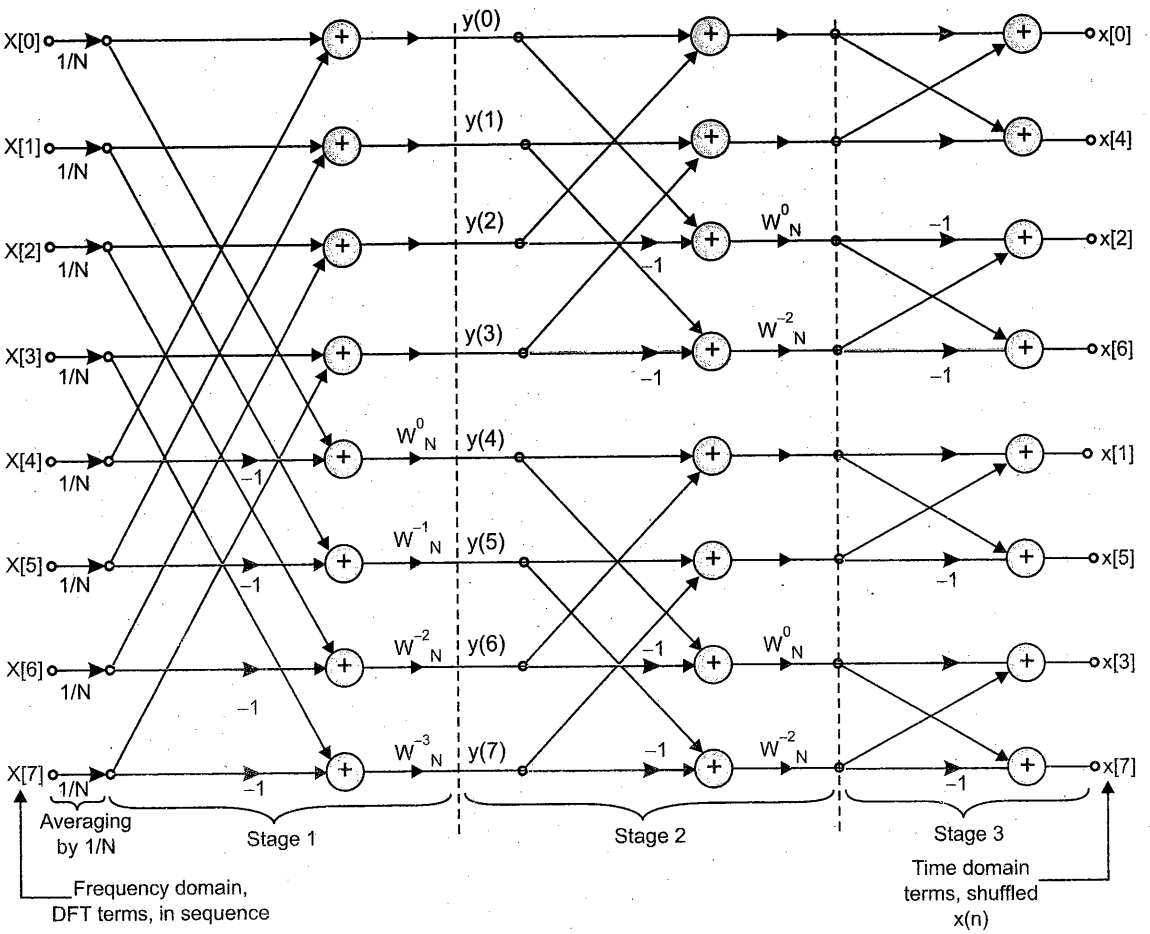


Fig. G-22

Let g (n) represent output of first stage and h (n) represent output of second stage. By changing sign of imaginary part of twiddle factor the values of twiddle factor are as follows :

$$W_8^0 = 1$$

$$W_8^1 = 0.707 + j\,0.707$$

$$W_8^2 = j$$

$$W_8^3 = -0.707 + j\,0.707$$

Output of first stage :

$$g(0) = \frac{1}{8}X(0)+\frac{1}{8}X(4) = \frac{1}{8}(0.5)+\frac{1}{8}(3) = 0.44$$

$$g(1) = \frac{1}{8}X(1)+\frac{1}{8}X(5) = \frac{1}{8}(2+j)+\frac{1}{8}(-j) = 0.25$$

$$g(2) = \frac{1}{8}X(2)+\frac{1}{8}X(6) = \frac{1}{8}(3+j2)+\frac{1}{8}(3-j2) = 0.75$$

$$g(3) = \frac{1}{8}X(3)+\frac{1}{8}X(7) = \frac{1}{8}(j)+\frac{1}{8}(2-j) = 0.25$$

$$g(4) = \left[\frac{1}{8}X(0)-\frac{1}{8}X(4)\right](+W_8^0) = \left[\frac{1}{8}(0.5)-\frac{1}{8}(3)\right]\cdot 1 = -0.31$$

$$g(5) = \left[\frac{1}{8}X(1)-\frac{1}{8}X(5)\right](W_8^1) = \left[\frac{1}{8}(2+j)-\frac{1}{8}(-j)\right](+0.707+j\,0.707)$$

$$= (0.25+j\,0.25)(+0.707+j\,0.707) = j\,0.35$$

$$g(6) = \left[\frac{1}{8}X(2)-\frac{1}{8}X(6)\right]W_8^2 = \left[\frac{1}{8}\cdot(3+j2)-\frac{1}{8}(3-j2)\right]j$$

$$= -0.5$$

$$g(7) = \left[\frac{1}{8}X(3)-\frac{1}{8}X(7)\right]W_8^3 = \left[\frac{1}{8}(j)-\frac{1}{8}(2-j)\right](-0.707+j\,0.707)$$

$$= \left(\frac{1}{4}j-\frac{1}{4}\right)(-0.707+j\,0.707) = -j\,0.35$$

Output of second stage :

$$h(0) = g(0)+g(2) = 0.44 + 0.75 = 1.19$$

$$h(1) = g(1)+g(3) = 0.25 + 0.25 = 0.5$$

$$h(2) = [g(0)-g(2)]W_8^0 = 0.44 - 0.75 = -0.31$$

$$h(3) = [g(1)-g(3)]W_8^2 = 0$$

$$h(4) = g(4)+g(6) = -0.31 - 0.5 = -0.81$$

$$h(5) = g(5)+g(7) = j\,0.35 - j\,0.35 = 0$$

$$h(6) = [g(4)-g(6)]W_8^0 = -0.31 + 0.5 = 0.19$$

$$h(7) = [g(5)-g(7)]W_8^2 = [j\,0.35 + j\,0.35]\,j = -0.7$$

Final output :

$$x(0) = h(0)+h(1) = 1.19 + 0.5 = 1.69$$

$$x(1) = h(4)+h(5) = -0.81$$

$$x(2) = h(2)+h(3) = -0.31$$

$$x(3) = [h(6)+h(7)]W_8^0 = (0.19-0.7)\cdot 1 = -0.51$$

$$x(4) = [h(0)-h(1)]\cdot W_8^0 = (1.19-0.5)\cdot 1 = 0.69$$

$$x(5) = [h(4)-h(5)]W_8^0 = -0.81$$

$$x(6) = [h(2)-h(3)]W_8^0 = -0.31$$

$$x(7) = [h(6)-h(7)]W_8^0 = (0.19+0.7)\cdot 1 = 0.89$$

Now sequence x (n) is,

$$x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$$

$$\therefore x(n) = \{1.69, -0.81, -0.31, -0.51, 0.69, -0.81, -0.31, 0.89\}$$

1.6 Computation of Linear Filtering and Correlation using FFT :

We know that linear filtering is same as linear convolution. We have studied FIR linear filtering using DFT. For that, overlap add and overlap save methods are used. Now, we will discuss how to obtain linear convolution using FFT algorithms. As an example, we will consider overlap save method. The procedure is as follows :

(1) Input data sequence x (n) is segmented into the block of L points.

(2) Impulse response of FIR filter is h (n). It has 'M' samples.

(3) Each block contains 'M – 1' samples from the previous sequence and 'L' new samples from the input sequence. Thus the total length of each input block is L + M – 1 = N.

(4) Using DIF FFT algorithm; the DFT of each input data block is performed. We know that DIF FFT algorithm does not require bit reversal of input sequence. But the output sequence is in the bit reversed order.

(5) DFT of impulse response, that is H (k) is obtained by using DIF FFT algorithm. So H (k) is also in bit reversed order.

(6) Now the output of m^{th} block is obtained by multiplying H (k) with $X_m(k)$. Here $X_m(k)$ is DFT of m^{th} input block.

$$\therefore \quad Y_m(k) = H(k)\cdot X_m(k)$$

Here $Y_m(k)$ sequence is also in bit reversed order.

(7) Now inverse DFT is obtained using DIF FFT algorithm. This requires bit reversed input $Y_m(k)$; which produces the output y (n) in the natural order.

Note : We know that correlation is nothing but the convolution with one of the sequences, folded. Thus by folding the sequence h (n) we can compute the linear filtering (convolution) using FFT. That means we can compute the correlation.

1.7 Computation of DFT by Linear Filtering Approach :

We have studied that Radix-2 FFT algorithms are used to compute 'N'-point DFT. In this case, $\frac{N}{2}\log_2 N$ complex multiplications and $N\log_2 N$ complex additions are required to compute N-point DFT.

But there are certain applications in which selected number of DFT values are required; but not the entire DFT. In such cases the direct computation of DFT is more efficient than FFT algorithms. The direct computation of DFT for selected values can be realized using linear filtering approach.

This type of computation can be performed using the algorithm, called as Goertzel algorithm.

1.7.1 Goertzel Algorithm :

In this algorithm the periodicity property of twiddle factor (W_N^k) is used.

Now recall the definition of DFT,

$$X(k) = \sum_{m=0}^{N-1} x(m) W_N^{km} \qquad ...(1)$$

Note that here we have used notation x (m) for input sequence.

Now the twiddle factor is given by,

$$W_N = e^{-\frac{j2\pi}{N}}$$

$$\therefore \quad W_N^{-kN} = \left(e^{-\frac{j2\pi}{N}}\right)^{-kN} = e^{+j2\pi k} = \cos 2\pi k + j \sin 2\pi k$$

$$= 1 + j\,0 = 1$$

Since $W_N^{-kN} = 1$ we can multiply Equation (1) by W_N^{-kN}

$$\therefore \quad X(k) = \sum_{m=0}^{N-1} x(m) W_N^{km} \cdot W_N^{-kN}$$

$$\therefore \quad X(k) = \sum_{m=0}^{N-1} x(m) W_N^{-k(N-m)} \qquad ...(2)$$

Now according to the equation of linear convolution,

$$y_k(n) = \sum_{m=-\infty}^{\infty} x(m) h_k(n-m) \qquad ...(3)$$

We know that $h_k(n)$ is an impulse response. Now let,

$$h_k(n) = W_N^{-kn} u(n) \quad ...(4)$$

Replacing n by 'n – m' we get,

$$h_k(n-m) = W_N^{-k(n-m)} u(n-m) \quad ...(5)$$

Put this value in Equation (3). Here $u(n-m)$ is unit step. So because of multiplication by $u(n-m)$ the limits of summation will become m = 0 to m = ∞.

$$\therefore \quad y_k(n) = \sum_{m=0}^{N-1} x(m) W_N^{-k(n-m)} \quad ...(6)$$

Now we will calculate $y_k(n)$ at n = N.

$$\therefore \quad y_k(n)\Big|_{n=N} = \sum_{m=0}^{N-1} x(m) W_N^{-k(N-m)} \quad ...(7)$$

Equations (2) and (7) are identical. Thus we can conclude that DFT, X(k) can be obtained as output of LTI system at n = N. ($y_k(n)$). But in this case, we will get X(k) only at one value of K. So we need to use parallel systems to obtain X(k) at different values of K.

Note : Goertzel Algorithm is efficient when X(k) is to be computed at points less than $\log_2^N$.

1.7.2 Computation of DFT of Two Real Sequences using Only One FFT Flow Graph :

Suppose $x_1(n)$ and $x_2(n)$ are two real valued sequences of length N. We will define a complex valued sequence x(n) as,

$$x(n) = x_1(n) + j\,x_2(n), \qquad 0 \le n \le N-1 \quad ...(1)$$

Since DFT operation is linear, we can write DFT of Equation (1) as,

$$X(k) = X_1(k) + j\,X_2(k) \quad ...(2)$$

In Equation (1), $x_1(n)$ represents real part of x(n) and $x_2(n)$ represents imaginary part of x(n). Thus we can write,

$$x_1(n) = \frac{x(n) + x^*(n)}{2} \quad ...(3)$$

$$\text{and} \quad x_2(n) = \frac{x(n) - x^*(n)}{2j} \quad ...(4)$$

The DFTs of Equations (3) and (4) can be written as,

$$X_1(k) = \frac{1}{2}\left\{DFT[x(n)] + DFT[x^*(n)]\right\} \quad ...(5)$$

$$\text{and} \quad X_2(k) = \frac{1}{2j}\left\{DFT[x(n)] - DFT[x^*(n)]\right\} \quad ...(6)$$

According to complex conjugate property we have,

$$x^*(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X^*((-k))_N = X^*(N-k)$$

$$\therefore \quad X_1(k) = \frac{1}{2}[X(k) + X^*(N-k)] \quad ...(7)$$

$$\text{and} \quad X_2(k) = \frac{1}{2j}[X(k) - X^*(N-k)] \quad ...(8)$$

Thus by performing single DFT on complex valued sequence $x(n)$ we can obtain DFT of two real sequences $x_1(n)$ and $x_2(n)$.

Prob. 1 : Find DFT of two real sequences using only one FFT flow graph.

$x_1(n) = \{1, 1, 1, 1\}$

$x_2(n) = \{2, 1, 2, 1\}$

Perform FFT only once.

Soln. :

Consider a sequence $x(n)$ such that,

$$x(n) = x_1(n) + j\,x_2(n)$$

$$\therefore \quad x(n) = \{1 + j2, 1 + j1, 1 + j2, 1 + j1\}$$

Here $x(0) = 1 + j2$, $x(1) = 1 + j1$, $x(2) = 1 + j2$, $x(3) = 1 + j1$.

We will obtain DFT using DITFFT, the flow graph is as shown in Fig. G-23(a).

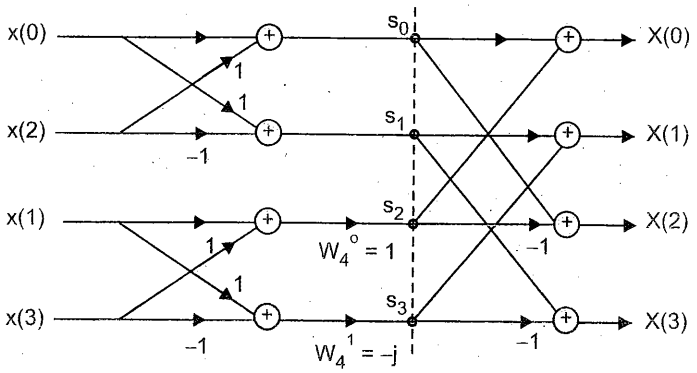


Fig. G-23(a)

$$s_0 = x(0)+x(2) = 1 + j2 + 1 + j2 = 2 + j4$$

$$s_1 = x(0)-x(2) = 1+j2-1-j2 = 0$$

$$s_2 = [x(1)+x(3)]W_4^0 = [1 + j1 + 1 + j1] \cdot 1 = 2 + j2$$

$$s_3 = [x(1)-x(3)]W_4^1 = [1+j1-1-j1](-j) = 0$$

The final output is,

$$X(0) = s_0+s_2 = 2 + j4 + 2 + j2 = 4 + j6$$

$$X(1) = s_1+s_3 = 0$$

$$X(2) = s_0-s_2 = 2+j4-2-j2 = j2$$

$$X(3) = s_1-s_3 = 0$$

$$\therefore \quad X(k) = \{4 + j6, 0, j2, 0\} \qquad ...(1)$$

Calculation of $X_1(k)$:

Now we have, $$X_1(k) = \frac{1}{2}[X(k)+X^*(N-k)] \qquad ...(2)$$

Here $N = 4$ and $X^*(k) = \{4-j6, 0, -j2, 0\}$ ...(3)

In Equation (2), $X^*(N-k) = X^*((-k))$ which represents circular folding of $X^*(k)$. It is shown in Fig. G-23(b).

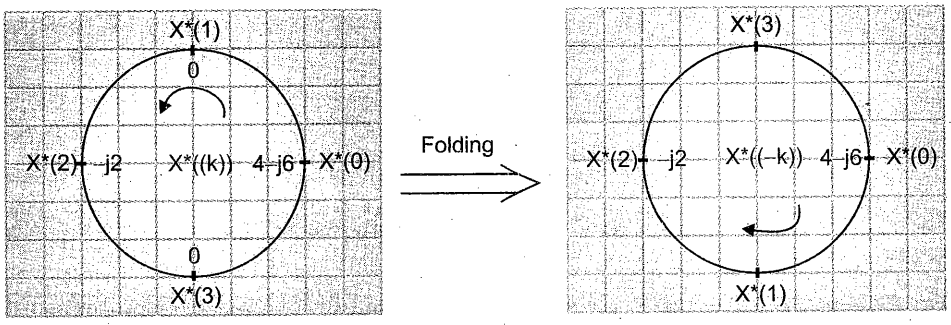


Fig. G-23(b)

$$\therefore \quad X^*(-k) = \{4-j6, 0, -j2, 0\} \qquad ...(4)$$

Using Equation (2) we can obtain DFT $X_1(k)$ as follows :

We have, $$X_1(k) = \frac{1}{2}[X(k)+X^*((-k))]$$

For $k = 0 \Rightarrow X_1(0) = \frac{1}{2}[X(0) + X^*(0)] = \frac{1}{2}[4 + j6 + 4 - j6] = 4$

For $k = 1 \Rightarrow X_1(1) = \frac{1}{2}[X(1) + X^*(1)] = \frac{1}{2}[0 + 0] = 0$

For $k = 2 \Rightarrow X_1(2) = \frac{1}{2}[X(2) + X^*((-2))] = \frac{1}{2}[+j2 - j2] = 0$

For $k = 3 \Rightarrow X_1(3) = \frac{1}{2}[X(3) + X^*((-3))] = \frac{1}{2}[0 + 0] = 0$

$$\therefore X_1(k) = \{4, 0, 0, 0\}$$

Calculation of $X_2(k)$:

We have,

$$X_2(k) = \frac{1}{2j}[X(k) - X^*(N-k)] = \frac{1}{2j}[X(k) - X^*((-k))]$$

For $k = 0 \Rightarrow X_2(0) = \frac{1}{2j}[X(0) - X^*(0)] = \frac{1}{2j}[4 + j6 - 4 + j6] = 6$

For $k = 1 \Rightarrow X_2(1) = \frac{1}{2j}[X(1) - X^*((-1))] = \frac{1}{2j}[0 + 0] = 0$

For $k = 2 \Rightarrow X_2(2) = \frac{1}{2j}[X(2) - X^*((-2))] = \frac{1}{2j}[j2 + j2] = 2$

For $k = 3 \Rightarrow X_2(3) = \frac{1}{2j}[X(3) - X^*((-3))] = \frac{1}{2j}[0 - 0] = 0$

$$\therefore X_2(k) = \{6, 0, 2, 0\}$$

1.8 Practical Considerations in FFT Implementation :

We have studied DITFFT and DIFFFT algorithms. While implementing these algorithms practically; several things should be taken into consideration.

1. These are number of stages; depending on the value of N. So accessing and storing of data at an intermediate stage should be properly done.
2. Once an appropriate data is obtained then actual implementation of butterfly computation should be considered.

Indexing :

We know that DITFFT requires input data in a bit reversed order; while DIFFFT produces output data in a bit reversed order. Actual input sequence is in natural order. So bit reversal algorithm is required to generate the data in required form.

Take an example of 8 point DIT FFT (Fig. G-9). Here input sequence x (n) is not in natural order. Input data is shuffled as x (0), x (4) , x (2), x (6), x (1), x (5), x (3), x (7). Similarly in case of 8-point DIF FFT (Fig. G-17); the output sequence is not in natural order. It is shuffled as X (0), X (4), X (2), X (6), X (1), X (5), X (3), X (7).

Now all inputs and outputs are stored in the memory of computer. The given input sequence x (n) will be in natural order. But for DIT FFT we want this sequence to be shuffled. Similarly for DIF FFT, output sequence X (k) should be shuffled. For this, bit reversal algorithm is used. The concept of bit reversal is shown in Table. G-2.

Table G-2

Memory address in decimal	B_2	B_1	B_0	→	B_0	B_2	B_1	→	B_0	B_1	B_2	New memory address in decimal
0 →	0	0	0		0	0	0		0	0	0	→ 0
1 →	0	0	1		1	0	0		1	0	0	→ 4
2 →	0	1	0		0	0	1		0	1	0	→ 2
3 →	0	1	1		1	0	1		1	1	0	→ 6
4 →	1	0	0		0	1	0		0	0	1	→ 1
5 →	1	0	1		1	1	0		1	0	1	→ 5
6 →	1	1	0		0	1	1		0	1	1	→ 3
7 →	1	1	1		1	1	1		1	1	1	→ 7
	Natural order						bit reversal					

As shown in flow chart :

Step I : Accept value of N from user.

Step II : Calculate number of bits (n) from N.

Step III : Accept 'n' numbers from user.

Step IV : Swap the number as shown in table using swapping of array technique.

Step V : Display the array which contains numbers in reverse order.

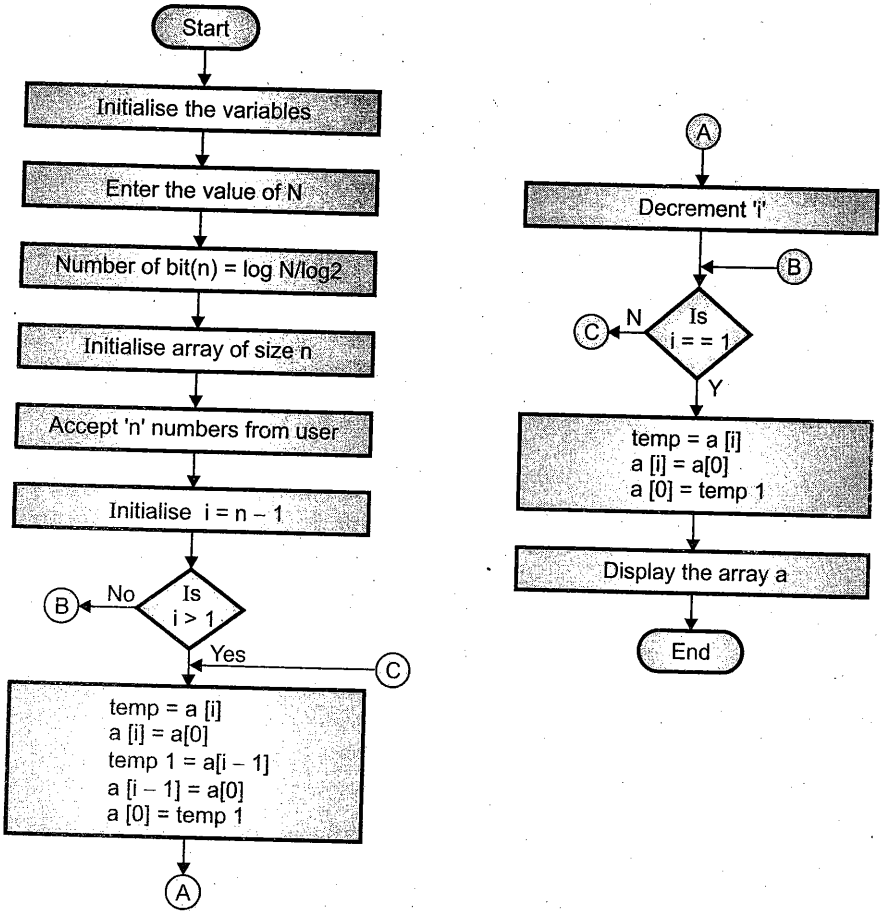


Fig. G-24 : Flow chart

While performing linear filtering using DFT; we have studied, how to avoid use of bit reversal algorithm.

To make coefficient use of memory; in-place computation should be done. We have studied how to perform in-place computation in FFT algorithms. But there are two disadvantages if we use in place computation.

1. The random access memory is required rather than the sequential memory.
2. Bit reversed data is required to generate either at input or at output. So indexing structure required to implement these algorithms is complicated.

Coefficients :

1. We know that the coefficients W_N^r are required to generate at every state of FFT algorithm. Here the range of variable r is r = 0, 1, ... $\frac{N}{2} - 1$. These are two methods to generate required values of W_N^r can be obtained.

By using this approach; the speed of operation is improved but it requires extra memory to store the values of W_N^r.

2. The calculation of W_N^r can be done as and when required. This method does not require any extra memory but it is less efficient. But if this method is to be used then it is preferred to use recursion formula. It is,

$$W_N^{ql} = W_N^q \cdot W_N^{q(l-1)}$$

This formula shows, how to obtain l^{th} coefficient from $(l-1)^{st}$ coefficient. We can use this formula because the required coefficients are all powers of complex number of the form W_N^q, where q depends on algorithm and the stage.

1.9 Chirp-Z Algorithm :

The chirp-Z algorithm is used to compute X(Z) on the other countours; in the Z-plane. It may include the unit circle.

If we want to calculate X(Z) at a set of points $[Z_k]$ then the equation of Z-transform can be written as,

$$X[Z_k] = \sum_{n=0}^{N-1} x(n) Z_k^{-n} \qquad ...(1)$$

Here let $k = 0, 1, ..., L-1$

and $n = 0, 1, ... N-1$

Now Z can be represented in the polar form with radius of countour 'r' and at N equally spaced points then

$$Z_k = r e^{\frac{j 2\pi kn}{N}}, \quad k = 0, 1, ..., N-1 \qquad ...(2)$$

Putting this value in Equation (1) we get,

$$X[Z_k] = \sum_{n=0}^{N-1} [x(n) r^{-n}] e^{-j 2\pi kn / N} \qquad ...(3)$$

Now let us say that Z_k fall on an arc which starts at some point $Z_o = r_o e^{j\theta_o}$ and spirals either in; towards the origin or out, away from the origin. Then Z_k can be defined as,

$$Z_k = r_o e^{j\theta_o} (R_o e^{j\phi_o})^k, \quad k = 0, 1, ..., L-1 \qquad ...(4)$$

Here if $R_o < 1$; the points fall on a contour that spirals towards the origin and if $R_o > 1$ then points fall on a countour that spirals away from the origin. If R = 1 then the countour is a circular

arc of radius r_o. And when $r_o = R_o = 1$ then the countour is an arc of unit circle.

Now if $r_o = R_o = 1$, $\theta = 0$, $\phi = \frac{2\pi}{N}$ and L = N then the countour is the entire unit circle and frequencies are those of the DFT.

Putting Equation (4) in Equation (1) we get,

$$X[Z^k] = \sum_{n=0}^{N-1} x(n)\left[r_o e^{j\theta_o}\right]^{-n} \cdot \left[R_o e^{j\phi_o}\right]^{-nk}$$

$$\text{Let } V = R_o e^{j\phi_o}$$

$$\therefore \quad X[Z_k] = \sum_{n=0}^{N-1} x(n)[r_o e^{j\theta_o}]^{-n} \cdot V^{-nk} \qquad ...(5)$$

Now the term nk can be expressed as,

$$nk = \frac{1}{2}[n^2 + k^2 - (k-n)^2] \qquad ...(6)$$

Putting this value in Equation (5) we get,

$$X[Z_k] = V^{-k^2/2} \sum_{n=0}^{N-1} \left[x(n)[r_o e^{j\theta_o}]^{-n} \cdot V^{-n^2/2} \cdot V^{(k-n)^2/2} \right] \qquad ...(7)$$

Let us define, $g(n) = x(n)[r_o e^{j\theta_o}]^{-n} V^{-n^2/2}$

$$\therefore X[Z_k] = V^{-k^2/2} \cdot \sum_{n=0}^{N-1} g(n) V^{(k-n)^2/2} \qquad ...(8)$$

Let $g(n)$ = Input sequence

and impulse response $h(n) = V^{n^2/2}$ ...(9)

$$\therefore \quad h(k-n) = V^{(k-n)^2/2} \qquad ...(10)$$

Thus Equation (8) becomes,

$$X[Z_k] = V^{-k^2/2} \sum_{n=0}^{N-1} g(n)h(k-n) \qquad ...(11)$$

Here the summation term represents the equation of linear convolution. That means it represents the output of digital filter y (k).

$$\therefore \quad X[Z_k] = V^{-k^2/2} \cdot y(k) \qquad \ldots(12)$$

Now we have $V^{n^2/2} = h(n)$ (from Equation (9))

$$\therefore \quad V^{-k^2/2} = [h(k)]^{-1} = \frac{1}{h(k)}$$

Thus Equation (12) becomes,

$$X[Z_k] = \frac{y(k)}{h(k)}$$

Thus the DFT at discrete point Z_k is the ratio of $\frac{y(k)}{h(k)}$. Now the Z-transform evaluated as in Equation (8) is called as, **Chirp-Z transform.**

Solved Problem :

Prob. 1 : Derive the signal flow graph for N = 16 point radix 4-decimation in time FFT algorithm in which the input sequence is in normal order and computations are done in place.

Soln. :

If number of data points N in DFT is a power of 4 (N = 4) then we can use Radix2 algorithm. But more efficient design can be implemented by Radix4 algorithm.

To implement a Radix4 decimation in time. FFT algorithm we will us Divide and Conquer Approach.

Here N-point DFT where N can be factored as a product of two integers.

$$\therefore \quad N = LM$$

Sequence $x(n)$, $0 \le n \le N-1$ can be stored in single dimensional array n or two dimensional array (l, m)

Selective mapping is done by,

$$n = Ml + m \quad \text{or}$$

$$n = l + mL$$

Similarly arrangement is used to store values of DFT i.e. $X(k)$ in one dimensional array k or two dimensional array (p, q) when $0 \le p \le L-1$ and $0 \le q \le M-1$

Selective mapping

$$k = M_{p+q}$$

The DFT matrix can be expressed as a double sum over the elements of the rectangular array multiplied by corresponding phase.

$\therefore$ Mapping is done by the given formula,

$$X(p,q) = \sum_{m=0}^{M-1} \sum_{l=0}^{L-1} x(l,m)\, W_N^{(M_{p+q})(mL+l)} \qquad ...(1)$$

Further simplified equation is,

$$X(p,q) = \sum_{l=0}^{L-1} \left\{ W_N^{lq} \left[\sum_{M=0}^{M-1} x(l,m) \right] \right\} W_L^{Lp} \qquad ...(2)$$

Here in Radix4 point DFT, L = 4 and M = N/4, l, p = 0, 1, 2, 3;

$$m, q = 0, 1, ... N/4 - 1$$

$$n = 4m + l,\ k = (N/4)p + q$$

From these parameters Equation (3) can be written from Equation (2).

$$X(p,q) = \sum_{l=0}^{3} \left[W_N^{lq} F(l,q) \right] W_4^{lp} \qquad p = 0, 1, 2, 3 \qquad ...(3)$$

where,

$$F(l,q) = \sum_{M=0}^{(N/4)-1} x(l,m)\, W_{N/4}^{mq}$$

where l = 0, 1, 2, 3

$$p = 0, 1, 2, ... \frac{N}{4} - 1$$

and $x(l,m) = x(4m + l)$

$$X(p,q) = X\left(\frac{N}{4}p + q\right)$$

when expressed in matrix form

$$\begin{bmatrix} X(0,q) \\ X(1,q) \\ X(2,q) \\ X(3,q) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} W_N^0 F(0,q) \\ W_N^q F(1,q) \\ W_N^{2q} F(2,q) \\ W_N^{3q} F(3,q) \end{bmatrix}$$

Using this algorithm for N = 16 the Radix4 FFT is obtained as follows :

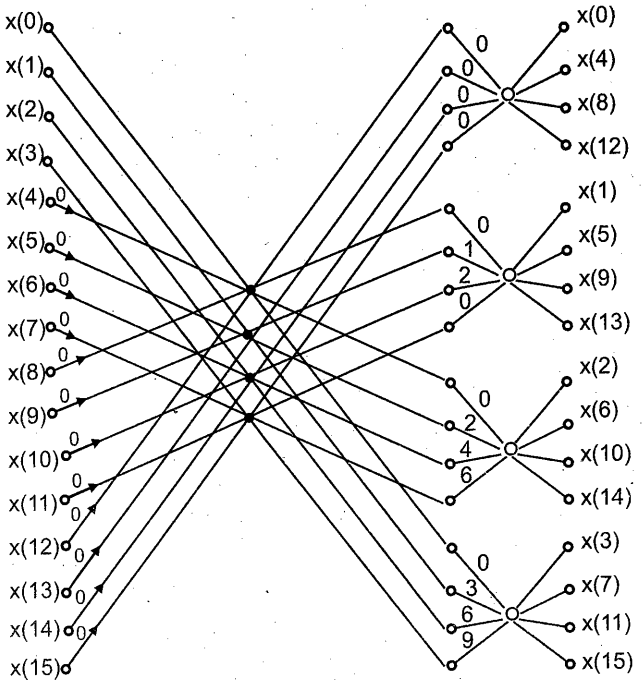


Fig. G-25

Theory Questions

Q. 1 How an N-point sequence x (n) can be obtained from its N-point DFT X (k) using an FFT algorithm.

Q. 2 Derive and explain the Goertzel algorithm used to compute DFT. Develop the signal flow graph for second order recursive computation of X [k].

Q. 3 Derive and explain N-point radix-2 DIF FFT algorithm. For N = 8 draw the signal flow graph.

Q. 4 Explain the procedure to find linear convolution of two causal, finite duration sequences using FFT algorithm.

Q. 5 Obtain second order recursive implementation of Goertzel Algorithm.

Q. 6 How one can obtain the inverse DFT to get an N-point sequence x (n) by means of FFT algorithm ?

Q. 7 Draw a signal flow graph of 8-point DFT for DIF FFT Algorithm Comment on the number of twiddle factor computations required in each stage and the computational complexity.

Q. 8 Determine the total number of twiddle factors required to compute and N-point DFT using Radix-2 DIT FFT algorithm and compare it with that of direct computation of N-point DFT.

Q. 9 Draw the signal flow graph for the computation of a 2-point DIT using Radix-2 DIT FFT algorithm.

Q. 10 Explain how an N-point DFT can be obtained by means of linear convolution and determine the computational complexity.

Q. 11 Derive and explain Radix-2 DIT FFT Algorithm (all stages) for N = 8. Draw signals flow graph and commet on the computational complexity and in-place computation.

Q. 12 Derive and explain Radix-2 DIF (Decimation in Frequency) FFT Algorithm for N = 8. Draw the signal flow graph for all the stages. Compare this algorithm with DIT FFT. Comment on the computational complexity and in-place computation.

Unsolved Numericals

Q. 1 Determine 8 point DFT of $x(n) = 1$ for $-3 \le n \le 3$ using DITFFT algorithm.

Ans. :

$$X(k) = \{7, -0.707 - j\,0.707, -j, 0.707 - j\,0.707, 1, 0.707 + j\,0.707, j, -0.707 + j\,0.707\}$$

Q. 2 Obtain 8 point DFT of a sequence using DIF FFT algorithm.

Ans. :

$$X(k) = \{3, -1.707 - j\,1.707, j, -0.293 + j\,0.293, 1, -0.293 - j\,0.293, -j, -1.707 + j\,1.707\}$$

Q. 3 If $x_1(n) = \{1, 2, 0, 1\}$ and $x_2(n) = \{1, 3, 3, 1\}$ obtain $x_1(n) \textcircled{N} x_2(n)$ using DITFFT algorithm.

Ans. :

$$x_1(n) \textcircled{N} x_2(n) = \{6, 8, 10, 8\}$$

Q. 4 The DFT $X(k)$ of a sequence is

$$X(k) = \{0, 2\sqrt{2}(1-j), 0, 0, 0, 0, 0, 2\sqrt{2}(1+j)\}$$

Determine corresponding time domain sequence and write its signal flow graph.

Ans. :

$$x(n) = \{1, 1, -1, -1, -1, 1, 1, -1\}$$

Q. 5 Using DIF FFT compulse DFT of a sequence $x(n) = \{44, 22, 33, 22\}$. What is the number of computations required in this algorithm.

Ans. :

$$X(k) = \{121, 33, 11, 11\}$$

Complex multiplications = 4

Complex additions = 8

Q. 6 If $x(n) = \cos\left(\frac{\pi n}{4}\right)$, $0 \le n \le 7$

Obtain $X(k)$ using DIFFFT algorithm.

Ans. :

$$X(k) = \{0, 3.99, 0, 0.0003, 0, 0.0003, 0, 3.99\}$$

Q. 7 Given $x(n) = n + 1$ and $N = 8$, determine $X(k)$ using DIFFFT.

Ans. :

$$X(k) = \{36, -4, -4+j4, -4-j4, -4+j9.656, -4-j1.656, -4+j1.656, -4-j1.656\}$$

Q. 8 Using FFT algorithm perform circular convolution of
$x_1(n) = \{1, 2, 3, 4\}$ $x_2(n) = \{4, 3, 2, 1\}$

Ans. : $x_1(n) \textcircled{N} x_2(n) = \{24, 22, 24, 30\}$

❑❑❑

Note

FIR Filter Structures

Section No.	Name of the Topic
1.1	Introduction
1.2	Direct Form Structure
1.3	Cascade Form Structure
1.4	Frequency Sampling Structure
1.5	Lattice Ladder Structure for FIR Filters
1.6	Digital Sinusoidal Generator
1.7	Structures for linear Phase FIR Filters

1.1 Introduction :

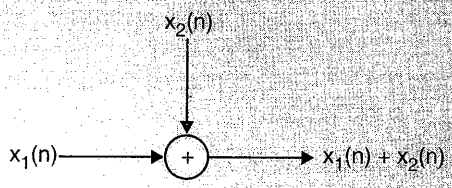
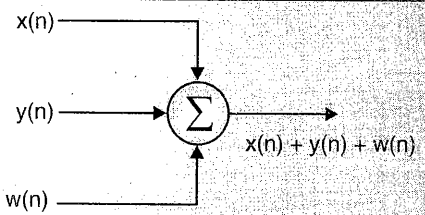
In order to realize (design) a digital filter; the given differential equation is broken into small equations. Then for each small equation; a structure using elementary blocks is drawn. Finally all these blocks are interconnected. The different types of structures used to realize the digital filter are as follows :

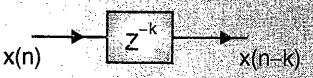
1. Direct form structure
2. Cascade form structure
3. Parallel form structure

1.1.1 Block Diagram Representation of Linear Constant Coefficient Difference Equations (LCCDE) :

Before studying each structure in detail; we will study the different basic blocks used to represent a discrete time system. These blocks are shown in Table H-1.

Table H-1

Sr. No.	Name of the block	Symbol
1.	Adder	 **Fig. H-1(a)**
2.	Summing device	 **Fig. H-1(b)**
3.	Constant multiplier	 **Fig. H-1(c)**

Sr. No.	Name of the block	Symbol
4.	Delay by k samples	 Fig. H-1(d)
5.	Advance by k samples	 Fig. H-1(e)

Advantages of representing the digital filter in block diagram form :

1. Just by observing the block diagram ; the computational algorithm can be easily written.
2. From the block diagram, the relationship between input and output can be determined.
3. The memory requirement and the computational complexity can be determined using block diagram structure.
4. From the transfer function ; variety of equivalent block diagram representation can be determined.

What is the meaning of canonic and non-canonic structures ?

If the number of delay elements in the block diagram is equal to the order of difference equation of a digital filter, then the realization structure is called as **canonic structure**. Otherwise it is called as **non-canonic structure**.

1.1.2 Transfer Function of FIR Filters :

We know that FIR stands for finite impulse response. The difference equation of FIR system is,

$$y(n) = \sum_{k=0}^{M-1} b_k x(n-k) \qquad \ldots(1)$$

Equation (1) shows that the system has length M as the limits of summation are from 0 to M – 1. These limits also indicates that the system is causal.

Taking Z-transform of Equation (1) we get,

$$Y(Z) = \sum_{k=0}^{M-1} b_k Z^{-k} X(k) \qquad \ldots(2)$$

Here we have used time shifting property of Z-transform.

$$\therefore \quad Z\{x(n-k)\} = Z^{-k} X(k)$$

Now the system transfer function is given by,

$$H(Z) = \frac{Y(Z)}{X(Z)}$$

Thus from Equation (2) we get,

$$H(Z) = \frac{Y(Z)}{X(Z)} = \sum_{k=0}^{M-1} b_k Z^{-k} \qquad ...(3)$$

Equation (3) is called as system transfer function of FIR filter.

1.2 Direct Form Structure :

The direct form realization of FIR filter can be obtained by using the equation of linear convolution. It is,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k) \qquad ...(1)$$

If we consider that there are 'M' samples then Equation (1) becomes,

$$y(n) = \sum_{k=0}^{M-1} h(k)\,x(n-k) \qquad ...(2)$$

Expanding Equation (2) we get,

$$y(n) = h(0)\,x(n) + h(1)\,x(n-1) + h(2)\,x(n-2) + \ldots + h(M-1)\,x(n-M+1) \qquad ...(3)$$

How to draw the structure ?

Now we will study how to draw structure (block diagram) for this filter. The first term in Equation (3) is $h(0)\,x(n)$. Here $x(n)$ is input and $h(0)$ is first sample of $h(n)$. So $h(0)$ is constant. That means $h(0)\,x(n)$ indicates that input $x(n)$ is multiplied by constant $h(0)$. It is represented as shown in Fig. H-2(a).

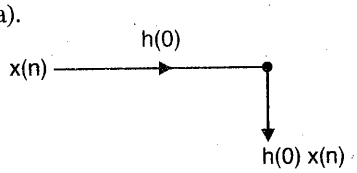


Fig. H-2(a) : $h(0)\,x(n)$

The second term in Equation (3) is $h(1)\,x(n-1)$. Here $x(n-1)$ indicates delay of input $x(n)$ by 1 sample. The term $h(1)\,x(n-1)$ is represented as shown in Fig. H-2(b).

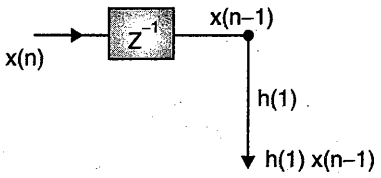


Fig. H-2(b) : h (1) x (n – 1)

Now, the third term is h (2) x (n – 2). Here x (n – 2) indicates delay of input x (n) by two samples. It is represented as shown in Fig. H-2(c).

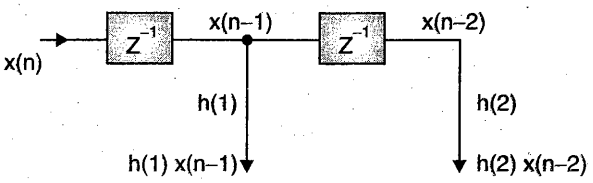


Fig. H-2(c) : h (2) x (n – 2)

Similarly we can draw the block diagram for remaining terms of Equation (3). The total structure for Equation (3) can be drawn by adding the block diagrams for all the terms. This structure is shown in Fig. H-3(a). It is called as direct form realization of FIR system.

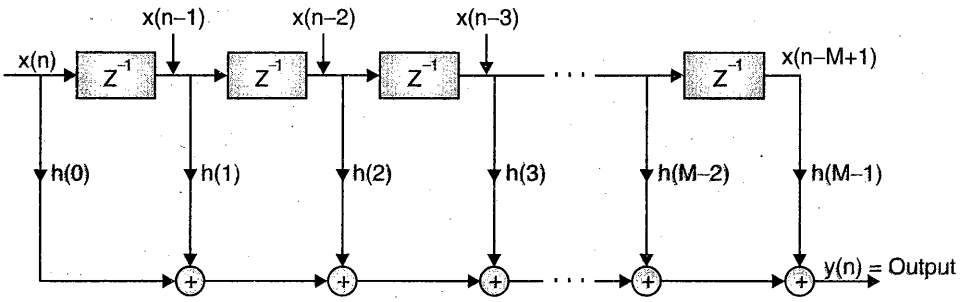


Fig. H-3(a) : Direct form realization of FIR system

Observations :

1. There are 'M – 1' delay blocks. So this is canonic structure.
2. Input signal is delayed 'M – 1' times. So to store this delayed input signal 'M – 1' memory locations are required.
3. This structure has 'M – 1' additions and 'M' multiplications.
4. Equation (3) shows that present input x (n) and past inputs (delayed inputs) e.g. x (n – 1), x (n – 2) etc. are multiplied by the corresponding sample of h (n) that is h (0), h (1)... etc. Thus output y (n) is weighted linear combination of present input and past inputs.

5. If the structure looks like, Fig. H-3(b) then it is called as pick-off node or tapped line. Now we can draw the same Fig. H-3(a) as shown in Fig. H-3(c). In this case it looks like tapped delay line. Therefore the direct form realization is also called as tapped or transversal delay line filter.

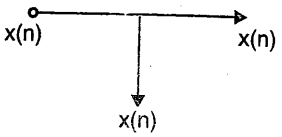


Fig. H-3(b) : Pick-off node

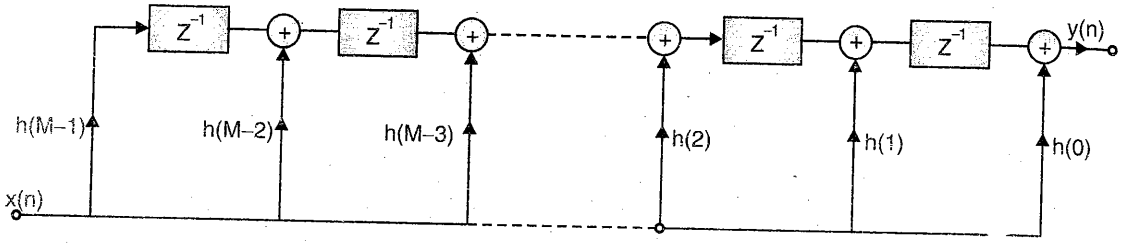


Fig. H-3(c) : Tapped delay line filter

1.3 Cascade Form Structure :

Cascade means the number of stages are connected in series. Now we have the system transfer function of the FIR system is given by,

$$H(Z) = \sum_{k=0}^{M-1} b_k Z^{-k} \qquad ...(1)$$

Generally the higher order FIR filter is realised by using a series connection of different FIR sections (cascade connection). Here each section is characterized by the second order transfer function. Then due to cascade connection ; the total transfer function [$H(Z)$] will be multiplication of all second order transfer functions.

$$\therefore \quad H(Z) = H_1(Z) \cdot H_2(Z) \cdot H_3(Z) \ldots\ldots H_k(Z) \qquad ...(2)$$

Here $H_k(Z)$ is the second order transfer function and it is given by,

$$H_k(Z) = b_{k0} + b_{k1} Z^{-1} + b_{k2} Z^{-2} \qquad ...(3)$$

But we know that, $H_k(Z) = \dfrac{Y_k(Z)}{X_k(Z)}$

$$\therefore \quad \frac{Y_k(Z)}{X_k(Z)} = b_{k0} + b_{k1} Z^{-1} + b_{k2} Z^{-2}$$

$$\therefore \quad Y_k(Z) = b_{k0} X_k(Z) + b_{k1} Z^{-1} X_k(Z) + b_{k2} Z^{-2} X_k(Z) \qquad ...(4)$$

Taking inverse Z-transform (IZT) of both sides we get,

$$y_k(n) = b_{k0} x_k(n) + b_{k1} x_k(n-1) + b_{k2} x_k(n-2) \qquad ...(5)$$

From Equation (5), we can draw the direct form realization of second order section as shown in Fig. H-4(a).

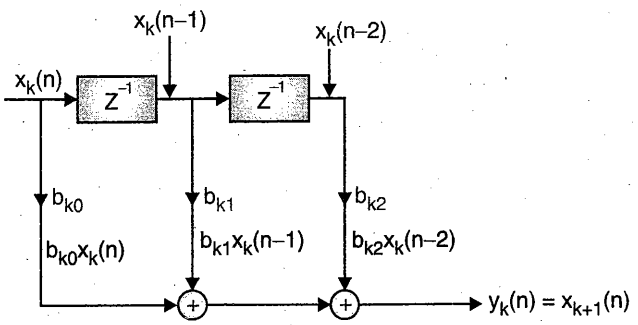


Fig. H-4(a) : Direct-form realization of second order section

Now the total structure is obtained by cascading all second order sections. The block schematic of cascade structure is shown in Fig. H-4(b).

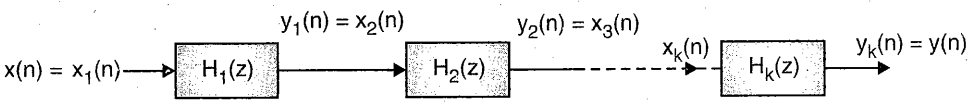


Fig. H-4(b) : Block diagram of cascade structure

Thus using Figs. H-4(a) and H-4(b), we can draw the cascade realization of FIR system as shown in Fig. H-4(c).

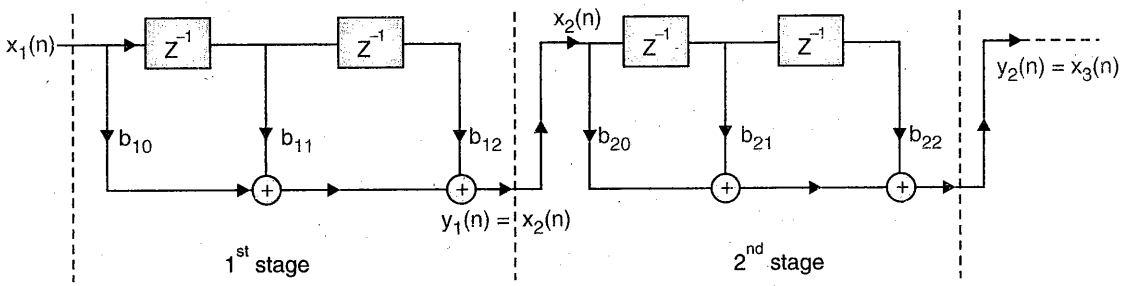


Fig. H-4(c) : Cascade realization of FIR system

Note : This cascade realization is obtained by putting K = 0 in first stage and K = 1 in second stage.

Observations :

1. This cascade form is canonic with respect to delay.
2. It has M – 1 adders and M multipliers for $(M-1)^{th}$ order FIR transfer function.

1.4 Frequency Sampling Structure :

The major advantage of designing FIR filters using frequency sampling structure is that; we can design FIR filter at the frequencies of interest and not at all frequencies. So the design complexity is reduced.

Now in frequency sampling structure; the desired frequency response is specified at the set of equally spaced frequencies as follows :

$$\omega = \omega_k = \frac{2\pi}{M}(k+\alpha) \qquad ...(1)$$

Here $k = 0, 1, 2, ... \frac{M-1}{2}$

M is an integer and $\alpha = 0$ or $\frac{1}{2}$

According to the definition of fourier transform we have,

$$H(\omega) = \sum_{n=0}^{M-1} h(n) e^{-j\omega n} \qquad ...(2)$$

This equation gives the frequency response of input signal.

Putting Equation (1) in Equation (2) we get,

$$H(\omega_k) = \sum_{n=0}^{M-1} h(n) e^{-j\frac{2\pi}{M}(k+\alpha)\cdot n}$$

$$\therefore \quad H(\omega_k) = \sum_{n=0}^{M-1} h(n) e^{-j2\pi(k+\alpha)n/M} \qquad ...(3)$$

This set given by Equation (3) are called as frequency samples of $H(\omega_k)$. Here $H(\omega_k)$ is same as DFT $H(k+\alpha)$. Now if $\alpha = 0$, then $H(\omega_k) = H(k)$ which corresponds to M point DFT.

$$\therefore \quad H(k+\alpha) = \sum_{n=0}^{M-1} h(n) e^{-j2\pi(k+\alpha)n/M} \qquad ...(4)$$

Using the definition of IDFT we can write,

$$h(n) = \frac{1}{M} \sum_{k=0}^{M-1} H(k+\alpha) e^{j2\pi(k+\alpha)n/M} \qquad ...(5)$$

Now we will obtain the equations in Z domain. According to the definition of Z-transform we have,

$$H(Z) = \sum_{n=0}^{M-1} h(n) Z^{-n} \qquad ...(6)$$

Putting Equation (5) in Equation (6),

$$H(Z) = \sum_{n=0}^{M-1} \left[\frac{1}{M} \sum_{k=0}^{M-1} H(k+\alpha) e^{j2\pi(k+\alpha)n/M} \right] Z^{-n}$$

Interchanging the order of summation we get,

$$H(Z) = \sum_{k=0}^{M-1} H(k+\alpha) \left[\frac{1}{M} \sum_{n=0}^{M-1} e^{j2\pi(k+\alpha)/M} \cdot Z^{-1} \right]^n \qquad ...(7)$$

Now we have the standard summation formula,

$$\sum_{n=0}^{N} a^n = \frac{1-a^{N+1}}{1-a}$$

Thus Equation (7) becomes,

$$H(Z) = \sum_{k=0}^{M-1} H(k+\alpha) \left[\frac{1}{M} \left\{ \frac{1-Z^{-M} e^{j2\pi(k+\alpha)}}{1-e^{j2\pi(k+\alpha)/M} \cdot Z^{-1}} \right\} \right] \qquad ...(8)$$

Here the term $e^{j2\pi(k+\alpha)}$ can be written as,

$$e^{j2\pi(k+\alpha)} = e^{j2\pi k} \cdot e^{j2\pi\alpha}$$

And $e^{j2\pi k} = \cos 2\pi k + j \sin 2\pi k = 1$

Thus Equation (8) can be simplified as,

$$H(Z) = \frac{1-Z^{-M} e^{j2\pi\alpha}}{M} \sum_{k=0}^{M-1} \frac{H(k+\alpha)}{1-e^{j2\pi(k+\alpha)/M} \cdot Z^{-1}} \qquad ...(9)$$

This equation shows that $H(Z)$ is characterised by set of frequency samples, $H(k+\alpha)$. Now we can write $H(Z)$ as the function of two cascade filters as,

$$H(Z) = H_1(Z) \cdot H_2(Z)$$

Let $$H_1(Z) = \frac{1}{M}\left(1 - Z^{-M} e^{j2\pi\alpha}\right) \qquad ...(10)$$

This is called as All-Zero system (filter) and zeros are located as,

$$Z_k = e^{j2\pi(k+\alpha)/M}$$

Similarly, $$H_2(Z) = \sum_{k=0}^{M-1} \frac{H(k+\alpha)}{1 - e^{j2\pi(k+\alpha)/M} \cdot Z^{-1}} \quad ...(11)$$

This consists of parallel banks of single pole.

Now $H_1(Z)$ and $H_2(Z)$ can be realised separately. Then $H(Z)$ is obtained by connecting $H_1(Z)$ and $H_2(Z)$ is cascade form as shown in Fig. H-5.

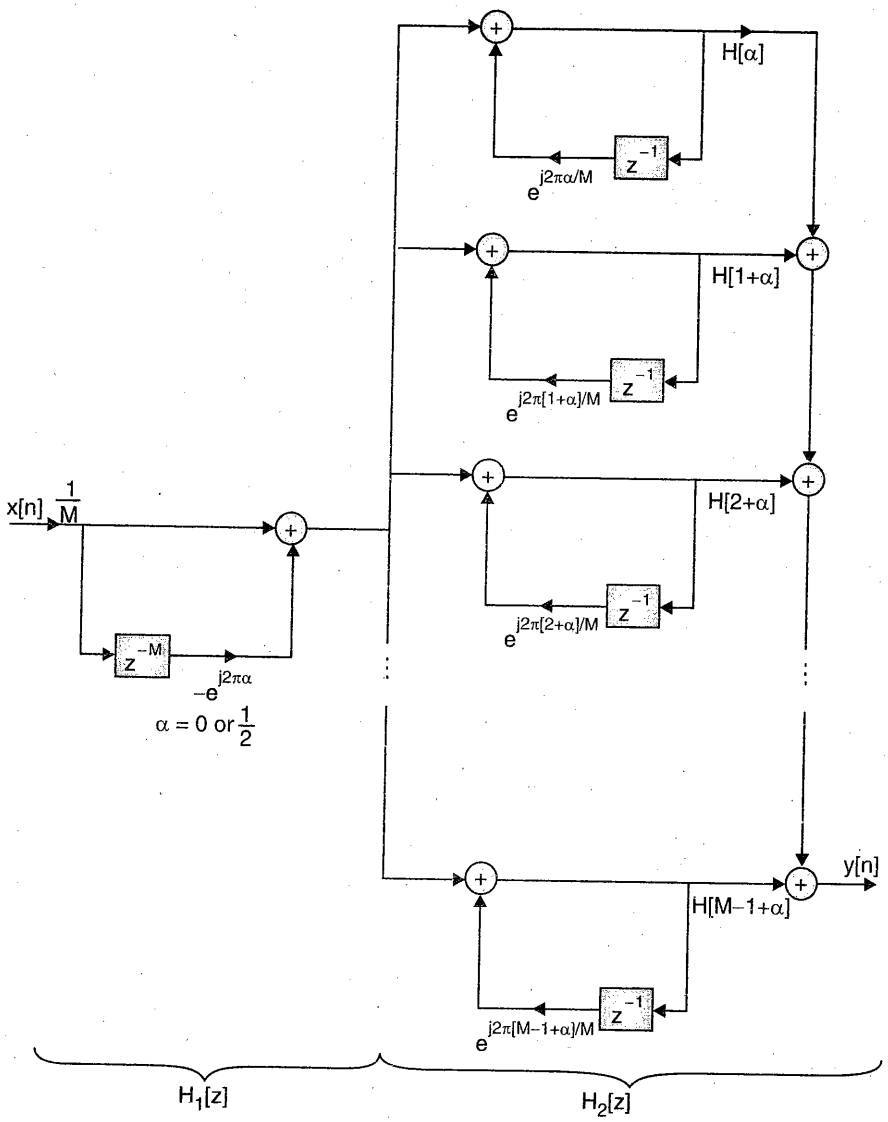


Fig. H-5 : Frequency sampling realization

1.5 Lattice Ladder Structure for FIR Filters :

Consider M^{th} order FIR system with the transfer function,

$$H_M(Z) = 1 + \sum_{k=1}^{M} \beta_M(k) Z^{-k}, \; M \geq 1 \qquad ...(1)$$

Here M denotes the degree of polynomial and β_M is coefficient.

Thus when M = 0 we get,

$$H_o(Z) = 1 \qquad ...(2)$$

Basically $H_M(Z)$ is the transfer function which can be written as,

$$H_M(Z) = \frac{\text{output}}{\text{input}} = \frac{Y(Z)}{X(Z)}$$

$$\therefore \quad Y(Z) = X(Z) H_M(Z) \qquad ...(3)$$

Putting Equation (1) in Equation (3) we get,

$$Y(Z) = X(Z)\left[1 + \sum_{k=1}^{M} \beta_M(k) Z^{-k}\right]$$

$$\therefore \quad Y(Z) = X(Z) + X(Z) \sum_{k=1}^{M} \beta_M(k) Z^{-k} \qquad ...(4)$$

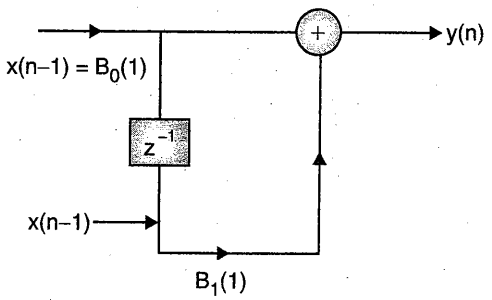
Taking IZT of both sides we get,

$$y(n) = x(n) + \sum_{k=1}^{M} \beta_M(k) \cdot x(n-k) \qquad ...(5)$$

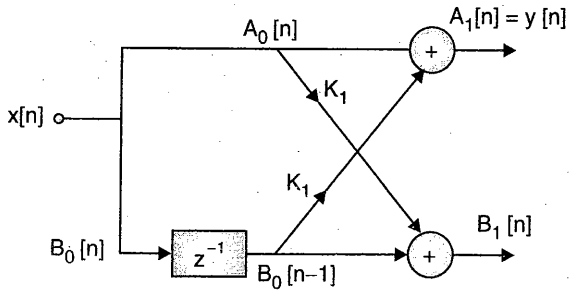
Let M = 1 then Equation (5) becomes,

$$y(n) = x(n) + \beta_1(1)\, x(n-1) \; \text{....(Here k = M = 1)} \qquad ...(6)$$

In the simplest way Equation (6) can be realized as shown in Fig. H-6(a).



(a)



(b) Single stage lattice structure

Fig. H-6

Now the same output can be obtained by using the structure shown in Fig. H-6(b). This structure is called as single stage lattice structure. This structure provides two outputs namely $A_1(n)$ and $B_1(n)$.

Here $A_1(n) = A_0(n) + k_1 B_0(n-1)$ ...(7(a))

A_0 and B_0 are constant multipliers.

In terms of $x(n)$ we can write,

$$A_1(n) = x(n) + k_1 x(n-1) \quad \text{...(7(b))}$$

Similarly, the other output can be written as,

$$B_1(n) = k_1 A_0(n) + B_0(n-1) \quad \text{...(8(a))}$$

In terms of $x(n)$ we can write,

$$B_1(n) = k_1 x(n) + x(n-1) \quad \text{...(8(b))}$$

We know that Equation (7(b)) is obtained by using single stage lattic structure. Equations (6) and (7(b)) are same if,

$$A_0(n) = x(n) \quad \text{and} \quad k_1 = \beta_1(1)$$

Similarly, Equations (6) and (7(a)) are matching if,

$$A_0(n) = x(n),\ \beta_1(1) = k_1 \text{ and } B_0(n-1) = x(n-1)$$

Here 'k' is called as reflection coefficient.

That means the same output can be obtained using single stage lattice structure.

Now for M = 2; Equation (5) becomes,

$$y(n) = x(n) + \beta_2(1)\,x(n-1) + \beta_2(2)\,x(n-2) \quad \ldots(9)$$

As M = 2 we have to cascade two stages to obtain two stage lattice structure as shown in Fig. H-6(c).

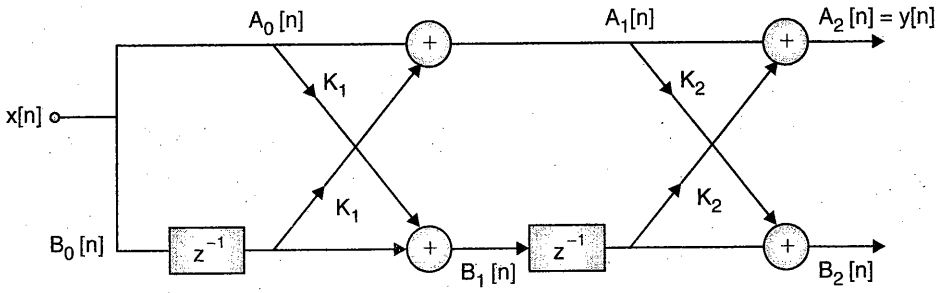


Fig. H-6(c) : Two stage lattice structure

From Fig. H-6(c) we can write,

Output of first stage :

$$A_1(n) = x(n) + k_1\,x(n-1) \quad \ldots(10)$$

and $$B_1(n) = k_1\,x(n) + x(n-1) \quad \ldots(11)$$

And the output of second stage is,

$$A_2(n) = A_1(n) + k_2\,B_1(n-1) \quad \ldots(12)$$

and $$B_2(n) = k_2\,A_1(n) + B_1(n-1) \quad \ldots(13)$$

Now putting Equation (10) in Equation (12) we get,

$$A_2(n) = x(n) + k_1\,x(n-1) + k_2\,B_1(n-1) \quad \ldots(14)$$

From Equation (11) we can write,

$$B_1(n-1) = k_1\,x(n-1) + x(n-2)$$

Putting this value in Equation (14) we get,

$$A_2(n) = x(n) + k_1\,x(n-1) + k_2\,[k_1\,x(n-1) + x(n-2)]$$

$$\therefore \quad A_2(n) = x(n) + k_1(1+k_2)\,x(n-1) + k_2\,x(n-2) \quad \ldots(15)$$

Observe that Equations (9) and (15) are matching.

Here, $\beta_2(2) = k_2$, $\beta_2(1) = k_1(1+k_2)$

or $k_2 = \beta_2(2)$, or $k_1 = \dfrac{\beta_2(1)}{1+\beta_2(2)}$

Here k_1 and k_2 are reflection coefficients. The same way we can increase the number of stages. Finally M^{th} stage lattice structure is obtained as shown in Fig. H-6(d).

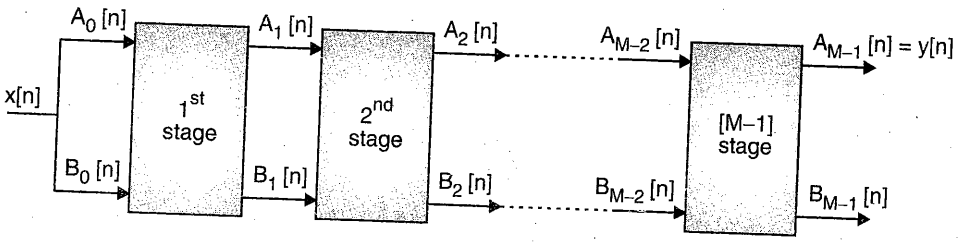


Fig. H-6(d) : M^{th} order lattice structure

1.6 Digital Sinusoidal Generator :

Digital sinusoidal oscillators are also called as digital sinusoidal generators. Consider a second order system whose transfer function is given by,

$$H(Z) = \frac{b_0}{1 + a_1 Z^{-1} + a_2 Z^{-2}} \qquad ...(1)$$

Here parameters a_1 and a_2 are given by,

$$a_1 = 2r\cos\omega_0 \text{ and } a_2 = r^2$$

where r is the radius of circle.

This system has complex conjugate poles at $p = re^{\pm j\omega_0}$ and its unit impulse response is given by,

$$h(n) = \frac{b_0 r^n}{\sin\omega_0} \sin(n+1)\omega_0 u(n) \qquad ...(2)$$

If the poles are placed on the unit circle (r = 1) and if $b_0 = A\sin\omega_0$ then Equation (2) becomes,

$$h(n) = \frac{A\sin\omega_0 \cdot (1)^n}{\sin\omega_0} \sin(n+1)\omega_0 u(n)$$

$$\therefore \quad h(n) = A\sin(n+1)\omega_0 u(n) \qquad ...(3)$$

This equation shows that the impulse response of second order system is sinusoidal. Thus this system is called as digital sinusoidal generator.

Now we can the draw the block schematic of digital sinusoidal oscillator as follows:

From Equation (1) we get,

$$H(Z) = \frac{Y(Z)}{X(Z)} = \frac{b_0}{1 + a_1 Z^{-1} + a_2 Z^{-2}}$$

$$\therefore \quad Y(Z)\left(1 + a_1 Z^{-1} + a_2 Z^{-2}\right) = b_0 X(Z)$$

$$\therefore \quad Y(Z) + a_1 Z^{-1} Y(Z) + a_2 Z^{-2} Y(Z) = b_0 X(Z)$$

$$\therefore \quad Y(Z) = b_0 X(Z) - a_1 Z^{-1} Y(Z) - a_2 Z^{-2} Y(Z) \qquad ...(4)$$

Taking IZT of both sides we get,

$y(n) = b_0 x(n) - a_1 y(n-1) - a_2 y(n-2)$. Thus the block schematic of digital sinusoidal oscillator is as shown in Fig. H-7.

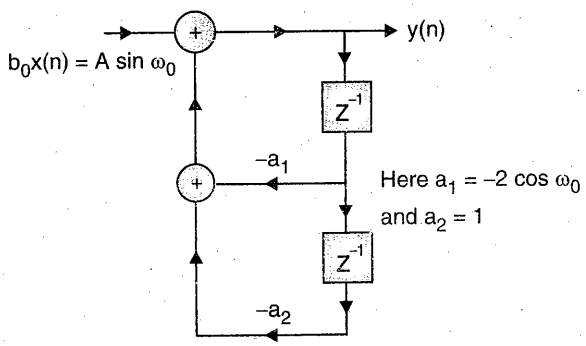


Fig. H-7

1.7 Structures for Linear Phase FIR Filters :

FIR filter is said to be having a linear phase structure if its unit impulse sequence is either symmetric or antisymmetric about some point in time. That means FIR filter has linear phase if it satisfies the condition.

$$h(n) = h(M-1-n), \qquad n = 0, 1 \ldots M-1 \qquad ...(1)$$

Here M = Number of samples

The transfer function of FIR filter is given by,

$$H(Z) = \sum_{n=0}^{M-1} h(n) Z^{-n} \qquad ...(2)$$

Let us split the summation into two parts.

$$\therefore \quad H(Z) = \sum_{n=0}^{\frac{M}{2}-1} h(n) Z^{-n} + \sum_{n=\frac{M}{2}}^{M-1} h(n) Z^{-n} \qquad ...(3)$$

But for linear phase we have,

$$h(n) = h(M-1-n) \Rightarrow n = M-1-n$$

Putting this condition in the second summation of Equation (3) we get,

$$H(Z) = \sum_{n=0}^{\frac{M}{2}-1} h(n)Z^{-n} + \sum_{n=0}^{\frac{M}{2}-1} h(M-1-n)Z^{-(M-1-n)} \quad ...(4)$$

Note : The limits of summation are changed as follows :

We have $n = M-1-\underset{=}{n}$

When $n = \frac{M}{2} \Rightarrow \frac{M}{2} = M-1-\underset{=}{n} \Rightarrow \underset{=}{n} = M-1-\frac{M}{2} = \frac{M}{2}-1$

When $n = M-1 \Rightarrow M-1 = M-1-\underset{=}{n} \Rightarrow \underset{=}{n} = 0$

Equation (4) can be written as,

$$H(Z) = \sum_{n=0}^{\frac{M}{2}-1} h(n)[Z^{-n} + Z^{-(M-1-n)}] \quad ...(5)$$

Case (i) For even M :

We know that $H(Z) = \frac{Y(Z)}{X(Z)}$; thus Equation (5) becomes,

$$\frac{Y(Z)}{X(Z)} = \sum_{n=0}^{\frac{M}{2}-1} h(n)\left[Z^{-n} + Z^{-(M-1-n)}\right]$$

$$\therefore \quad Y(Z) = \left\{\sum_{n=0}^{\frac{M}{2}-1} h(n)[Z^{-n} + Z^{-(M-1-n)}]\right\} X(Z)$$

Now expanding the summation we get,

$$Y(Z) = h(0)\left[1 + Z^{-(M-1)}\right]X(Z) + h(1)\left[Z^{-1} + Z^{-(M-2)}\right]X(Z)$$

$$+ .... + h\left(\frac{M}{2}-1\right)\left[Z^{-\left(\frac{M}{2}-1\right)} + Z^{-\frac{M}{2}}\right]X(Z)$$

$$\therefore \quad Y(Z) = h(0)\left[X[Z]+Z^{-(M-1)}\cdot X(Z)\right]+h(1)\left[Z^{-1}X(Z)+Z^{-(M-2)}X(Z)\right]$$

$$+ \ldots + h\left(\frac{M}{2}-1\right)\left[Z^{-\left(\frac{M}{2}-1\right)}X(Z)+Z^{-\frac{M}{2}}X(Z)\right] \qquad \ldots(6)$$

Taking IZT of both sides we get,

$$y(n) = h(0)\left\{x(n)+x[n-(M-1)]\right\}+h(1)\left\{x(n-1)+x[n-(M-2)]\right\}$$

$$+ \ldots + h\left(\frac{M}{2}-1\right)\left\{x\left[\left(n-\left(\frac{M}{2}-1\right)\right)\right]+x\left(n-\frac{M}{2}\right)\right\} \qquad \ldots(7)$$

Let $M = 6$. Thus Equation (7) becomes,

$$y(n) = h(0)\left\{x(n)+x(n-5)\right\}+h(1)[x(n-1)+x(n-4)]+h(2)[x(n-2)+x(n-3)] \quad \ldots(8)$$

The realization of Equation (8) is shown in Fig. H-8(a).

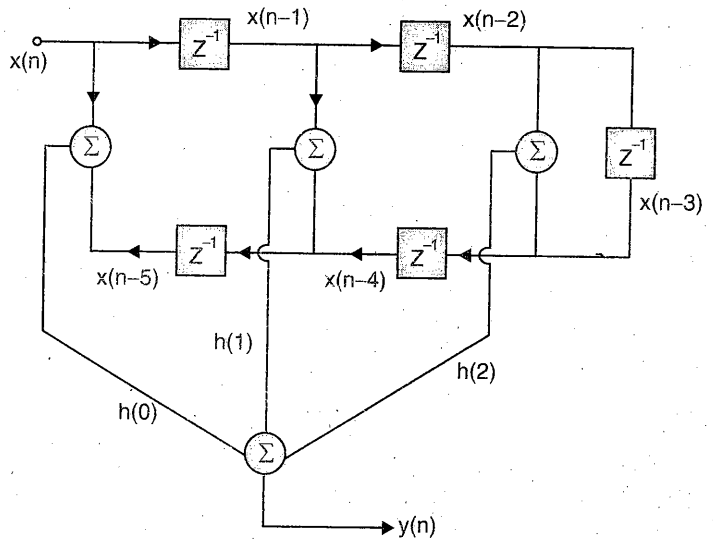


Fig. H-8(a) : Linear phase FIR structure for M = 6 (even)

Case (ii) : For odd M :

When M is odd (let M = 5) we can simply write the difference equation as,

$$y(n) = h(0)x(n)+h(1)x(n-1)+h(2)x(n-2)+h(3)x(n-3)+h(4)x(n-4) \quad \ldots(9)$$

But we have the condition for symmetry,

$$h(n) = h(M-1-n) \qquad \ldots(10)$$

Here M = 5. Thus from Equation (10) we get,

For $n = 0 \Rightarrow h(0) = h(4)$

For $n = 1 \Rightarrow h(1) = h(3)$

For $n = 2 \Rightarrow h(2) = h(2)$

For $n = 3 \Rightarrow h(3) = h(1)$

For $n = 4 \Rightarrow h(4) = h(0)$

Thus Equation (9) becomes,

$$y(n) = h(0)x(n)+h(1)x(n-1)+h(2)x(n-2)+h(1)x(n-3)+h(0)x(n-4)$$

$$\therefore \quad y(n) = h(0)[x(n)+x(n-4)]+h(1)[x(n-1)+x(n-3)]+h(2)x(n-2) \quad ...(11)$$

The realization of Equation (11) is shown in Fig. H-8(b).

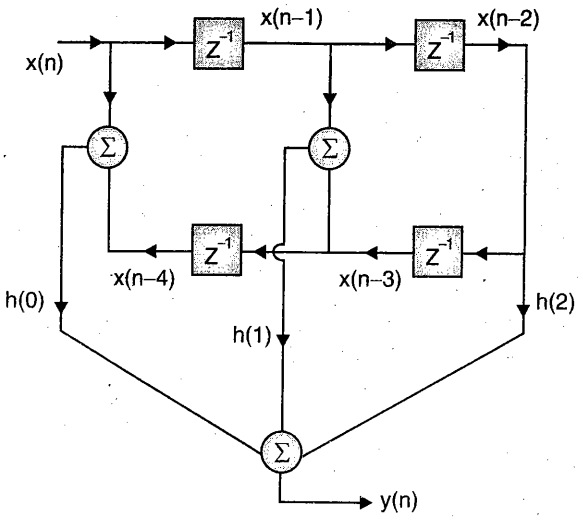


Fig. H-8(b) : Linear phase FIR structure for M = 5 (odd)

Note : Similar to case (i), equation of H(Z) for odd M can be expressed as,

$$H(Z) = h\left(\frac{M-1}{2}\right) Z^{\frac{-(M-1)}{2}} + \sum_{n=0}^{\frac{M-3}{2}} h(n)\left[Z^{-n}+Z^{-(M-1-n)}\right]$$

Then by using same procedure we can obtain the required realization.

Solved Problems :

Prob. 1 : Obtain linear phase realization of

$$H(Z) = 1+\frac{Z^{-1}}{4}+\frac{Z^{-2}}{4}+Z^{-3}$$

Soln. : Given equation is,

$$H(Z) = 1+\frac{Z^{-1}}{4}+\frac{Z^{-2}}{4}+Z^{-3}$$

$$\therefore \quad h(0) = 1, \quad h(1) = \frac{1}{4}, \quad h(2) = \frac{1}{4}, \quad h(3) = 1$$

Here $M = 4$ that means M is even.

Now we have the condition for symmetric response.

$$h(n) = h(M-1-n), \quad n = 0, 1, \ldots M-1$$

For $n = 0 \Rightarrow h(0) = h(3)$

For $n = 1 \Rightarrow h(1) = h(2)$

This realization is similar to Fig. H-8(a). So for M = 4; the realization is shown in Fig. H-9.

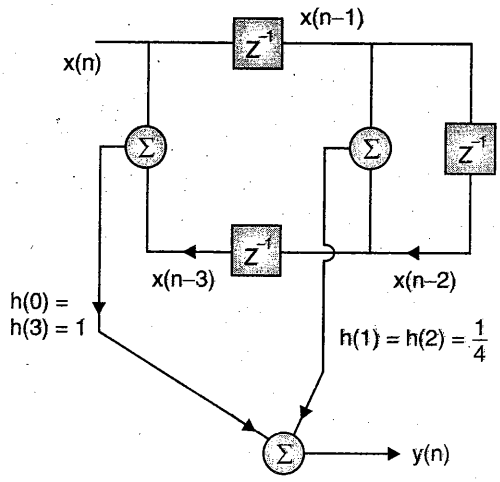


Fig. H-9

Prob. 2 : Realize a linear phase FIR filter with the following impulse response.

$$h(n) = \delta(n) + \frac{1}{2}\delta(n-1) - \frac{1}{4}\delta(n-2) + \delta(n-4) + \frac{1}{2}\delta(n-3)$$

Soln. : We can write the impulse response as,

$$h(n) = \left\{1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{2}, 1\right\}$$
$$\uparrow$$

$$\therefore \quad h(0) = 1, h(1) = \frac{1}{2}, \; h(2) = -\frac{1}{4}, h(3) = \frac{1}{2}, h(4) = 1$$

Here $M = 5$

Now according to the condition of symmetry,

$$h(n) = h(M-1-n), \qquad n = 0, 1, \ldots M-1$$

For $n = 0 \Rightarrow h(0) = h(4)$

For $n = 1 \Rightarrow h(1) = h(3)$

For $n = 2 \Rightarrow h(2) = h(2)$

This is the case for M = 5 (odd). The realization is same as shown in Fig. H-8(b).

Prob. 3 : An FIR filter is given by

$$y(n) = x(n) + \frac{4}{5}x(n-1) + \frac{3}{2}x(n-2) + \frac{2}{3}x(n-3)$$

Find the lattice structure coefficients.

Soln. :

$$y(n) = x(n) + \frac{4}{5}x(n-1) + \frac{3}{2}x(n-2) + \frac{2}{3}x(n-3)$$

Taking Z-transform,

$$Y(Z) = X(Z) + \frac{4}{5}Z^{-1}X(Z) + \frac{3}{2}Z^{-2}X(Z) + \frac{2}{3}Z^{-3}X(Z)$$

$$\therefore Y(Z) = X(Z)\left[1 + \frac{4}{5}z^{-1} + \frac{3}{2}Z^{-2} + \frac{2}{3}Z^{-3}\right]$$

$$\therefore \frac{Y(Z)}{X(Z)} = H(Z) = 1 + \frac{4}{5}Z^{-1} + \frac{3}{2}Z^{-2} + \frac{2}{3}Z^{-3} \quad ...(1)$$

As we know

$$A_m(Z) = A_{m-1}(Z) + K_m Z^{-1} B_{m-1}(Z) \quad ...(2)$$

$$B_m(Z) = K_m A_m(Z) + Z^{-1} B_{m-1}(Z) \quad ...(3)$$

$$A_{m-1}(Z) = \frac{A_m(Z) - K_m B_m(Z)}{1 - K_m^2} \quad ...(4)$$

In above example, using step down recursion, as order of filter m = 3,

$$A_3(Z) = 1 + \frac{4}{5}Z^{-1} + \frac{3}{2}Z^{-2} + \frac{2}{3}Z^{-3}$$

$$\therefore \alpha_3(3) = \frac{2}{3}$$

as K_3 = coefficient of lattice structure

$$\therefore K_3 = \alpha_3(3) = \frac{2}{3} \quad ...(5)$$

$$A_s\ B_3(Z) = Z^{-3}A_3(Z^{-1})$$

$$= Z^{-3}\left[1 + \frac{4}{5}Z^1 + \frac{3}{2}Z^2 + \frac{2}{3}Z^3\right]$$

$$B_3(Z) = Z^{-3} + \frac{4}{5}Z^{-2} + \frac{3}{2}Z^{-1} + \frac{2}{3}$$

$$B_3(Z) = \frac{2}{3} + \frac{3}{2}Z^{-1} + \frac{4}{5}Z^{-2} + Z^{-3} \quad ....(6)$$

Now, $$A_{m-1}(Z) = \frac{A_m(Z) - K_m B_m(Z)}{1 - K_m^2}$$

$$\therefore A_2(Z) = \frac{A_3(Z) - K_3 B_3(Z)}{1 - K_3^2}$$

$$= \frac{\left[1 + \frac{4}{5}Z^{-1} + \frac{3}{2}Z^{-2} + \frac{2}{3}Z^{-3}\right] - \frac{2}{3}\left[\frac{2}{3} + \frac{3}{2}Z^{-1} + \frac{4}{5}Z^{-2} + 2^{-3}\right]}{1 - \left(\frac{2}{3}\right)^2}$$

$$= \frac{1 + \frac{4}{5}Z^{-1} + \frac{3}{2}Z^{-2} + \frac{2}{3}Z^{-3} - \frac{4}{9} - Z^{-1} - \frac{8}{15}Z^{-2} - \frac{2}{3}Z^{-3}}{1 - \left(\frac{4}{9}\right)}$$

Collecting same terms together

$$A_2(Z) = \frac{\left(1 - \frac{4}{9}\right) + Z^{-1}\left(\frac{4}{5} - 1\right) + Z^{-2}\left(\frac{3}{2} - \frac{8}{15}\right)}{\frac{5}{9}}$$

$$= \frac{\frac{5}{9} - \frac{1}{5}Z^{-1} + \frac{29}{30}Z^{-2}}{\frac{5}{9}}$$

$$A_2(Z) = 1 - \frac{9}{25}Z^{-1} + \frac{87}{50}Z^{-2}$$

$$\therefore \alpha_2(2) = \frac{87}{50}$$

$$\therefore K_2 = \alpha_2(2) = \frac{87}{50} \quad \text{...(7)}$$

Now, $$B_2(Z) = Z^{-2}\left[A_2(Z^{-1})\right]$$

$$= Z^{-2}\left[1 - \frac{9}{25}Z^1 + \frac{87}{50}Z^2\right]$$

$$B_2(Z) = Z^{-2} - \frac{9}{25}Z^{-1} + \frac{87}{50}$$

$$B_2(Z) = \frac{87}{50} - \frac{9}{25}Z^{-1} + Z^{-2} \quad \text{...(8)}$$

Now,
$$A_1(Z) = \frac{A_2(Z) - K_2 B_2(Z)}{1 - K_2^2}$$

$$= \frac{\left(1 - \frac{9}{25}Z^{-1} + \frac{87}{50}Z^{-2}\right) - \frac{87}{50}\left(\frac{87}{50} - \frac{9}{25}Z^{-1} + Z^{-2}\right)}{1 - \left(\frac{87}{50}\right)^2}$$

$$A_1(Z) = \frac{1 - \frac{9}{25}Z^{-1} + \frac{87}{50}Z^{-2} - \left(\frac{87}{50}\right)^2 + \left(\frac{87 \times 9}{50 \times 25}\right)Z^{-1} - \frac{87}{50}Z^{-2}}{1 - \left(\frac{87}{5}\right)^2}$$

Collecting same terms together,

$$A_1(Z) = \frac{1 - \left(\frac{87}{50}\right)^2 - Z^{-1}\left(\frac{783}{1250} - \frac{9}{25}\right)}{1 - \frac{7569}{2500}}$$

$$= \frac{(1 - 3.0276) - Z^{-1}(0.6264 - 0.36)}{1 - 3.0276}$$

$$= \frac{-2.0276 - 0.2664\,Z^{-1}}{-2.0276}$$

$$= 1 - \frac{0.2664\,Z^{-1}}{-2.0276}$$

$$A_1(Z) = 1 + 0.1313869\,Z^{-1}$$

$$\therefore \alpha_1(1) = 0.1314$$

$$\therefore K_1 = 0.1314$$

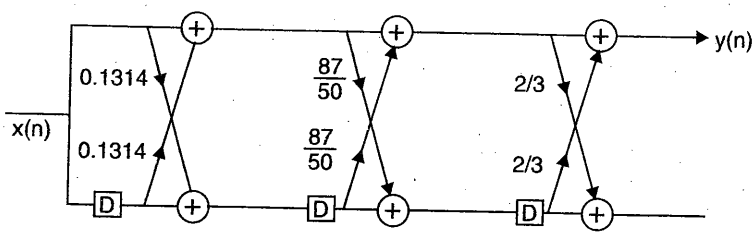


Fig. H-10

Theory Questions

Q. 1 Compare various design methods of linear phase FIR filter.

Q. 2 Write short note on : Frequency sampling techniques.

Q. 3 How to draw direct form structure ?

Q. 4 Explain cascade form realization of FIR filter ?

Q. 5 Write short note on :

(a) Frequency sampling structure

(b) Lattice ladder structure

Unsolved Numericals

Q. 1 Represent the system function using linear phase FIR structure

$$H(Z) = 1 + \frac{1}{2}[Z + Z^{-1}]$$

Ans. :

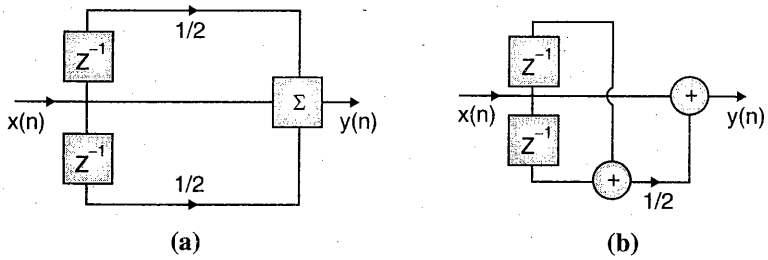


Fig. H-11

Q. 2 Represent system function using linear phase FIR structures

$$H(Z) = \frac{1}{2} + Z^{-1} + \frac{Z^{-2}}{2}$$

Ans. :

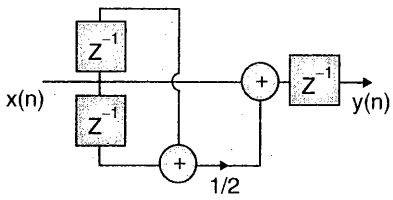


Fig. H-12

Q. 3 Draw cascade realization of $H(Z) = 1 + \frac{3}{2}Z^{-1} + \frac{3}{2}Z^{-2} - Z^{-3}$

Hint : $H(Z) = H_1(Z) \cdot H_2(Z)$; $H_1(Z) = (1 - Z^{-1})$ and $H_2(Z) = \left(1 - \frac{1}{2}Z^{-1} + Z^{-2}\right)$

Ans. :

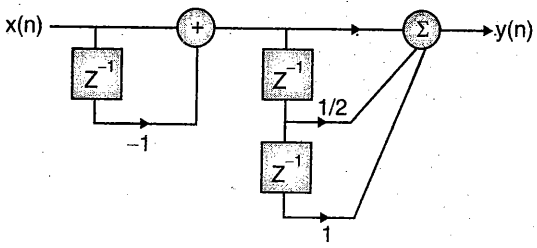


Fig. H-13

❑❑❑

Structures for IIR Filters

Section No.	Name of the Topic
1.1	Introduction
1.2	Direct Form Structure for IIR Systems
1.3	Cascade Form Structures
1.4	Parallel Form Structure
1.5	Representation of Structures using Signal Flow Graphs
1.6	Transposed Structures
1.7	Feedback in IIR Systems
1.8	Lattice Structure for IIR Filters
1.9	Lattice-ladder Structure of IIR Filter
1.10	Ideal Filters and Approximations

1.1 Introduction :

We know that IIR stands for infinite impulse response. Generally IIR systems are recursive type. A recursive system means; feedback connection is present from output side to the input side. For the realization of IIR systems present, past, future samples of input and past values of output are required. For the realization of IIR systems, following structures are used.

1. Direct form structure
2. Cascade form structure
3. Parallel form structure

1.2 Direct Form Structure for IIR Systems :

Direct form structure is again divided into two types :

A. Direct form-I structure

B. Direct form-II structure.

The general difference equation for discrete time LTI system is given as,

$$H(Z) = \frac{\sum_{k=0}^{M} b_k Z^{-k}}{1 + \sum_{k=1}^{N} a_k Z^{-k}} \quad \ldots(1)$$

Now let $H_1(Z)$ = Numerator term of Equation (1)

$$\therefore \quad H_1(Z) = \sum_{k=0}^{M} b_k Z^{-k} \quad \ldots(2)$$

$$\text{And let,} \quad H_2(Z) = \frac{1}{1 + \sum_{k=1}^{N} a_k Z^{-k}} \quad \ldots(3)$$

Putting Equations (2) and (3) in Equation (1) we get,

$$H(Z) = H_1(Z) \cdot H_2(Z) \quad \ldots(4)$$

This equation shows that $H(Z)$ can be represented as the cascade connection of $H_1(Z)$ and $H_2(Z)$ as shown in Fig. I-1(a).

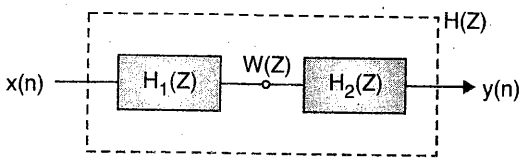


Fig. I-1(a) : Cascade connection of $H_1(Z)$ and $H_2(Z)$

We know that $H_1(Z)$ is the transfer function of the numerator term. Numerator always contain zeros of the system. So $H_1(Z)$ is called as all zero system. Now $H_2(Z)$ is transfer function of denominator. Denominator always contain poles of the system. So $H_2(Z)$ is called as all pole system.

(A) Direct form-I structure :

The direct form-I structure is obtained by cascading (connecting in series), the structure for $H_1(Z)$ and $H_2(Z)$. First we will draw the direct form structure for $H_1(Z)$.

Direct form structure for $H_1(Z)$:

Recall the equation of $H_1(Z)$ (Equation (2)). It is,

$$H_1(Z) = \sum_{k=0}^{M} b_k Z^{-k} \qquad ...(5)$$

But we know that, $H(Z) = \dfrac{\text{Output}\,[Y(Z)]}{\text{Input}\,[X(Z)]}$

To avoid the confusion we will write,

$H_1(Z) = \dfrac{W(Z)}{X(Z)}$; where $W(Z)$ is output of first stage.

$$\frac{W(Z)}{X(Z)} = \sum_{k=0}^{M} b_k Z^{-k} \qquad ...(6)$$

$$\therefore \quad \frac{W(Z)}{X(Z)} = b_0 Z^0 + b_1 Z^{-1} + b_2 Z^{-2} + ... + b_M Z^{-M}$$

But $Z^0 = 1$

$$\therefore \quad W(Z) = b_0 X(Z) + b_1 Z^{-1} X(Z) + b_2 Z^{-2} X(Z) + ... + b_M Z^{-M} X(Z) \qquad ...(7)$$

Taking inverse Z-transform (IZT) of Equation (7) we get,

$$y_1(n) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + ... + b_M x(n-M) \qquad ...(8)$$

Here b_0, b_1, b_2 ... b_M are the coefficients. The direct form realization of Equation (8) is shown in Fig. I-1(b).

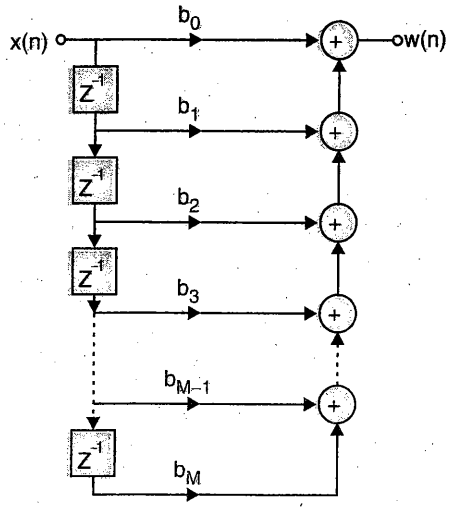


Fig. I-1(b) : Direct form realization of $H_1(Z)$ (All zero system)

Direct form structure for $H_2(Z)$:

Recall the equation for $H_2(Z)$. It is,

$$H_2(Z) = \frac{1}{1 + \sum_{k=1}^{N} a_k Z^{-k}} \qquad ...(9)$$

We have, $$H_2(Z) = \frac{\text{Output } [Y(Z)]}{\text{Input } [X(Z)]} \qquad ...(10)$$

Now $H_2(Z)$ represents second stage of Fig. I-1(a). Input of second stage is the output of first stage. Thus input of $H_2(Z)$ is $W(Z)$ and output of $H_2(Z)$ is the output of overall system which is $Y(Z)$. Thus Equation (10) becomes,

$$H_2(Z) = \frac{Y(Z)}{W(Z)}$$

Putting this value in Equation (9) we get,

$$\frac{Y(Z)}{W(Z)} = \frac{1}{1 + \sum_{k=1}^{N} a_k Z^{-k}}$$

$$\therefore \quad Y(Z)\left[1 + \sum_{k=1}^{N} a_k Z^{-k}\right] = W(Z)$$

$$\therefore \quad Y(Z) + Y(Z)\left[\sum_{k=1}^{N} a_k Z^{-k}\right] = W(Z)$$

$$\therefore \quad Y(Z) = -\left[\sum_{k=1}^{N} a_k Z^{-k}\right] Y(Z) + W(Z) \qquad ...(11)$$

Expanding the summation we get,

$$Y(Z) = -\left[a_1 Z^{-1} + a_2 Z^{-2} + \ldots + a_N Z^{-N}\right] Y(Z) + W(Z)$$

$$\therefore \quad Y(Z) = -a_1 Z^{-1} Y(Z) - a_2 Z^{-2} Y(Z) \ldots - a_N Z^{-N} Y(Z) + W(Z) \qquad ...(12)$$

Taking IZT of Equation (12) we get,

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) \ldots - a_N y(n-N) + w_n \qquad ...(13)$$

Here $-a_1$, $-a_2$ ... $-a_N$ are the coefficients.

The direct form implementation of Equation (13) is shown in Fig. I-1(c).

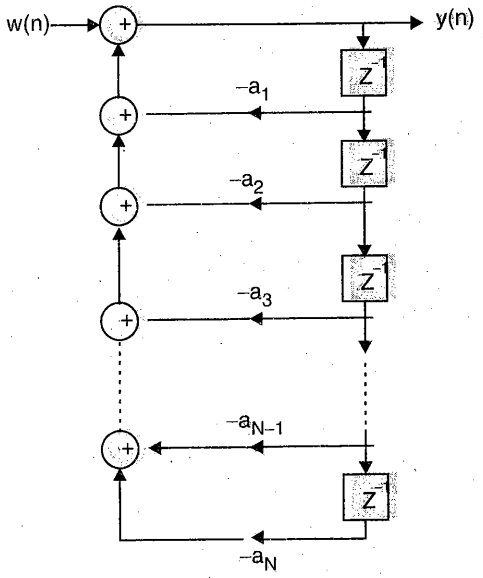


Fig. I-1(c) : Direct form realization of $H_2(Z)$ (All pole system)

Here all R.H.S. terms of Equation (13) except $w(n)$ are the delayed output terms. Thus this is a feedback (recursive) connection.

Now **direct form - I** structure is obtained by cascading $H_1(Z)$ and $H_2(Z)$. Thus direct form-I realization of IIR system is obtained by connecting Fig. I-1(b) and Fig. I-1(c) in series. It is shown in Fig. I-1(d).

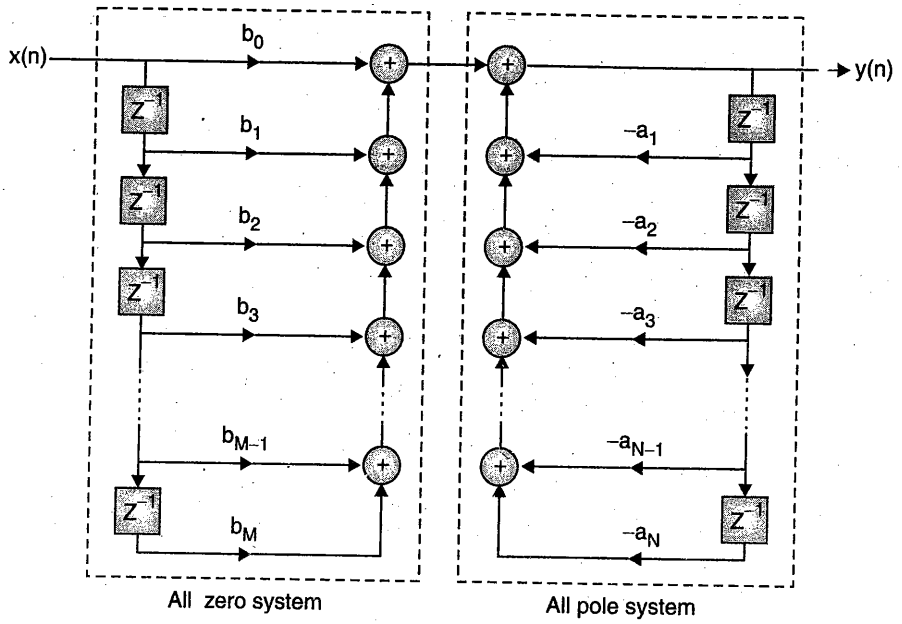


Fig. I-1(d) : Direct form-I realization of IIR system

Computation complexity :

The computational complexity of direct form-I structure is as follows :

1. Number of multiplications = $M + N + 1$
2. Number of additions = $M + N$
3. Number of memory locations = $M + N + 1$.

(B) Direct form-II structure :

Observe Fig. I-1(d). Here, first block represents all zero system and second block represents all pole system. Thus, in direct form- I structure ; zeros of $H(Z)$ are realised first and the poles of $H(Z)$ are realized second.

Now we are studying these structures for LTI (linear time invariant) systems. Since the systems are linear we can interchange the positions of $H_1(Z)$ and $H_2(Z)$. This will give us the direct form-II structure. Thus in direct form-II structure; poles are realised first and zeros second.

We have, $$H(Z) = H_1(Z) \cdot H_2(Z) \quad ...(1)$$

We will interchange the equations of $H_1(Z)$ and $H_2(Z)$.

$$\therefore \quad H_1(Z) = \frac{1}{1 + \sum_{k=1}^{N} a_k Z^{-k}} \quad ...(2)$$

and $$H_2(Z) = \sum_{k=0}^{M} b_k Z^{-k} \quad ...(3)$$

The cascade connection of $H(Z) = H_1(Z) \cdot H_2(Z)$ is shown in Fig. I-2(a).

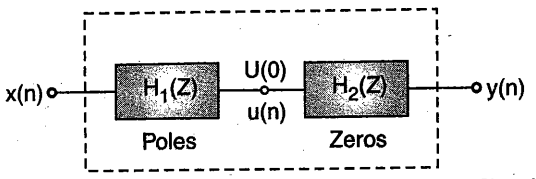


Fig. I-2(a) : Decomposition of direct form-II realization

Here $U(Z)$ represents the output of first block.

(1) All pole system :

We can write, $$H_1(Z) = \frac{\text{Output}}{\text{Input}} = \frac{U(Z)}{X(Z)} \quad ...(4)$$

Comparing Equations (2) and (3) we get,

$$\frac{U(Z)}{X(Z)} = \frac{1}{1 + \sum_{k=1}^{N} a_k Z^{-k}}$$

$$\therefore \quad U(Z)\left[1 + \sum_{k=1}^{N} a_k Z^{-k}\right] = X(Z)$$

$$\therefore \quad U(Z) + U(Z)\left[\sum_{k=1}^{N} a_k Z^{-k}\right] = X(Z)$$

$$\therefore \quad U(Z) = X(Z) - U(Z)\left[\sum_{k=1}^{N} a_k Z^{-k}\right]$$

Expanding the summation we get,

$$U(Z) = X(Z) - a_1 Z^{-1} U(Z) - a_2 Z^{-2} U(Z) .... - a_N Z^{-N} U(Z) \quad ...(5)$$

Taking IZT of Equation (5) we get,

$$u(n) = x(n) - a_1 x(n-1) - a_2 x(n-2) ..... - a_N x(n-N) \quad ...(6)$$

The direct form implementation of Equation (6) is shown in Fig. I-2(b).

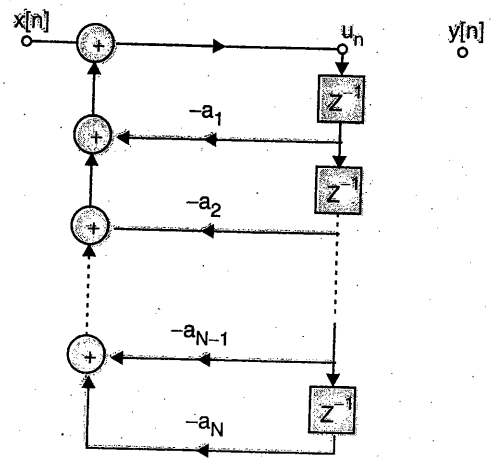


Fig. I-2(b) : Direct form realization of $H_1(Z)$ (All pole system)

(2) All zero system :

As shown in Fig. I-2(a), the output of second stage is the overall output of the system which is $Y(Z)$. While input to this stage is the output of first stage, which is $U(Z)$. Thus we can write,

$$H_2(Z) = \frac{\text{Output}}{\text{Input}} = \frac{Y(Z)}{U(Z)} \qquad ...(7)$$

Comparing Equations (3) and (7) we get,

$$\frac{Y(Z)}{U(Z)} = \sum_{k=0}^{M} b_k Z^{-k} \qquad ...(8)$$

$$\therefore \quad Y(Z) = U(Z)\left[\sum_{k=0}^{M} b_k Z^{-k}\right]$$

Expanding the summation we get,

$$Y(Z) = U(Z)\left[b_0 Z^0 + b_1 Z^{-1} + b_2 Z^{-2} + \ldots + b_M Z^{-M}\right]$$

Here $Z^0 = 1$; multiplying by $U(Z)$ we get,

$$Y(Z) = b_0 U(Z) + b_1 Z^{-1} U(Z) + b_2 Z^{-2} U(Z) + \ldots + b_M Z^{-M} U(Z) \qquad ...(9)$$

Taking IZT of Equation (9) we get,

$$y(n) = b_0 u(n) + b_1 u(n-1) + b_2 u(n-2) + \ldots + b_m u(n-M) \qquad ...(10)$$

The direct form implementation of Equation (10) is shown in Fig. I-2(c).

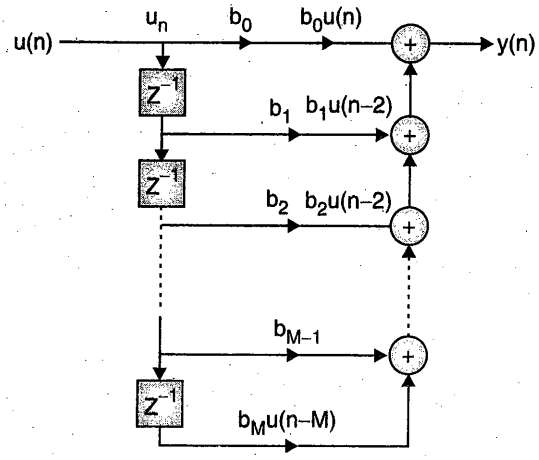


Fig. I-2(c) : Direct form realization of $H_2(Z)$ (All zero system)

Now direct form-II structure is obtained by cascading $H_1(Z)$ and $H_2(Z)$. That means it is obtained by connecting Fig. I-2(b) and Fig. I-2(c) in series. This is shown in Fig. I-2(d).

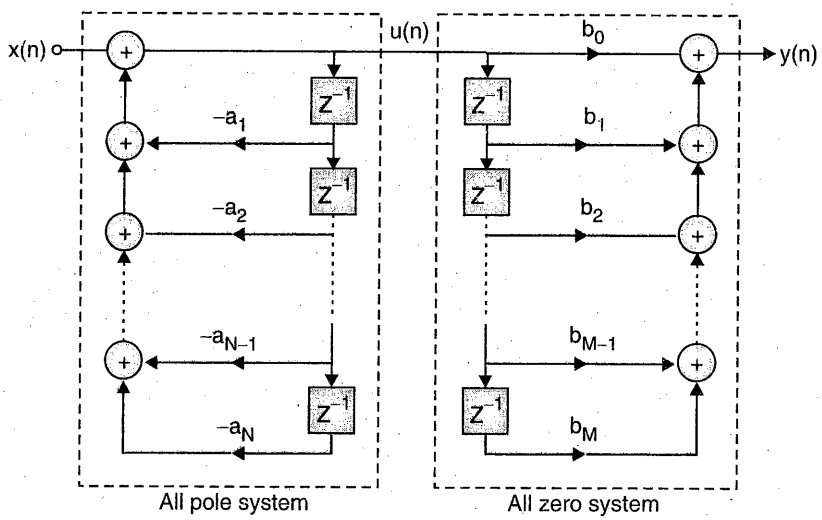


Fig. I-2(d) : Direct form-II realization of IIR system

In Fig. I-2(d), separate delay elements are used. Generally common delay elements are used in direct form-II structure. By using common delays, direct form II structure is drawn as shown in Fig. I-2(e).

Note that here we have considered N = M.

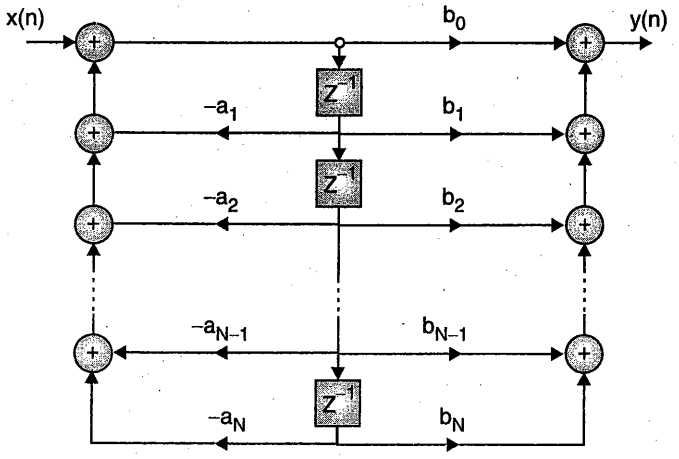


Fig. I-2(e) : Direct form-II realization using common delay elements

Computation complexity :

The computational complexity for direct form-II realization of IIR is as follows :

1. Number of multiplications = $M + N + 1$.
2. Number of additions = $M + N$.
3. The number of delay elements are reduced in direct form-II structure, compared to direct form-I structure. That means the memory locations are reduced. The memory locations required for direct form-II structure are $\{M, N\}$.

Why these structure are called as direct form structures ?

The direct form-I and direct form-II structures are obtained directly from the corresponding transfer functions without any rearrangements. So these structure is called as direct form structures.

Advantage : The only advantage of direct form realization is its implementation which is easy. We prefer direct form-II structure compared to direct form-I structure; because less memory locations are required.

Disadvantage : Both direct form structures are sensitive to the effects of quantization errors in the coefficients. So in the practical applications ; these structures are not preferred.

1.3 Cascade Form Structures :

To obtain the cascade form realization; the numerator and denominator of given transfer function $H(Z)$ is factored into the product of second order terms.

Then the total transfer function $H(Z)$ is expressed as,

$$H(Z) = H_1(Z) \cdot H_2(Z) \ldots\ldots H_k(Z) \qquad \ldots(1)$$

Here $H_1(Z)$, $H_2(Z)$ ... $H_k(Z)$ are second order polynomials. Then each subtransfer function (H_1 , H_2 ... etc.) can be realised using direct form-I or direct form-II structures. The total transfer function is obtained by connecting all second order subsystems in series as shown in Fig. I-3(a).

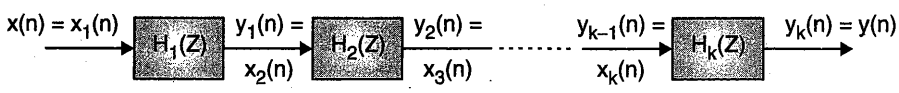


Fig. I-3(a) : Cascade form realization

Now we have the general difference equation for discrete time LTI system given by,

$$H(Z) = \frac{\sum_{k=0}^{M} b_k Z^{-k}}{1 + \sum_{k=1}^{N} a_k Z^{-k}} \qquad ...(2)$$

Since $H_1(Z)$, $H_2(Z)$ are second order polynomials ; we can write the second order differential equation for $H_k(Z)$ by putting M = N = 2 in Equation (2).

$$\therefore \quad H_k(Z) = \frac{\sum_{k=0}^{2} b_k Z^{-k}}{1 + \sum_{k=1}^{2} a_k Z^{-k}}$$

$$\therefore \quad H_k(Z) = \frac{b_0 Z^0 + b_1 Z^{-1} + b_2 Z^{-2}}{1 + a_1 Z^{-1} + a_2 Z^{-2}}$$

But $Z^0 = 1$

$$\therefore \quad H_k(Z) = \frac{b_0 + b_1 Z^{-1} + b_2 Z^{-2}}{1 + a_1 Z^{-1} + a_2 Z^{-2}} \qquad ...(3)$$

We can obtain direct form-II structure of Equation (3), similar to Fig. I-2(e). Now, Fig. I-2(e) shows the direct form-II realization for N = M ; that means for M stages. In this case we have second order equation. So by using Fig. I-2(e) we can draw the direct form structure for Equation (3) by putting M = N = 2. This structure is shown in Fig. I-3(b).

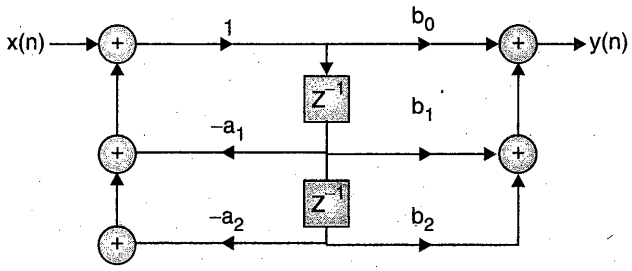


Fig. I-3(b) : Direct form-II realization of second order subsystem

So all such subsystems should be connected in series to obtain the cascade form realization of IIR system.

1.4 Parallel Form Structure :

We have the general difference equation for IIR systems given by,

$$H(Z) = \frac{\sum_{k=0}^{M} b_k Z^{-k}}{1 + \sum_{k=1}^{N} a_k Z^{-k}} \quad ...(1)$$

By using partial fraction expansion we can express overall transfer function $H(Z)$ as,

$$H(Z) = C + H_1(Z) + H_2(Z) + ... + H_k(Z) \quad ...(2)$$

Here 'C' is constant and $H_1(Z)$, $H_2(Z)$... $H_k(Z)$ are second order subsystems. The general block schematic of parallel form realization structure for IIR system is as shown in Fig. I-4.

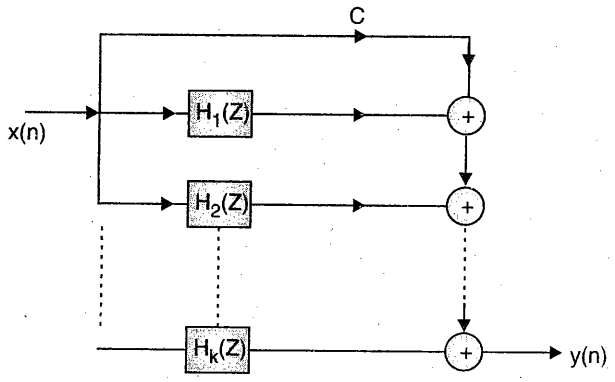


Fig. I-4 : Parallel form realization for IIR system

Here $H_1(Z)$, $H_2(Z)$...etc. can be realised by using direct form-I or direct form-II structures. Then all these structures are connected in parallel as shown in Fig. I-4 to obtain the parallel form realization for IIR system.

Applications :

The parallel form realization is generally used for high speed filtering applications. Since this is a parallel connection ; the processing of filtering operation is performed parallely.

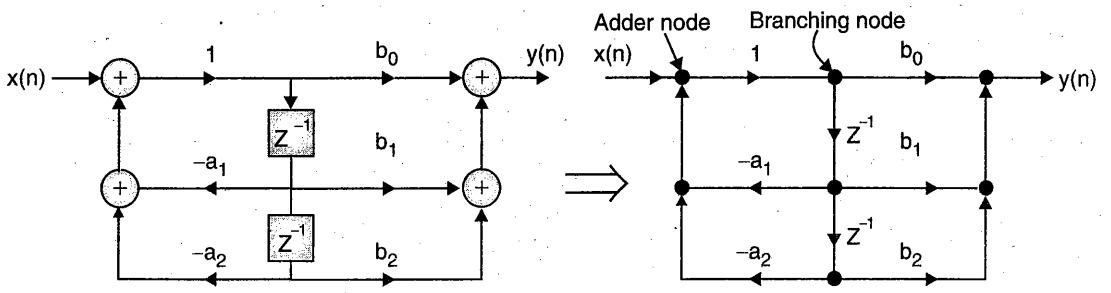
1.5 Representation of Structures using Signal Flow Graphs :

Basically a signal flow graph is graphical representation of the block diagram structure. Both the signal flow graph and the block diagram structure provide the same information.
For example, we will consider the second order subsystem. We know that the transfer function of

second order subsystem is given by,

$$H(Z) = \frac{b_0 + b_1 Z^{-1} + b_2 Z^{-2}}{1 + a_1 Z^{-1} + a_2 Z^{-2}} \quad ...(1)$$

We have already drawn the direct form-II realization of second order subsystem (Fig. H-7(b)). For the reference it is again drawn as shown in Fig. I-5(a) and its signal flow graph representation is shown in Fig. I-5(b).



(a) **(b)**

Fig. I-5 : Direct form-II realization and its signal flow graph

How to draw a signal flow graph ?

From the given block diagram realization; it is simple to draw the signal flow graph. The procedure is as follows :

1. Replace all adders by adder nodes.
2. Whenever there are different branches ; draw the branching node.
3. Keep the directions of arrows and the corresponding coefficients as it is.
4. Replace every delay element by simple transmittance branch. For that branch, write Z^{-1} to indicate the delay operation.

There is 1:1 correspondence between the signal flow graph and block diagram representation. And the signal flow graph representation is much simple compared to block diagram representation.

1.6 Transposed Structures :

If two digital filter structures have the same transfer function then they are called as equivalent structures. By using the transpose operation, we can obtain equivalent structure from a given realization structure.

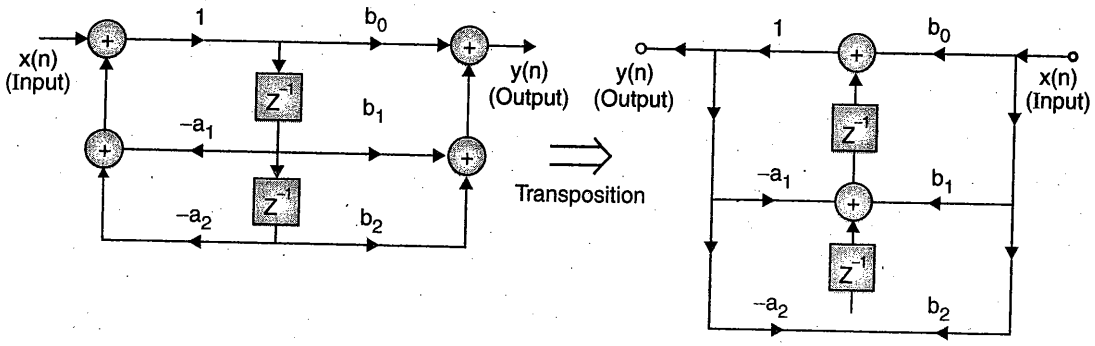
Transposition or flow graph reversal theorem :

If we reverse the directions of all branch transmittances and interchange input and output in the flow graph (or given structure) then the system transfer function remains unchanged. It is called as transposition theorem.

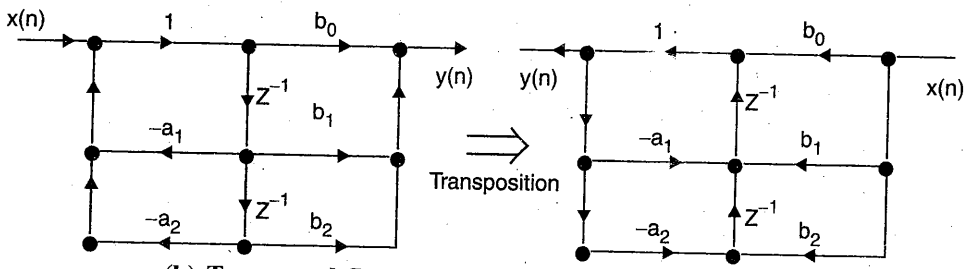
Procedure to obtain transposed structure and transposed flow graph :

1. Reverse all signal flow graph directions.
2. Change branching nodes into adders and vice-versa.
3. Interchange input and output.

The transposed structure for second order subsystem is shown in Fig. I-6(a) and the corresponding transposed flow graph is shown in Fig. I-6(b).



(a) Transposed structure for second order subsystem



(b) Transposed flow graph for second order subsystem

Fig. I-6

1.7 Feedback in IIR Systems :

We have studied that, in case of IIR systems, feedback connection is present. That means output signal is fed back to the input side. This is also called as a loop. Generally this feedback connection is required to generate infinitely long impulse responses.

First we will consider a system where feedback is not used. Consider the difference equation,

$$y(n) = x(n) + x(n-1) + x(n-2) \quad ...(1)$$

The block diagram representing Equation (1) is shown in Fig. I-7(a).

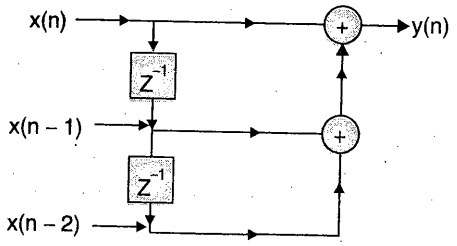


Fig. I-7(a)

As shown in Fig. I-7(a), a feedback path is not present. All the signals simply travels in the forward direction towards output. In this case an impulse response is no longer than the total number of delays in the system. Thus we can conclude that, if loops (feedback connections) are not present then the system function has only zeros (except for pole at z = 0). To prove this, we will take Z transform of Equation (1),

$$Y(Z) = X(Z) + Z^{-1}X(Z) + Z^{-2}X(Z)$$

$$\therefore \quad Y(Z) = X(Z)[1 + Z^{-1} + Z^{-2}]$$

$$\therefore \quad \frac{Y(Z)}{X(Z)} = H(Z) = 1 + Z^{-1} + Z^{-2}$$

Multiplying and dividing by Z^2 we get,

$$\therefore \quad H(Z) = \frac{Z^2 + Z + 1}{Z^2} \qquad ...(2)$$

Equation (2) shows that the system has only zeros except the poles at origin. The poles are not present at any other locations.

Now consider a system with feedback connection. Let us consider a difference equation,

$$y(n) = 2y(n-1) + x(n) \qquad ...(3)$$

Equation (3) indicates that, the delayed output, $y(n-1)$ is multiplied by constant value '2' and it is added with input $x(n)$. The block diagram representation and signal flow graph is shown in Fig. I-7(b).

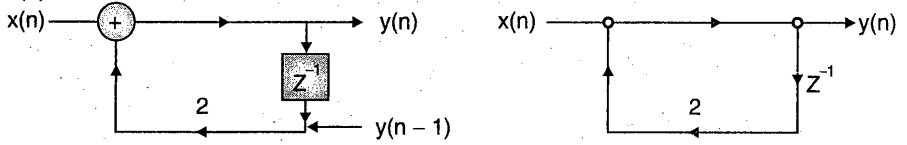


Fig. I-7(b)

If input is an impulse sequence then the single input sample continually recirculates in the feedback loop. Every time an amplitude increases due to multiplication by constant 2.

Now taking Z transform of both sides of Equation (3) we get,

$$Y(Z) = 2Z^{-1}Y(Z) + X(Z)$$

$$\therefore \quad Y(Z) - 2Z^{-1}Y(Z) = X(Z)$$

$$\therefore \quad Y(Z)[1 - 2Z^{-1}] = X(Z)$$

$$\therefore \quad \frac{Y(Z)}{X(Z)} = H(Z) = \frac{1}{1 - 2Z^{-1}}$$

$$\therefore \quad H(Z) = \frac{Z}{Z - 2} \qquad ...(4)$$

Impulse response h(n) is obtained by taking inverse Z transform of Equation (4).

$$\therefore \quad h(n) = 2^n u(n) \qquad ...(5)$$

Equation (5) indicates that, because of the feedback connection an infinitely long impulse response is obtained.

Equation (4) indicates that the system function has a pole other than origin. This indicates the presence of feedback loop. But a network with feedback connection will not produce infinitely long impulse response if the poles of system function cancels with zeros.

Solved Problems :

Prob. 1 : Obtain direct form I and II realization of a system described by,

$$y(n)-\frac{3}{4}y(n-1)+\frac{1}{8}y(n-2)=x(n)+\frac{1}{2}x(n-1).$$

Soln. :

Calculation of H (Z) :

Given equation is, $$y(n)-\frac{3}{4}y(n-1)+\frac{1}{8}y(n-2) = x(n)+\frac{1}{2}x(n-1) \quad ...(1)$$

Taking Z-transform of both sides we get,

$$Y(Z)-\frac{3}{4}Z^{-1}Y(Z)+\frac{1}{8}Z^{-2}Y(Z) = X(Z)+\frac{1}{2}Z^{-1}X(Z)$$

$$\therefore \quad Y(Z)\left[1-\frac{3}{4}Z^{-1}+\frac{1}{8}Z^{-2}\right] = X(Z)\left[1+\frac{1}{2}Z^{-1}\right] \quad ...(2)$$

Now the transfer function H (Z) is,

$$H(Z) = \frac{Y(Z)}{X(Z)}$$

Thus from Equation (2) we get,

$$H(Z) = \frac{1+\frac{1}{2}Z^{-1}}{1-\frac{3}{4}Z^{-1}+\frac{1}{8}Z^{-2}} \quad ...(3)$$

We will obtain direct form-I and II structures using cascade connection. This cascade connection is shown in Fig. I-8(a).

Fig. I-8(a) : $H(Z) = H_1(Z) \cdot H_2(Z)$

Direct form-I structure :

We have, $$H(Z) = H_1(Z) \cdot H_2(Z)$$

Let $$H_1(Z) = 1+\frac{1}{2}Z^{-1}$$

But $$H_1(Z) = \frac{\text{Output}}{\text{Input}} = \frac{W(Z)}{X(Z)}$$

$$\therefore \quad \frac{W(Z)}{X(Z)} = 1+\frac{1}{2}Z^{-1}$$

$$\therefore \quad W(Z) = X(Z)\left[1+\frac{1}{2}Z^{-1}\right]$$

$$\therefore \quad W(Z) = X(Z) + \frac{1}{2}Z^{-1}X(Z) \qquad ...(4)$$

Taking inverse Z-transform (IZT) of Equation (4) we get,

$$w(n) = x(n) + \frac{1}{2}x(n-1) \qquad ...(5)$$

The direct form-I realization of Equation (5) is as shown in Fig. I-8(b).

Now let, $H_2(Z) = \dfrac{1}{1 - \frac{3}{4}Z^{-1} + \frac{1}{8}Z^{-2}}$

But $H_2(Z) = \dfrac{\text{Output}}{\text{Input}} = \dfrac{Y(Z)}{W(Z)}$

$$\therefore \quad \frac{Y(Z)}{W(Z)} = \frac{1}{1 - \frac{3}{4}Z^{-1} + \frac{1}{8}Z^{-2}}$$

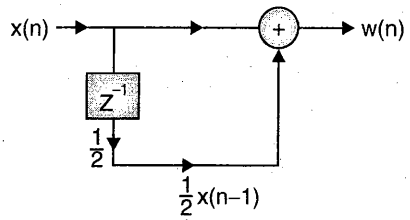


Fig. I-8(b) : Direct form-I realization of $H_1(Z)$

$$\therefore \quad Y(Z)\left[1 - \frac{3}{4}Z^{-1} + \frac{1}{8}Z^{-2}\right] = W(Z)$$

$$\therefore \quad Y(Z) - \frac{3}{4}Z^{-1}Y(Z) + \frac{1}{8}Z^{-2}Y(Z) = W(Z)$$

$$\therefore \quad Y(Z) = W(Z) + \frac{3}{4}Z^{-1}Y(Z) - \frac{1}{8}Z^{-2}Y(Z) \qquad ...(6)$$

Taking IZT of Equation (6),

$$y(n) = w(n) + \frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) \qquad ...(7)$$

The direct form-I realization of Equation (7) is shown in Fig. I-8(c).

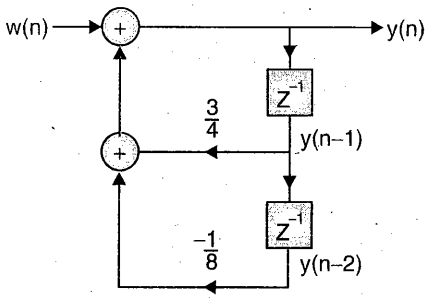


Fig. I-8(c) : Direct form-I realization of $H_2(Z)$

Now direct form-I realization of total transfer function H (Z) is obtained by cascading Fig. I-8(b) and Fig. I-8(c). This is shown in Fig. I-8(d).

Direct form-II realization :

The direct form-II realization is obtained by interchanging the positions of $H_1(Z)$ and $H_2(Z)$. That means by interchanging block-1 and block-2 of Fig. I-8(d). It is shown in Fig. I-8(e).

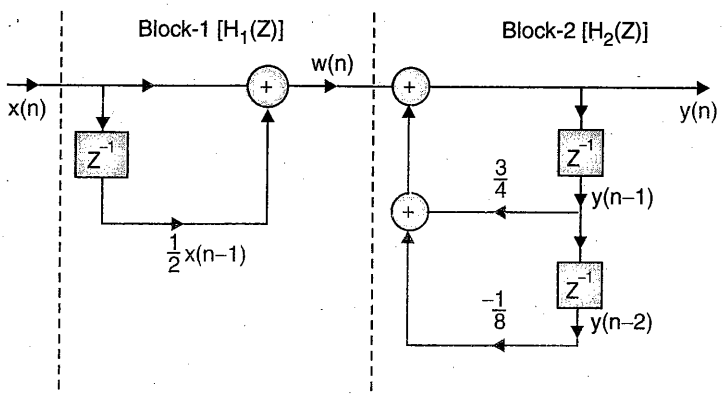


Fig. I-8(d) : Direct form-I realization of H (Z)

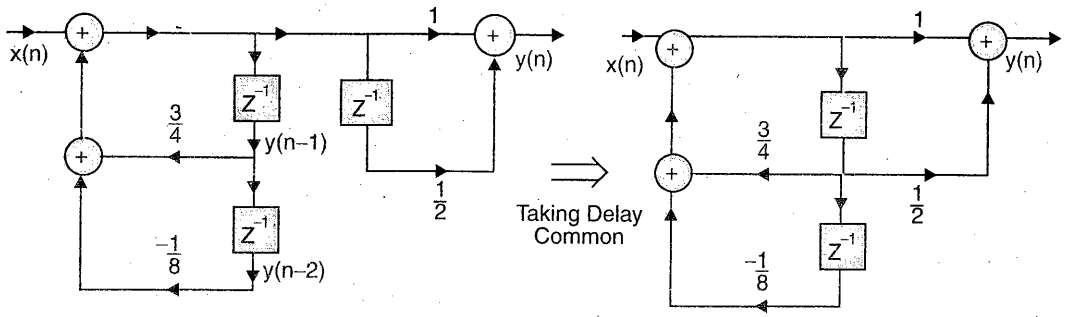


Fig. I-8(e) : Direct form-II realization

Prob. 2 : Consider a causal LTI system with system function,

$$H(Z) = \frac{1 + \frac{1}{5}Z^{-1}}{\left(1 - 0.5\,Z^{-1} + \frac{1}{3}Z^{-2}\right)\left(1 + 0.25\,Z^{-1}\right)}$$

Draw the signal flow graph for implementation of the system using direct form-II.

Soln. : The given transfer function is,

$$H(Z) = \frac{1+\frac{1}{5}Z^{-1}}{\left(1-0.5\,Z^{-1}+\frac{1}{3}Z^{-2}\right)(1+0.25\,Z^{-1})} \quad ..(1)$$

It can be simplified as follows :

$$H(Z) = \frac{1+\frac{1}{5}Z^{-1}}{1-0.5\,Z^{-1}+\frac{1}{3}Z^{-2}+0.25\,Z^{-1}-0.125\,Z^{-2}+0.0833\,Z^{-3}}$$

$$\therefore \quad H(Z) = \frac{1+\frac{1}{5}Z^{-1}}{1-0.25\,Z^{-1}+0.208\,Z^{-2}+0.0833\,Z^{-3}}$$

For the direct form-II structure, poles are realized first and then the zeros are realized. We can write,

$$H(Z) = H_1(Z)\cdot H_2(Z)$$

Here $H_1(Z)$ = All pole system $= \dfrac{1}{1-0.25\,Z^{-1}+0.208\,Z^{-2}+0.0833\,Z^{-3}}$

and $H_2(Z)$ = All zero system $= 1+0.2\,Z^{-1}$

consider $H_1(Z)$ terms.

We have $H_1(Z) = \dfrac{\text{Output}}{\text{Input}} = \dfrac{W(Z)}{X(Z)}$

$$\therefore \quad \frac{W(Z)}{X(Z)} = \frac{1}{1-0.25\,Z^{-1}+0.208\,Z^{-2}+0.0833\,Z^{-3}}$$

$$\therefore\ W(Z)\left[1-0.25\,Z^{-1}+0.208\,Z^{-2}+0.0833\,Z^{-3}\right] = X(Z)$$

$$\therefore\ W(Z)-0.25\,Z^{-1}W(Z)+0.208\,Z^{-2}W(Z)+0.0833\,Z^{-3}W(Z) = X(Z)$$

$$\therefore\ W(Z) = X(Z)+0.25\,Z^{-1}W(Z)-0.208\,Z^{-2}W(Z)-0.0833\,Z^{-3}W(Z) \quad ...(2)$$

Taking IZT of Equation (2) we get,

$$w(n) = x(n)+0.25\,w(n-1)-0.208\,w(n-2)-0.0833\,w(n-3) \quad ...(3)$$

The direct form realization of Equation (3) is shown in Fig. I-9(a).

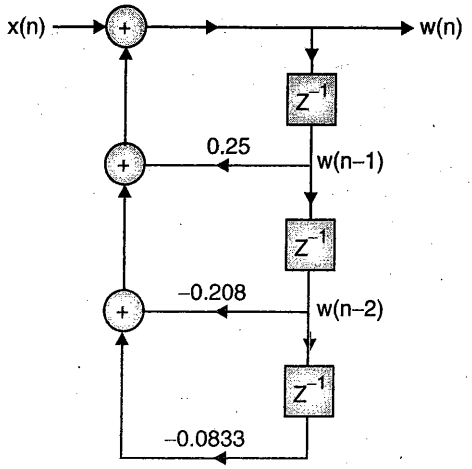


Fig. I-9(a) : Direct form realization of $H_1(Z)$

Now consider $H_2(Z)$ that means all zero system.

$$H_2(Z) = 1 + 0.2\,Z^{-1}$$

$$\text{But} \quad H_2(Z) = \frac{\text{Output}}{\text{Input}} = \frac{Y(Z)}{W(Z)}$$

$$\therefore \quad \frac{Y(Z)}{W(Z)} = 1 + 0.2\,Z^{-1}$$

$$\therefore \quad Y(Z) = W(Z) + 0.2\,Z^{-1}\,W(Z) \qquad \ldots(4)$$

Taking IZT of Equation (4) we get,

$$y(n) = w(n) + 0.2\,w(n-1) \qquad \ldots(5)$$

The direct form realization of Equation (5) is shown in Fig. I-9(b).

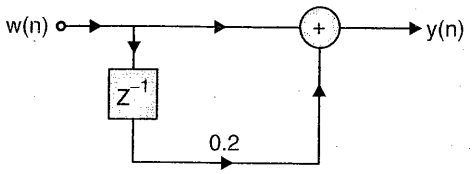


Fig. I-9(b) : Direct form realization of $H_2(Z)$

Now direct form-II realization is obtained by connecting Fig. I-9(a) and Fig. I-9(b) in series (cascade connection). It is shown in Fig. I-9(c).

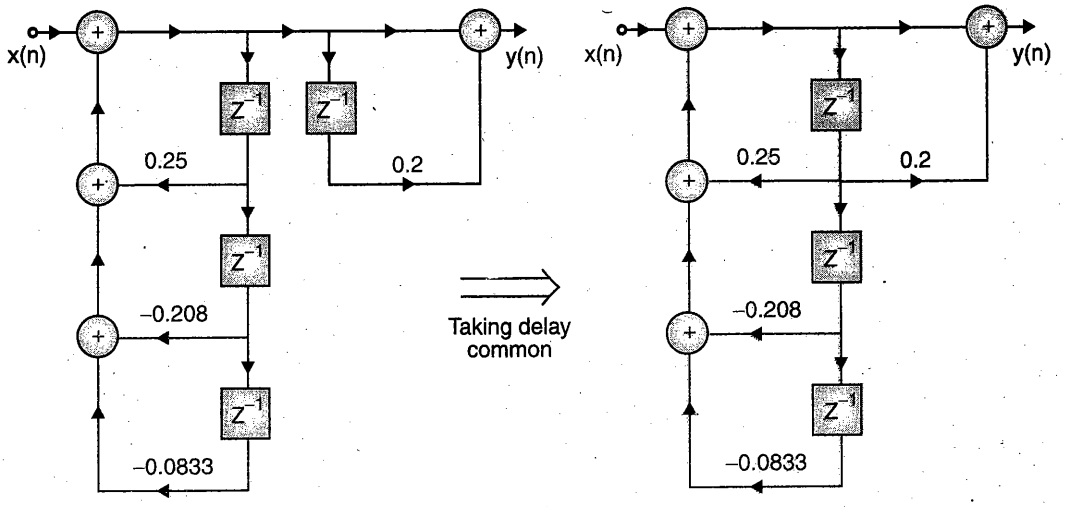


Fig. I-9(c) : Direct form-II realization

Note : We can also obtain direct form-II realization by first drawing direct form-I structure. To draw direct form-I structure take $H_1(Z)$ = all zero system. In this example take $H_1(Z) = 1 + 0.2\,Z^{-1}$ and take $H_2(Z)$ = all pole system, so here $H_2(Z) = \dfrac{1}{1 - 0.25\,Z^{-1} + 0.208\,Z^{-2} + 0.0833\,Z^{-3}}$. Then obtain direct form-I structure. By interchanging the positions of $H_1(Z)$ and $H_2(Z)$; obtain direct form-II structure. But this is tedious method. So, whenever it is asked to obtain direct form-II structure, the take $H_1(Z)$ = all pole system and $H_2(Z)$ = all zero system. Then cascade $H_1(Z)$ and $H_2(Z)$ to obtain direct form-II structure.

Now the signal flow graph is as shown in Fig. I-9(d).

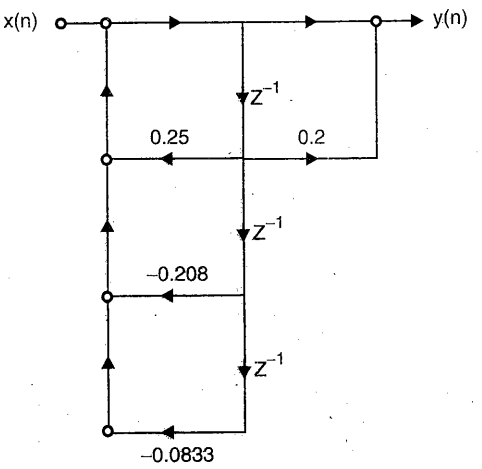


Fig. I-9(d) : Signal flow graph

Shortcut method to draw direct From-II structure :

In problem (1) and (2) we have obtained direct form-II structure by considering the cascade connection. That means by using $H(Z) = H_1(Z) \cdot H_2(Z)$. Now we will study the shortcut method to obtain direct form-II structure. By using this shortcut method we will obtain direct form-II structure for problem (2).

In problem (2) the given $H(Z)$ is, $$H(Z) = \frac{1 + \frac{1}{5}Z^{-1}}{1 - 0.5Z^{-1} + 0.208Z^{-2} + 0.083Z^{-3}}$$

The different steps are as follows :

Step I :

Write down three terms namely input $x(n)$, middle term $w(n)$ and output $y(n)$ as shown in Fig. I-10(a).

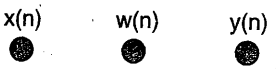


Fig. I-10(a) : Step-I

Step II :

Check the maximum power of Z inverse present in the numerator or denominator. It represents the maximum delay. For the given $H(Z)$ it is '-3'. This indicates that total '3' delay boxes are required. Draw three Z^{-1} boxes below $w(n)$ as shown in Fig. I-10(b).

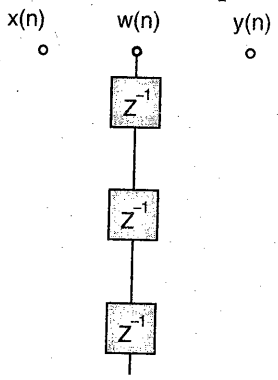


Fig. I-10(b) : Step-II

Step III : Now consider the numerator term. Check the constant multiplier value. It is 0.2 and it is multiplied by Z^{-1}. So write this value at the R.H.S. and below first Z^{-1} box as shown in Fig. I-10(c).

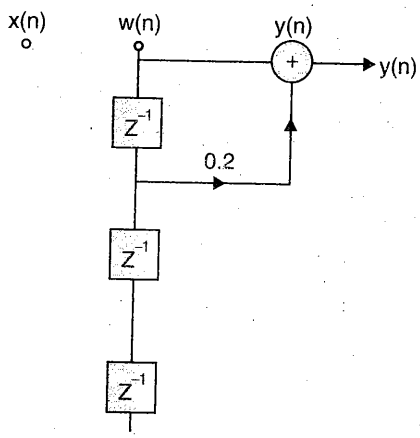


Fig. I-10(c) : Step-III

Then using the adder complete the connection as shown in Fig. I-10(c).

Step IV : Now consider the denominator term. Write down the constant multiplier values at the L.H.S. part of diagram. While writing these values, change the sign. The original constant multipliers are '– 0.5' for Z^{-1}, 0.208 for Z^{-2} and 0.083 for Z^{-3}. So change the signs of these values and write down at the L.H.S. near the corresponding delay box. Then by using adders complete the connection as shown in Fig. I-10(d).

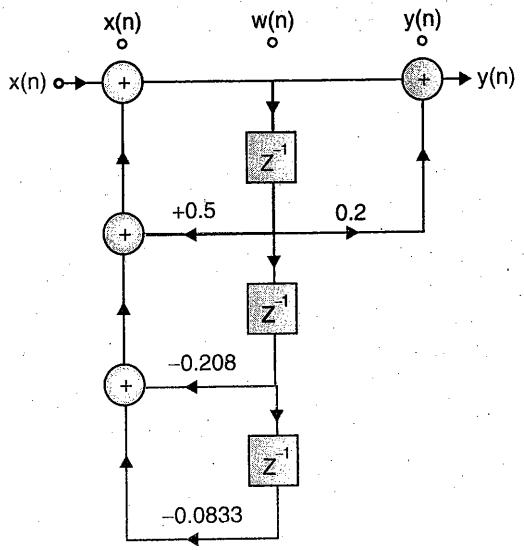


Fig. I-10(d) : Step-IV

This diagram is same as Fig. I-10(c). So by using this simple method; we can directly draw the direct form-II structure.

Now applying this method we will obtain direct form-II structure for problem No. (1). In this problem we have obtained the equation of H (Z) as,

$$H(Z) = \frac{1+\frac{1}{2}Z^{-1}}{1-\frac{3}{4}Z^{-1}+\frac{1}{8}Z^{-2}}.$$

Here maximum value of Z inverse is '2'. So two delay elements are required. For the numerator the constant multiplier value is $\frac{1}{2}$. While for the denominator term constant multiplier values are $-\frac{3}{4}$ for Z^{-1} and $\frac{1}{8}$ for Z^{-2}. After changing the signs we get $\frac{3}{4}$ for Z^{-1} and $-\frac{1}{8}$ for Z^{-2}. So we can draw direct form-II structure as shown in Fig. I-10(e).

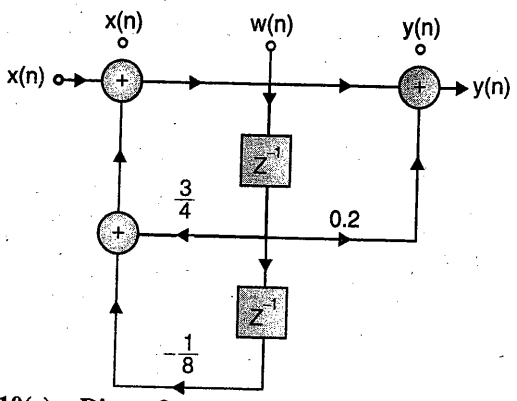


Fig. I-10(e) : Direct form-II structure for problem No. (1)

Prob. 3 : Develop direct form-II realization of the transfer function.

$$H(Z) = \frac{3 + 3.6\,Z^{-1} + 0.6\,Z^{-2}}{1 + 0.1\,Z^{-1} - 0.2\,Z^{-2}}$$

Is it advantageous than direct form-I ?

Soln. : For the direct form-II structure we have to realize the poles first and then zeros.

$$\therefore \quad \text{Let} \quad H_1(Z) = \text{All pole system} = \frac{1}{1 + 0.1\,Z^{-1} - 0.2\,Z^{-2}}$$

$$\text{Now} \quad H_1(Z) = \frac{\text{Output}}{\text{Input}} = \frac{W(Z)}{X(Z)}$$

$$\therefore \quad \frac{W(Z)}{X(Z)} = \frac{1}{1 + 0.1\,Z^{-1} - 0.2\,Z^{-2}}$$

$$\therefore \quad W(Z)\,[1 + 0.1\,Z^{-1} - 0.2\,Z^{-2}] = X(Z)$$

$$\therefore \quad W(Z) + 0.1\,Z^{-1}\,W(Z) - 0.2\,Z^{-2}\,W(Z) = X(Z)$$

$$\therefore \quad W(Z) = X(Z) - 0.1\,Z^{-1}\,W(Z) + 0.2\,Z^{-2}\,W(Z) \qquad \ldots(1)$$

Taking IZT of Equation (1),

$$w(n) = x(n) - 0.1\,w(n-1) + 0.2\,w(n-2) \qquad \ldots(2)$$

The direct form implementation of Equation (2) is shown in Fig. I-11(a)

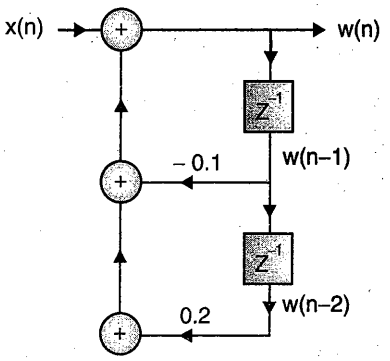


Fig. I-11(a) : Direct form realization of $H_1(Z)$

Now $H_2(Z)$ = All zero system = $3 + 3.6\,Z^{-1} + 0.6\,Z^{-2}$

But $H_2(Z) = \dfrac{\text{Output}}{\text{Input}} = \dfrac{Y(Z)}{W(Z)}$

$\therefore \quad \dfrac{Y(Z)}{W(Z)} = 3 + 3.6\,Z^{-1} + 0.6\,Z^{-2}$

$\therefore \quad Y(Z) = 3\,W(Z) + 3.6\,Z^{-1}\,W(Z) + 0.6\,Z^{-2}\,W(Z)$ ...(3)

Taking IZT of Equation (3) we get,

$$y(n) = 3\,w(n) + 3.6\,w(n-1) + 0.6\,w(n-2) \quad ...(4)$$

The direct form implementation of Equation (4) is shown in Fig. I-11(b).

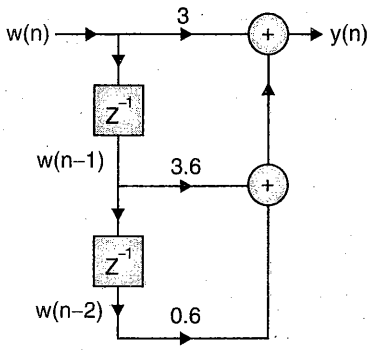


Fig. I-11(b) : Direct form realization of $H_2(Z)$

The direct form-II realization is obtained by cascading $H_1(Z)$ and $H_2(Z)$. It is shown in Fig. I-11(c).

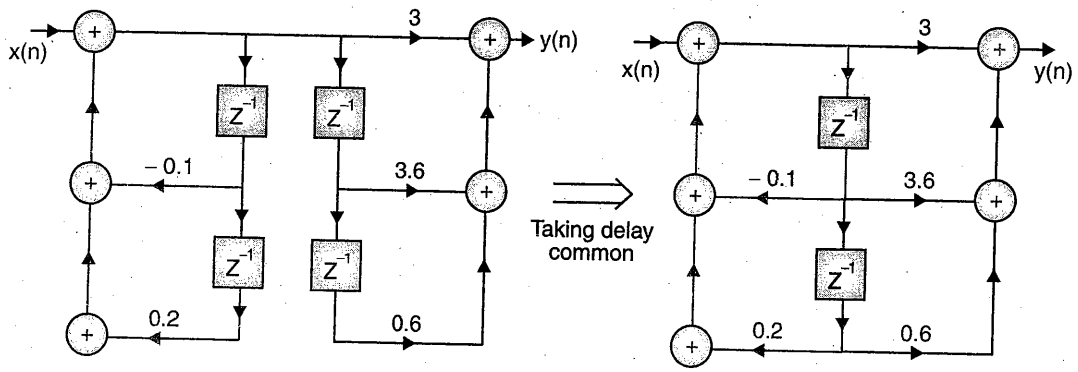


Fig. I-11(c) : Direct form-II realization

Now compared to direct form-I structure; nearly half number of delays are required for direct form-II realization. That means, it requires minimum number of storage elements. So it is advantageous than direct form-I structure.

Quick check :

We can check the answer by using shortcut method to obtain direct form-II realization.

Here maximum delays are 2. The coefficients of numerator are 3.6 for Z^{-1} and 0.6 for Z^{-2}. Now changing the signs, we get the coefficients of denominator as -0.1 for Z^{-1} and $+0.2$ for Z^{-2}. Thus direct form-II realization is shown in Fig. I-11(d).

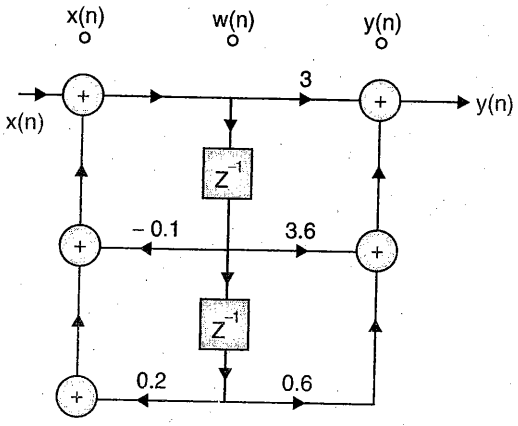


Fig. I-11(d) : Direct form-II realization using shortcut method

Thus Fig. I-11(c) and Fig. I-11(d) are same.

Prob. 4 : Obtain direct form-II implementation of LTI system with transfer function.

$$H(Z) = \frac{1+\frac{5}{6}Z^{-1}+\frac{1}{6}Z^{-2}}{1-\frac{1}{2}Z^{-1}-\frac{1}{2}Z^{-2}}$$

Soln. : For direct form-II structure. All pole system is realized first and then all zero system is realized.

$$\therefore \quad \text{Let} \quad H_1(Z) = \frac{1}{1-\frac{1}{2}Z^{-1}-\frac{1}{2}Z^{-2}}$$

$$\text{But} \quad H_1(Z) = \frac{W(Z)}{X(Z)}$$

$$\therefore \quad \frac{W(Z)}{X(Z)} = \frac{1}{1-\frac{1}{2}Z^{-1}-\frac{1}{2}Z^{-2}}$$

$$\therefore W(Z)-\frac{1}{2}Z^{-1}W(Z)-\frac{1}{2}Z^{-2}W(Z) = X(Z)$$

$$\therefore \quad W(Z) = X(Z)+\frac{1}{2}Z^{-1}W(Z)+\frac{1}{2}Z^{-2}W(Z) \qquad ...(1)$$

Taking IZT of Equation (1),

$$w(n) = x(n)+\frac{1}{2}w(n-1)+\frac{1}{2}w(n-2) \qquad ...(2)$$

The direct form realization of Equation (2) is shown in Fig. I-12(a).

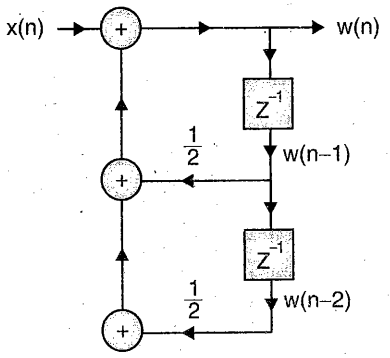


Fig. I-12(a) : Direct form realization of $H_1(Z)$

Now consider all zero system.

$$\therefore \quad H_2(Z) = 1+\frac{5}{6}Z^{-1}+\frac{1}{6}Z^{-2}$$

$$\text{But} \quad H_2(Z) = \frac{\text{Output}}{\text{Input}} = \frac{Y(Z)}{W(Z)}$$

$$\therefore \quad \frac{Y(Z)}{W(Z)} = 1+\frac{5}{6}Z^{-1}+\frac{1}{6}Z^{-2}$$

$$\therefore \quad Y(Z) = W(Z)+\frac{5}{6}Z^{-1}W(Z)+\frac{1}{6}Z^{-2}W(Z) \qquad ...(3)$$

Taking IZT of Equation (3),

$$y(n) = w(n)+\frac{5}{6}w(n-1)+\frac{1}{6}w(n-2) \qquad ...(4)$$

The direct form implementation of Equation (4) is shown in Fig. I-12(b).

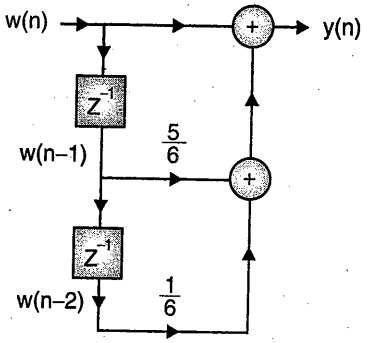


Fig. I-12(b) : Direct form realization of $H_2(Z)$

Now direct form-II realization is obtained by cascading $H_1(Z)$ and $H_2(Z)$ as shown in Fig. I-12(c).

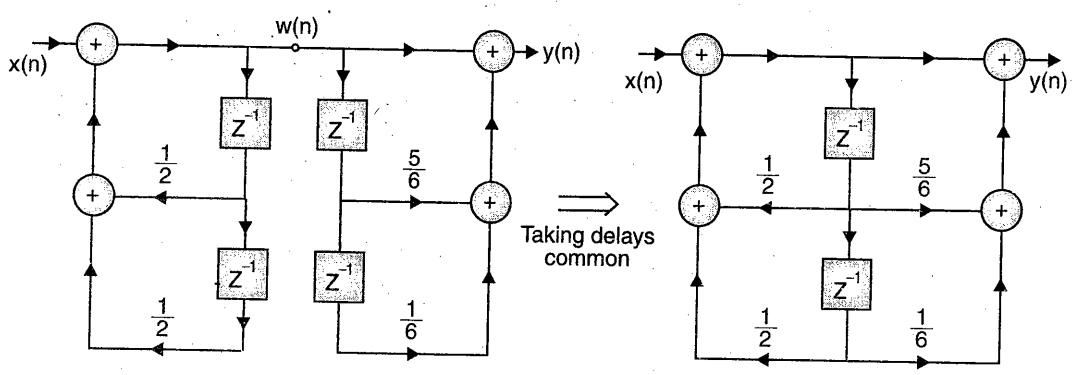


Fig. I-12(c) : Direct form-II realization

Note : We can easily check the answer using shortcut method.

Prob. 5 : A certain discrete time LTI filter has the following data :
Poles are at 0.2 and 0.6.
Zeros are at – 0.4 and origin.
Gain of filter is 5.
Show direct form-II realization and cascade form realization.

Soln. :

Given Zeros $\Rightarrow$ $Z_1 = 0$ and $Z_2 = -0.4$

Poles $\Rightarrow$ $P_1 = 0.2$ and $P_2 = 0.6$

Gain = $k = 5$

The transfer function of system is expressed as,

$$H(Z) = k \cdot \frac{(Z-Z_1)(Z-Z_2)}{(Z-P_1)(Z-P_2)}$$

$$\therefore \quad H(Z) = \frac{5Z(Z+0.4)}{(Z-0.2)(Z-0.6)} \qquad ...(1)$$

$$\therefore \quad H(Z) = \frac{5Z^2+2Z}{Z^2-0.8Z+0.12}$$

Converting into negative powers of Z we get,

$$H(Z) = \frac{5+2Z^{-1}}{1-0.8Z^{-1}+0.12Z^{-2}} \qquad ...(2)$$

Direct form-II realization :

Using shortcut method, direct form-II realization is shown in Fig. I-13.

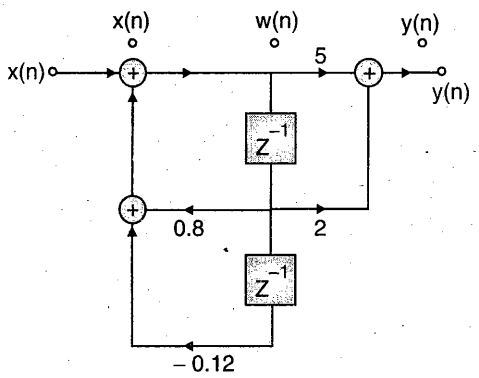


Fig. I-13

Cascade form realization :

According to Equation (1),

$$H(Z) = \frac{5Z(Z+0.4)}{(Z-0.2)(Z-0.6)}$$

$$\therefore \quad H(Z) = \frac{5Z}{Z-0.2} \cdot \frac{Z+0.4}{Z-0.6} \qquad ...(3)$$

$$\text{Let} \quad H_1(Z) = \frac{5Z}{Z-0.2} = \frac{5}{1-0.2Z^{-1}}$$

$$\text{and} \quad H_2(Z) = \frac{Z+0.4}{Z-0.6} = \frac{1+0.4Z^{-1}}{1-0.6Z^{-1}}$$

Using shortcut method $H_1(Z)$ and $H_2(Z)$ are realized as shown in Fig. I-14 and I-15.

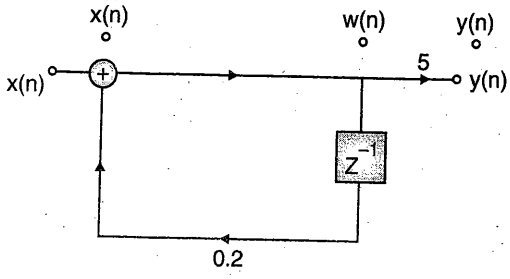


Fig. I-14

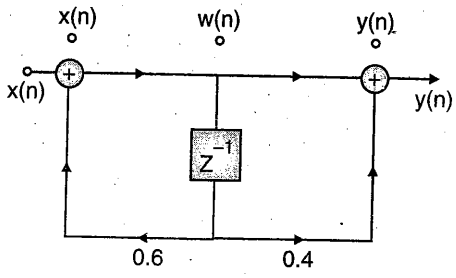


Fig. I-15

The cascade realization is shown in Fig. I-16.

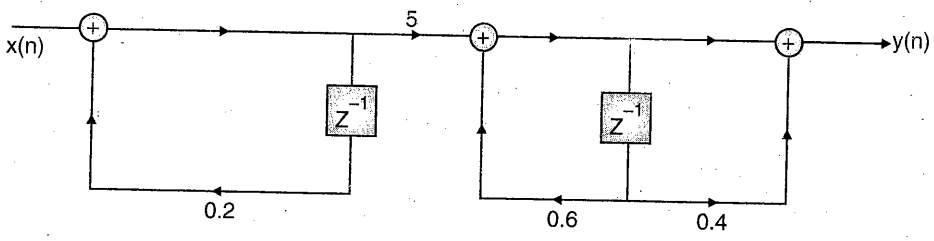


Fig. I-16

Prob. 6 : Determine direct-form-II realization for each of the following LTI system

(i) $2y(n)+y(n-1)-4y(n-3)=x(n)+3x(n-1)$

(ii) $y(n)=x(n)-x(n-1)+2x(n-2)-3x(n-4)$

Soln. :

(i) Given,

$$2y(n)+y(n-1)-4y(n-3) = x(n)+3x(n-1)$$

Taking Z-transform of both sides.

$$2Y(Z)+Z^{-1}Y(Z)-4Z^{-3}Y(Z) = X(Z)+3Z^{-1}X(Z)$$

$$\therefore \quad Y(Z)\left[2+Z^{-1}-4Z^{-3}\right] = X(Z)[1+3Z^{-1}]$$

$$\therefore \quad \frac{Y(Z)}{X(Z)} = H(Z) = \frac{1+3Z^{-1}}{2+Z^{-1}-4Z^{-3}}$$

Using short-cut method, the direct form-II structure is drawn as shown in Fig. I-17.

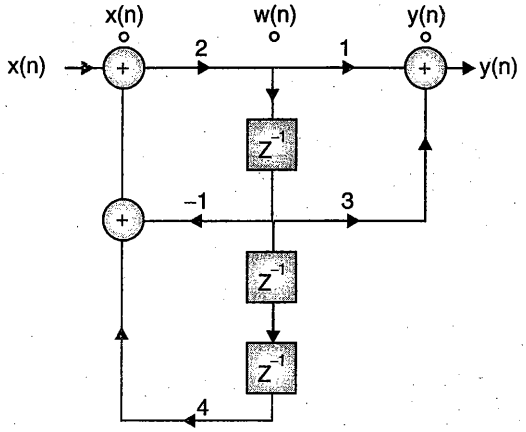


Fig. I-17

(i) We have, $y(n) = x(n) - x(n-1) + 2x(n-2) - 3x(n-4)$

Taking Z-transform of both sides we get,

$$Y(Z) = X(Z) - Z^{-1}X(Z) + 2Z^{-2}X(Z) - 3Z^{-4}X(Z)$$

$$\therefore \quad Y(Z) = X(Z)[1 - Z^{-1} + 2Z^{-2} - 3Z^{-4}]$$

$$\therefore \quad \frac{Y(Z)}{X(Z)} = H(Z) = 1 - Z^{-1} + 2Z^{-2} - 3Z^{-4}$$

Using short-cut method, the direct form-II realization is obtained as shown in Fig. I-18.

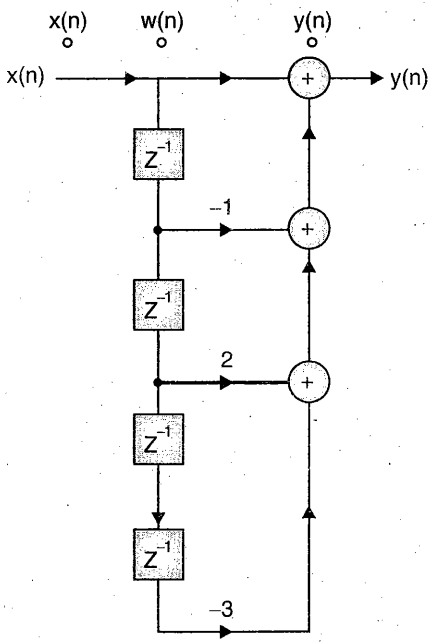


Fig. I-18

Prob. 7 : The transfer function of discrete time causal system is given by,

$$H(Z) = \frac{1 - Z^{-1}}{1 - 0.2\,Z^{-1} - 0.15\,Z^{-2}}$$

Draw cascade and parallel realization.

Soln. : The given transfer function is,

$$H(Z) = \frac{1 - Z^{-1}}{1 - 0.2\,Z^{-1} - 0.15\,Z^{-2}} \quad ..(1)$$

Cascade realization :

First we will obtain the roots of denominator, by using the equation,

$$\text{roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here a = 1, b = – 0.2 and c = – 0.15

$$\therefore \quad \text{roots} = \frac{0.2 \pm \sqrt{0.04 + 0.6}}{2} = \frac{0.2 \pm 0.8}{2} = 0.5, -0.3$$

Thus $(1 - 0.2\,Z^{-1} - 0.15\,Z^{-2}) = (1 - 0.5\,Z^{-1})(1 + 0.3\,Z^{-1})$

Putting this value in Equation (1) we get,

$$H(Z) = \frac{1 - Z^{-1}}{(1 - 0.5\,Z^{-1})(1 + 0.3\,Z^{-1})} \quad ...(2)$$

For the cascade realization we have to express H (Z) as,

$$H(Z) = H_1(Z) \cdot H_2(Z)$$

Let $H_1(Z) = \frac{1 - Z^{-1}}{1 - 0.5\,Z^{-1}}$ ..(3)

and let $H_2(Z) = \frac{1}{1 + 0.3\,Z^{-1}}$ ...(4)

Using shortcut method, we can obtain direct form-II structures for $H_1(Z)$ and $H_2(Z)$ as shown in Fig. I-19(a) and I-19(b) respectively.

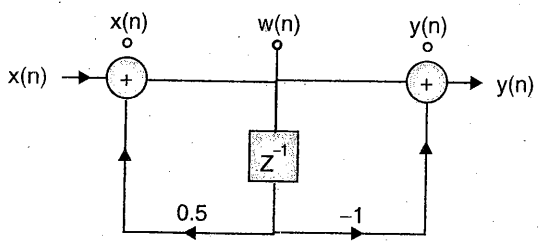


Fig. I-19(a) : Direct form-II realization of $H_1(Z)$

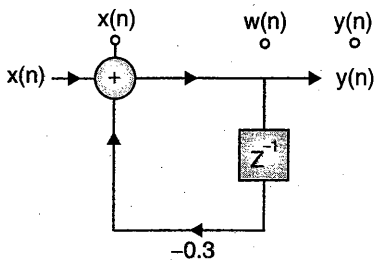


Fig. I-19(b) : Direct form-II realization of $H_2(Z)$

Now for the cascade connection we will have to connect $H_1(Z)$ and $H_2(Z)$ in series. This series connection is shown in Fig. I-19(c).

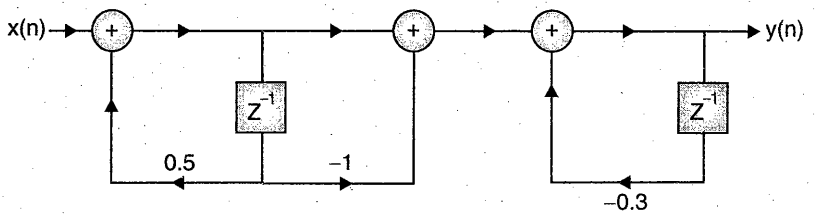


Fig. I-19(c) : Cascade realization

Parallel realization :

For the parallel form realization we have to use partial fraction expansion (PFE). It is similar to PFE which we have studied in Z transformation.

The given transfer function is,

$$H(Z) = \frac{1 - Z^{-1}}{1 - 0.2\,Z^{-1} - 0.15\,Z^{-2}}$$

To convert this equation in terms of positive powers of Z; multiply numerator and denominator by Z^2.

$$\therefore \quad H(Z) = \frac{Z^2(1 - Z^{-1})}{Z^2(1 - 0.2\,Z^{-1} - 0.15\,Z^{-2})}$$

$$\therefore \quad H(Z) = \frac{Z^2 - Z}{Z^2 - 0.2\,Z - 0.15} = \frac{Z(Z-1)}{Z^2 - 0.2\,Z - 0.15}$$

$$\therefore \quad \frac{H(Z)}{Z} = \frac{Z-1}{Z^2 - 0.2\,Z - 0.15} \qquad \ldots(5)$$

Equation (5) is in the proper form. The roots of denominator are 0.5 and − 0.3. That means poles are at, $P_1 = 0.5$ and $P_2 = -0.3$

$$\therefore \quad \frac{H(Z)}{Z} = \frac{Z-1}{(Z-0.5)(Z+0.3)} \qquad \ldots(6)$$

In the partial fraction expansion form Equation (6) can be written as,

$$\frac{H(Z)}{Z} = \frac{A_1}{(Z-0.5)} + \frac{A_2}{(Z+0.3)} \quad ...(7)$$

Now we will calculate the values of A_1 and A_2.

$$A_1 = (Z-p_1)\frac{H(Z)}{Z}\Big|_{Z=p_1}$$

$$\therefore \quad A_1 = \frac{(Z-0.5)(Z-1)}{(Z-0.5)(Z+0.3)}\Big|_{Z=0.5}$$

$$\therefore \quad A_1 = \frac{0.5-1}{0.5+0.3}$$

$$\therefore \quad A_1 = -0.625 \quad ...(8)$$

$$\text{Now} \quad A_2 = (Z-p_2)\frac{H(Z)}{Z}\Big|_{Z=p_2}$$

$$\therefore \quad A_2 = \frac{(Z+0.3)(Z-1)}{(Z-0.5)(Z+0.3)}\Big|_{Z=-0.3}$$

$$\therefore \quad A_2 = \frac{-0.3-1}{-0.3-0.5}$$

$$\therefore \quad A_2 = 1.625 \quad ..(9)$$

Putting the values of A_1 and A_2 in Equation (7),

$$\frac{H(Z)}{Z} = \frac{-0.625}{Z-0.5} + \frac{1.625}{Z+0.3}$$

$$\therefore \quad H(Z) = \frac{-0.625\,Z}{Z-0.5} + \frac{1.625\,Z}{Z+0.3} \quad ..(10)$$

Remember that, to realize the given transfer function $H(Z)$; it should be in terms of negative powers of Z. So we will convert Equation (10) in terms of negative powers of Z as follows :

$$H(Z) = \frac{Z^{-1}(-0.625\,Z)}{Z^{-1}(Z-0.5)} + \frac{Z^{-1}(1.625\,Z)}{Z^{-1}(Z+0.3)}$$

$$\therefore \quad H(Z) = \frac{-0.625}{1-0.5\,Z^{-1}} + \frac{1.625}{1+0.3\,Z^{-1}} \quad ..(11)$$

The general structure of parallel form realization is as shown in Fig. 1-19(d). The equation is

$$H(Z) = H_1(Z) + H_2(Z).$$

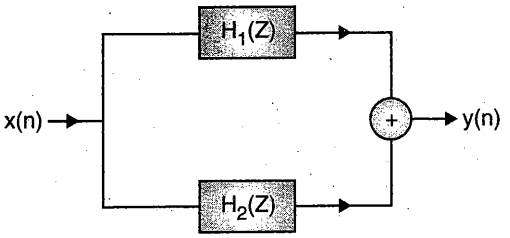
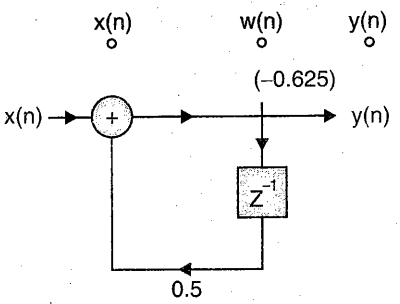


Fig. I-19(d) : General structure of parallel form realization

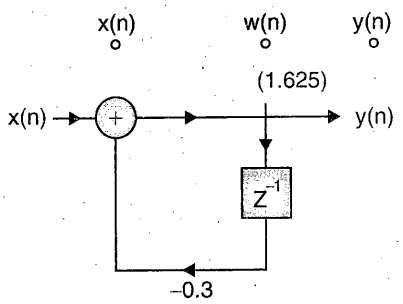
$$\text{Now let} \quad H_1(Z) = \frac{-0.625}{1-0.5\,Z^{-1}} \qquad ...(12)$$

$$\text{and} \quad H_2(Z) = \frac{1.625}{1+0.3\,Z^{-1}} \qquad ...(13)$$

Using shortcut method; the direct form-II realization of $H_1(Z)$ and $H_2(Z)$ is obtained as shown in Fig. I-19(e) and Fig. I-19(f) respectively.



(e) Direct form-II realization of $H_1(Z)$



(f) : Direct form-II realization of $H_2(Z)$

Fig. I-19

To obtain the parallel form realization; we have to connect Fig. I-19(e) and Fig. I-19(f) in parallel as shown in Fig. I-19(g).

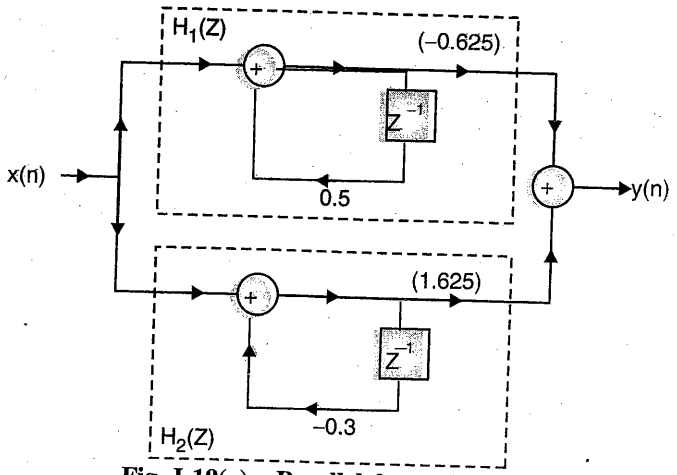


Fig. I-19(g) : Parallel form realization

Prob. 8 : A system has an impulse response,

$h(n) = (0.5)^n u(n) + n(0.2)^n u(n)$

show parallel realization of system.

Soln. : First we will obtain transfer function $H(Z)$.

We have, $$h(n) = (0.5)^n u(n) + n(0.2)^n u(n) \quad ...(1)$$

Recall the standard Z-transform pairs,

$$\alpha^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha} \quad \text{and} \quad n a^n u(n) \overset{Z}{\longleftrightarrow} \frac{aZ}{(Z-a)^2}$$

Thus Z-transform of Equation (1) can be written as,

$$H(Z) = \frac{Z}{Z-0.5} + \frac{0.2Z}{(Z-0.2)^2} \quad ...(2)$$

$$\text{Let} \quad H_1(Z) = \frac{Z}{Z-0.5} = \frac{1}{1-0.5Z^{-1}}$$

The direct form-II realization of $H_1(Z)$ is shown in Fig. I-20(a).

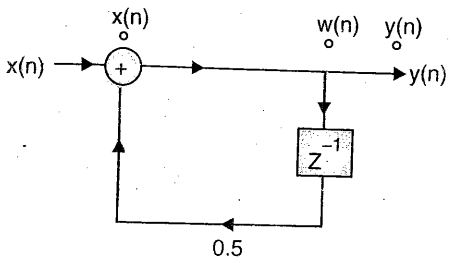


Fig. I-20(a)

$$\text{Let} \quad H_2(Z) = \frac{0.2\,Z}{(Z-0.2)^2}$$

$$= \frac{0.2\,Z}{Z^2 - 0.4\,Z + 0.04}$$

Multiplying numerator and denominator by Z^{-2} we get,

$$H_2(Z) = \frac{0.2\,Z^{-1}}{1 - 0.4\,Z^{-1} + 0.04\,Z^{-2}}$$

The direct form-II realization of $H_2(Z)$ is shown in Fig. I-20(b).

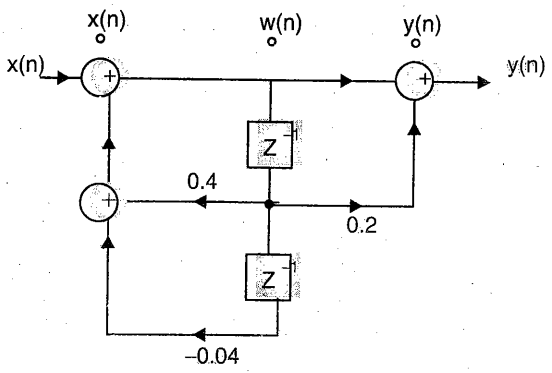


Fig. I-20(b)

The general structure parallel form realization is shown in Fig. I-20(c).

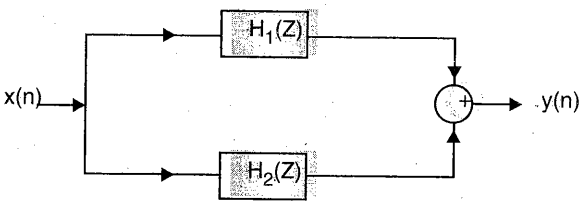


Fig. I-20(c)

Thus using Fig. I-20(a) and Fig. I-20(b); we can draw the parallel form realization of $H(Z)$ as shown in Fig. I-20(d).

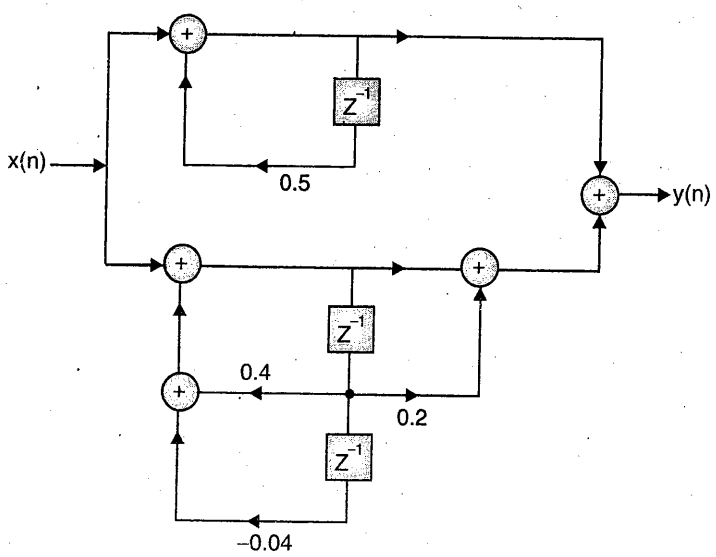


Fig. I-20(d)

Prob. 9 : Develop parallel form realization for the digital filter with transfer function.

$$H(Z) = \frac{1 + 2Z^{-1} + Z^{-2}}{1 - 0.75Z^{-1} + 0.125Z^{-2}}$$

Using first order systems.

Soln. : To obtain the parallel form realization we have to use partial fraction expansion.

The given transfer function is,

$$H(Z) = \frac{1 + 2Z^{-1} + Z^{-2}}{1 - 0.75Z^{-1} + 0.125Z^{-2}} \quad ...(1)$$

To convert if in terms of positive power of Z; multiply numerator and denominator by Z^2.

$$\therefore \quad H(Z) = \frac{Z^2 + 2Z + 1}{Z^2 - 0.75Z + 0.125}$$

$$\therefore \quad \frac{H(Z)}{Z} = \frac{Z^2 + 2Z + 1}{Z(Z^2 - 0.75Z + 0.125)} \quad ...(2)$$

Now we will obtain roots of $Z^2 - 0.75Z + 0.125$

Here a = 1, b = − 0.75 and c = 0.125

$$\therefore \quad \text{roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore \quad \text{roots} = \frac{0.75 \pm \sqrt{(-0.75)^2 - (4 \times 0.125)}}{2} = \frac{0.75 \pm 0.25}{2}$$

$$\therefore \quad \text{roots} = 0.5, 0.25$$

Thus Equation (2) becomes,

$$\frac{H(Z)}{Z} = \frac{Z^2 + 2Z + 1}{Z(Z-0.5)(Z-0.25)} \quad ..(3)$$

Thus poles are at $p_1 = 0$, $p_2 = 0.5$ and $p_3 = 0.25$

In the partial fraction expansion form, Equation (3) can be written as,

$$\frac{H(Z)}{Z} = \frac{A_1}{Z} + \frac{A_2}{Z-0.5} + \frac{A_3}{Z-0.25} \quad ...(4)$$

Now $A_1 = \frac{H(Z)}{Z} \times (Z - p_1) \Big|_{Z=p_1} = \frac{Z^2 + 2Z + 1}{Z(Z-0.5)(Z-0.25)} \times Z \Big|_{Z=0}$

$$\therefore A_1 = \frac{1}{(-0.5)(-0.25)}$$

$$\therefore A_1 = 8 \quad ...(5)$$

$$A_2 = \frac{H(Z)}{Z} \times (Z - p_2) \Big|_{Z=p_2} = \frac{Z^2 + 2Z + 1}{Z(Z-0.5)(Z-0.25)} \times (Z - 0.5) \Big|_{Z=0.5}$$

$$\therefore A_2 = \frac{Z^2 + 2Z + 1}{Z(Z-0.25)} \Big|_{Z=0.5} \quad \therefore A_2 = \frac{(0.5)^2 + 2(0.5) + 1}{0.5(0.5 - 0.25)}$$

$$\therefore A_2 = 18 \quad ...(6)$$

and $A_3 = \frac{H(Z)}{Z} \times (Z - p_3) \Big|_{Z=p_3}$

$$\therefore A_3 = \frac{Z^2 + 2Z + 1}{Z(Z-0.5)(Z-0.25)} \times (Z - 0.25) \Big|_{Z=0.25}$$

$$\therefore A_3 = \frac{(0.25)^2 + 2(0.25) + 1}{0.25(0.25 - 0.5)}$$

$$\therefore A_3 = -25 \quad ...(7)$$

Putting the values of A_1, A_2 and A_3 in Equation (4) we get,

$$\frac{H(Z)}{Z} = \frac{8}{Z} + \frac{18}{(Z-0.5)} - \frac{25}{(Z-0.25)}$$

$$\therefore H(Z) = 8 + \frac{18Z}{Z-0.5} - \frac{25Z}{Z-0.25} \quad ...(8)$$

To realize $H(Z)$; we want the expression in terms of negative powers of Z.

$$\therefore \quad H(Z) = 8 + \frac{Z^{-1}(18Z)}{Z^{-1}(Z-0.5)} - \frac{Z^{-1}(25Z)}{Z^{-1}(Z-0.25)}$$

$$\therefore \quad H(Z) = 8 + \frac{18}{1-0.5Z^{-1}} - \frac{25}{1-0.25Z^{-1}} \qquad ..(9)$$

Let $H_1(Z) = 8$

$$H_2(Z) = \frac{18}{1-0.5Z^{-1}}$$

and $$H_3(Z) = \frac{-25}{1-0.25Z^{-1}}$$

Using shortcut method; direct form-II realization of $H_2(Z)$ and $H_3(Z)$ is obtained as shown in Fig. I-21(a) and I-21(b) respectively.

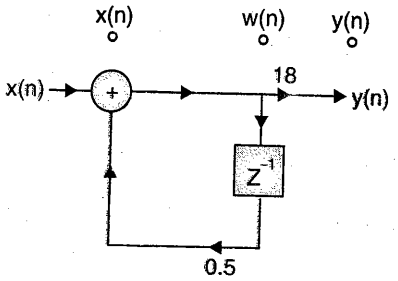


Fig. I-21(a) : Direct form-II realization of $H_2(Z)$

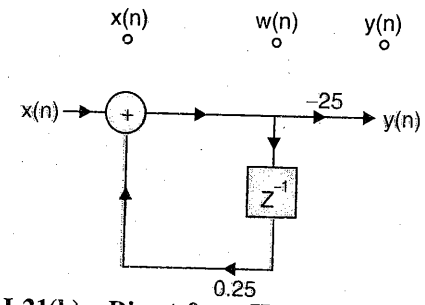


Fig. I-21(b) : Direct form-II realization of $H_3(Z)$

Now $H_1(Z) = 8$; which is constant. So it can be expressed as,

$$H_1(Z) = \frac{Y_1(Z)}{X_1(Z)}$$

$$\therefore \quad Y_1(Z) = H_1(Z) \cdot X_1(Z) = 8X_1(Z)$$

Taking IZT of both sides we get,

$$y_1(n) = 8x_1(n) \qquad ...(10)$$

Using Fig. I-21(a), Fig. I-21(b) and Equation (10); the parallel form realization is drawn as shown in Fig. I-21(c).

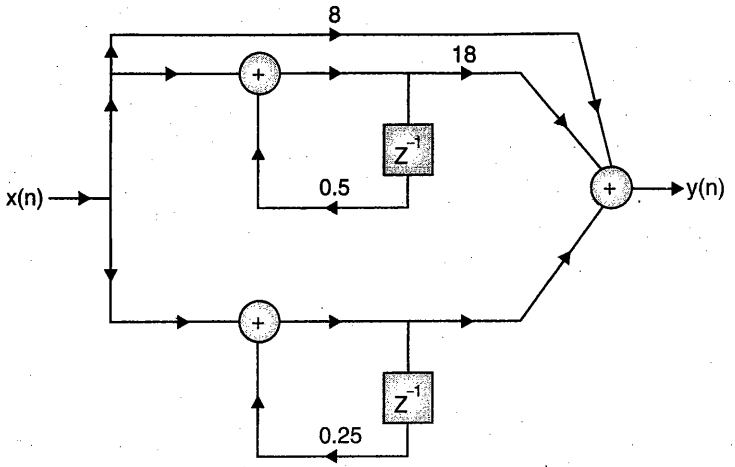


Fig. I-21(c) : Parallel form realization

Prob. 10 : System has transfer function as shown in Fig. I-22.

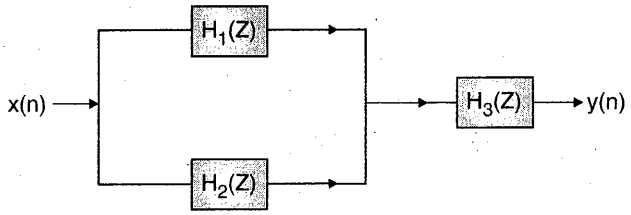


Fig. I-22

$H_1(Z)$ and $H_2(Z)$ has poles at + 1 and − 1.

$H_1(Z)$ has two zeros at origin, $H_2(Z)$ has only one zero at origin. The transfer $H_3(Z)$ is given by,

$$H_3(Z) = 1 + 1.5\,Z^{-1} - 1.5\,Z^{-2} - Z^{-3}$$

Show realization diagram of overall transfer function.

Soln. : First we will obtain expressions of $H_1(Z)$ and $H_2(Z)$.

For $H_1(Z)$:

$$\text{Zeros} \Rightarrow Z_1 = 0 \text{ and } Z_2 = 0; \text{ two zeros at origin}$$

$$\text{Poles} \Rightarrow P_1 = 1 \text{ and } P_2 = -1$$

$$\text{Now } H_1(Z) = \frac{(Z - Z_1)(Z - Z_2)}{(Z - P_1)(Z - P_2)}$$

$$\therefore \quad H_1(Z) = \frac{Z^2}{(Z-1)(Z+1)} \qquad ...(1)$$

For $H_2(Z)$:

$$\text{Zeros} \Rightarrow Z_1 = 0$$

$$\text{Poles} \Rightarrow P_1 = 1 \text{ and } P_2 = -1$$

$$\text{Now} \quad H_2(Z) = \frac{(Z-Z_1)}{(Z-P_1)(Z-P_2)}$$

$$\therefore \quad H_2(Z) = \frac{Z}{(Z-1)(Z+1)} \qquad ...(2)$$

Realization of $H_1(Z)$:

$$\text{We have} \quad H_1(Z) = \frac{Z^2}{(Z-1)(Z+1)} = \frac{Z^2}{Z^2-1}$$

$$\therefore \quad H_1(Z) = \frac{1}{1-Z^{-2}}$$

Direct form-II realization of $H_1(Z)$ is shown in Fig. I-23(a).

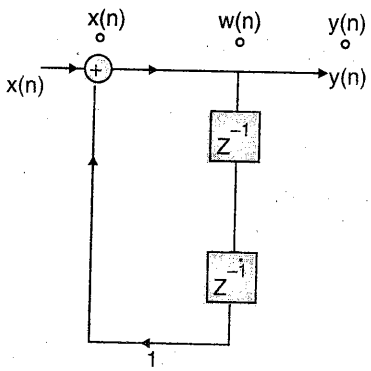


Fig. I-23(a)

Realization of $H_2(Z)$:

$$\text{We have,} \quad H_2(Z) = \frac{Z}{(Z-1)(Z+1)} = \frac{Z}{Z^2-1}$$

$$\therefore \quad H_2(Z) = \frac{Z^{-1}}{1-Z^{-2}}$$

Direct form-II realization of $H_2(Z)$ is shown in Fig. I-23(b).

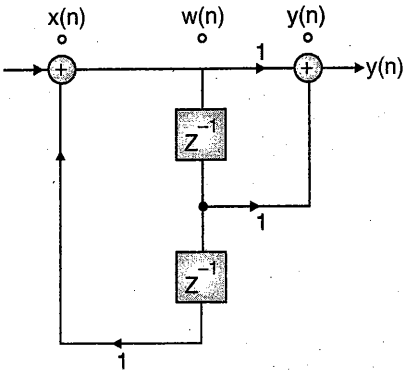


Fig. I-23(b)

Realization of $H_3(Z)$:

We have, $H_3(Z) = 1 + 1.5Z^{-1} - 1.5Z^{-2} - Z^{-3}$

Direct form-II realization of $H_3(Z)$ is shown in Fig. I-23(c).

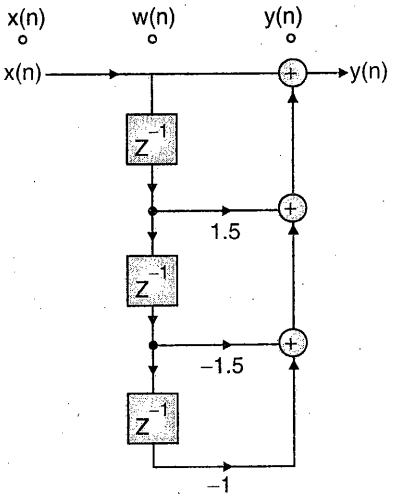


Fig. I-23(c)

Realization of overall transfer function :

As shown in Fig. I-22, $H_1(Z)$ and $H_2(Z)$ are connected in parallel and this parallel combination is cascaded with $H_3(Z)$. Thus realization of overall transfer function is shown in Fig. I-23(d).

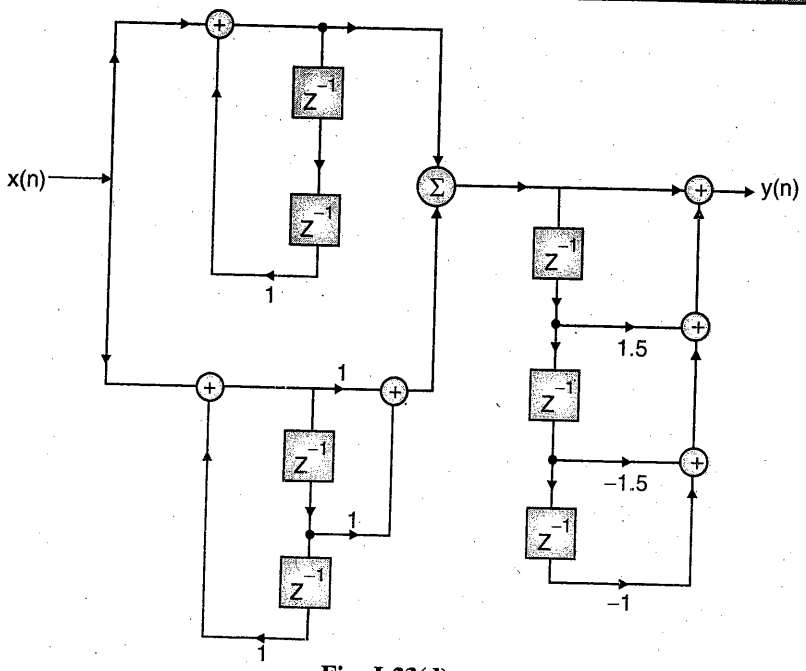
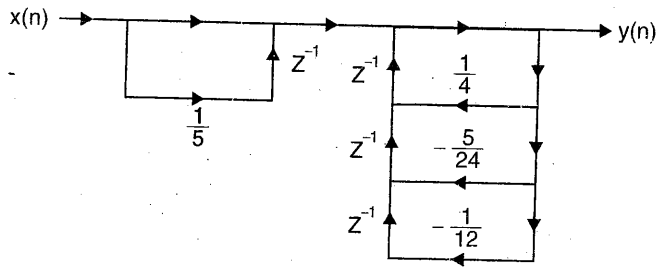
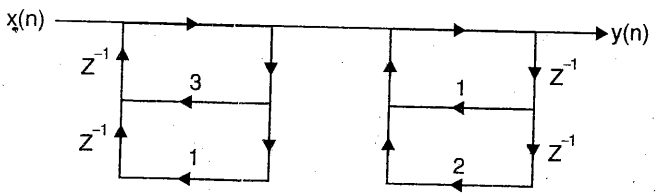


Fig. I-23(d)

Prob. 11 : For the graphs of Figs. I-24(a) and (b) write difference equation and system function.



(a) Given graph



(b) Given graph

Fig. I-24

Soln. :

For graph (a) :

By looking at the graph; we can easily conclude that it is a cascade connection. We know that in direct form-II structure poles are realized first and then zeros are realized. That means first part of Fig. I-24(a) indicates all pole system and second part indicates all zero system.

We can easily write the difference equation for the graph shown in Fig. I-24(a). It is,

$$y(n) = x(n)+\frac{1}{5}x(n-1)+\frac{1}{4}y(n-1)-\frac{5}{24}y(n-2)-\frac{1}{12}y(n-3) \quad ...(1)$$

This is the difference equation for the given graph. Now system transfer function is denoted by $H(Z)$. It is obtained by taking Z transform of Equation (1).

$$\therefore \quad Y(Z) = X(Z)+\frac{1}{5}Z^{-1}X(Z)+\frac{1}{4}Z^{-1}Y(Z)-\frac{5}{24}Z^{-2}Y(Z)-\frac{1}{12}Z^{-3}Y(Z)$$

$$\therefore \quad Y(Z) = X(Z)\left[1+\frac{1}{5}Z^{-1}\right]+Y(Z)\left[\frac{1}{4}Z^{-1}-\frac{5}{24}Z^{-2}-\frac{1}{12}Z^{-3}\right]$$

$$\therefore \quad Y(Z)-Y(Z)\left[\frac{1}{4}Z^{-1}-\frac{5}{24}Z^{-2}-\frac{1}{12}Z^{-3}\right] = X(Z)\left[1+\frac{1}{5}Z^{-1}\right]$$

$$\therefore \quad Y(Z)\left[1-\frac{1}{4}Z^{-1}-\frac{5}{24}Z^{-2}-\frac{1}{12}Z^{-3}\right] = X(Z)\left[1+\frac{1}{5}Z^{-1}\right] \quad ...(2)$$

The system transfer function $H(Z)$ is given by,

$$H(Z) = \frac{Y(Z)}{X(Z)}$$

Thus from Equation (2) we get,

$$H(Z) = \frac{Y(Z)}{X(Z)} = \frac{1+\frac{1}{5}Z^{-1}}{1-\frac{1}{4}Z^{-1}-\frac{5}{24}Z^{-2}-\frac{1}{12}Z^{-3}} \quad ...(3)$$

For graph (b) :

Now consider the graph shown in Fig. I-24(b). From this graph we cannot directly write the difference equation because first part of Fig. I-24(b) indicates the feedback operation (observe the directions of arrows). So we have to write the equations by considering $w(n)$.

$$\therefore \quad w(n) = x(n)+3w(n-1)+1w(n-2) \quad ...(4)$$

$$\text{and} \quad y(n) = w(n)+1y(n-1)+2y(n-2) \quad ...(5)$$

Taking Z transform of Equation (4),

$$W(Z) = X(Z)+3Z^{-1}W(Z)+Z^{-2}W(Z)$$

$$\therefore \quad W(Z)-3Z^{-1}W(Z)-Z^{-2}W(Z) = X(Z)$$

$$\therefore \quad W(Z)\left[1-3Z^{-1}-Z^{-2}\right] = X(Z)$$

$$\therefore \quad \frac{W(Z)}{X(Z)} = \frac{1}{1-3Z^{-1}-Z^{-2}} \quad ...(6)$$

Now taking Z transform of Equation (5),

$$Y(Z) = W(Z)+Z^{-1}Y(Z)+2Z^{-2}Y(Z)$$

$$\therefore \quad Y(Z)-Z^{-1}Y(Z)-2Z^{-2}Y(Z) = W(Z)$$

$$\therefore \quad Y(Z)\left[1-Z^{-1}-2Z^{-2}\right] = W(Z)$$

$$\therefore \quad \frac{Y(Z)}{W(Z)} = \frac{1}{1-Z^{-1}-2Z^{-2}} \qquad ...(7)$$

Now system transfer function H(Z) can be expressed as,

$$H(Z) = \frac{Y(Z)}{X(Z)} = \frac{Y(Z)}{W(Z)} \times \frac{W(Z)}{X(Z)}$$

$$\therefore \quad H(Z) = \frac{Y(Z)}{X(Z)} = \frac{1}{(1-Z^{-1}-2Z^{-2})(1-3Z^{-1}-Z^{-2})}$$

$$\therefore \quad H(Z) = \frac{Y(Z)}{X(Z)} = \frac{1}{1-3Z^{-1}-Z^{-2}-Z^{-1}+3Z^{-2}+Z^{-3}-2Z^{-2}+6Z^{-3}+2Z^{-4}}$$

$$\therefore \quad H(Z) = \frac{Y(Z)}{X(Z)} = \frac{1}{1-4Z^{-1}+7Z^{-3}+2Z^{-4}} \qquad ...(8)$$

Equation (8) gives the transfer function of overall system. To obtain the difference equation of overall system; we will take IZT of Equation (8). Rearranging Equation (8) we get,

$$\therefore \quad Y(Z)(1-4Z^{-1}+7Z^{-3}+2Z^{-4}) = X(Z)$$

$$\therefore \quad Y(Z)-4Z^{-1}Y(Z)+7Z^{-3}Y(Z)+2Z^{-4}Y(Z) = X(Z)$$

$$\therefore \quad Y(Z) = X(Z)+4Z^{-1}Y(Z)-7Z^{-3}Y(Z)-2Z^{-4}Y(Z)$$

Now taking IZT of both sides,

$$y(n) = x(n)+4y(n-1)-7y(n-3)-2y(n-4) \qquad ...(9)$$

Equation (9) represents the difference equation of overall system.

Prob. 12 : Using first order sections, obtain cascade realization for

$$H(Z) = \frac{\left(1+\frac{1}{2}Z^{-1}\right)\left(1+\frac{1}{4}Z^{-1}\right)}{\left(1-\frac{1}{2}Z^{-1}\right)\left(1-\frac{1}{4}Z^{-1}\right)\left(1-\frac{1}{8}Z^{-1}\right)}$$

Soln. : We know that for cascade realization the given transfer function can be expressed as,

$$H(Z) = H_1(Z)\cdot H_2(Z)\cdot H_3(Z)...$$

Thus the given transfer function can be expressed as,

$$H(Z) = \frac{\left(1+\frac{1}{2}Z^{-1}\right)}{\left(1-\frac{1}{2}Z^{-1}\right)}\cdot\frac{\left(1+\frac{1}{4}Z^{-1}\right)}{\left(1-\frac{1}{4}Z^{-1}\right)}\cdot\frac{1}{\left(1-\frac{1}{8}Z^{-1}\right)}$$

$$\text{Here} \quad H_1(Z) = \frac{1+\frac{1}{2}Z^{-1}}{1-\frac{1}{2}Z^{-1}} = \frac{Y_1(Z)}{X(Z)}$$

$$\therefore \quad Y_1(Z)\left(1-\frac{1}{2}Z^{-1}\right) = X(Z)\left(1+\frac{1}{2}Z^{-1}\right)$$

$$\therefore \quad Y_1(Z)-\frac{1}{2}Z^{-1}Y_1(Z) = X(Z)+\frac{1}{2}Z^{-1}X(Z)$$

$$\therefore \quad Y_1(Z) = X(Z)+\frac{1}{2}Z^{-1}X(Z)+\frac{1}{2}Z^{-1}Y_1(Z)$$

Taking IZT of both sides,

$$y_1(n) = x(n)+\frac{1}{2}x(n-1)+\frac{1}{2}y_1(n-1) \qquad \ldots(1)$$

It is realized as shown in Fig. I-25(a).

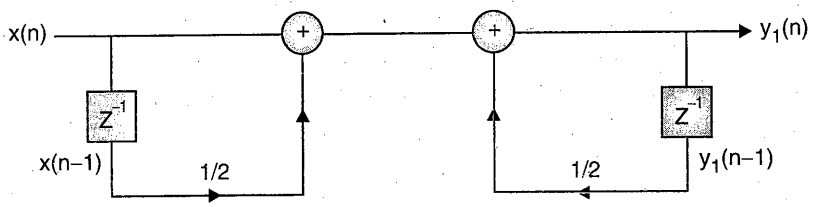


Fig. I-25(a)

Its direct form-II realization is as shown in Fig. I-25(b).

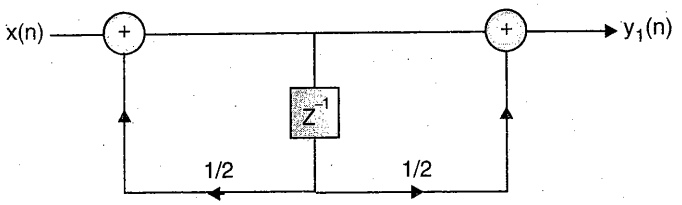


Fig. I-25(b) : Direct form-II

$$\text{Now} \quad H_2(Z) = \frac{1+\frac{1}{4}Z^{-1}}{1-\frac{1}{4}Z^{-1}} = \frac{Y_2(Z)}{X_2(Z)}$$

$$\therefore \quad Y_2(Z)\left(1-\frac{1}{4}Z^{-1}\right) = X_2(Z)\left(1+\frac{1}{4}Z^{-1}\right)$$

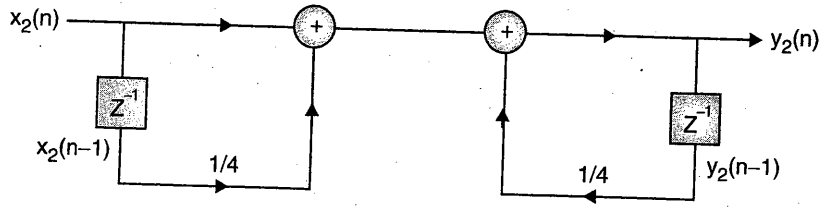
$$\therefore \quad Y_2(Z)-\frac{1}{4}Z^{-1}Y_2(Z) = X_2(Z)+\frac{1}{4}Z^{-1}X_2(Z)$$

$$\therefore \quad Y_2(Z) = X_2(Z)+\frac{1}{4}Z^{-1}X_2(Z)+\frac{1}{4}Z^{-1}Y_2(Z)$$

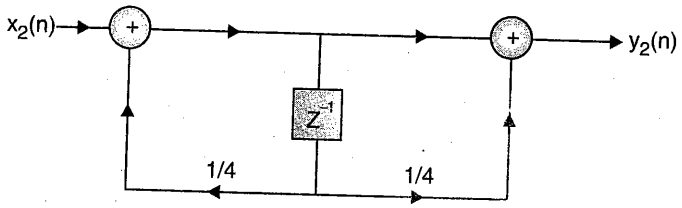
Taking IZT of both sides,

$$y_2(n) = x_2(n) + \frac{1}{4}x_2(n-1) + \frac{1}{4}y_2(n-1) \quad \ldots(2)$$

Its realization is as shown in Figs. I-25(c) and I-25(d).



(c) Direct form-I



(d) Direct form-II

Fig. I-25

Now $H_3(Z) = \dfrac{1}{1-\frac{1}{8}Z^{-1}} = \dfrac{Y(Z)}{X_3(Z)}$

$$\therefore \quad Y(Z)\left(1-\frac{1}{8}Z^{-1}\right) = X_3(Z)$$

$$\therefore \quad Y(Z) - \frac{1}{8}Z^{-1}Y(Z) = X_3(Z)$$

$$\therefore \quad Y(Z) = X_3(Z) + \frac{1}{8}Z^{-1}Y_3(Z)$$

Taking IZT of both sides,

$$y(n) = x_3(n) + \frac{1}{8}y(n-1) \quad \ldots(3)$$

Its realization is as shown in Fig. I-25(e).

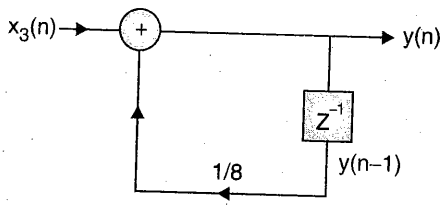


Fig. I-25(e)

Now cascade realization is obtained by cascading the direct form- II structures of $H_1(Z)$, $H_2(Z)$ and $H_3(Z)$ as shown in Fig. I-25(f).

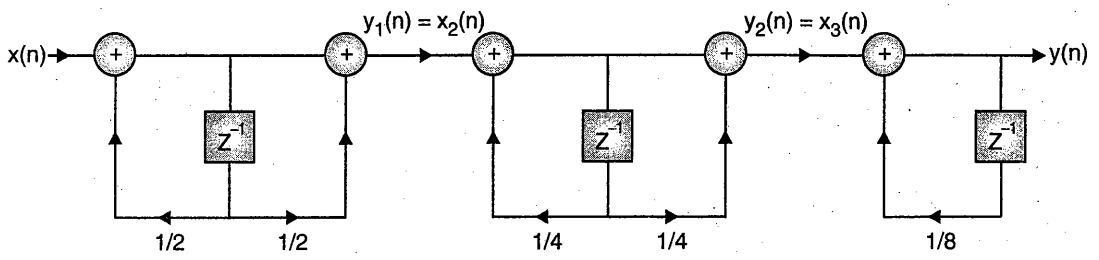


Fig. I-25(f)

Prob. 13 : Determine the parallel realization of the IIR digital filter transfer functions.

(i) $H(Z) = \dfrac{3(2Z^2+5Z+4)}{(2Z+1)(Z+2)}$

(ii) $H(Z) = \dfrac{3Z(5Z-2)}{\left(Z+\frac{1}{2}\right)(3Z-1)}$

Soln. :

(i) Given $H(Z) = \dfrac{3(2Z^2-5Z+4)}{(2Z+1)(Z+2)}$

Multiplying numerator and denominator by $\frac{1}{2}$ we get,

$$H(Z) = \frac{\frac{3}{2}(2Z^2+5Z+4)}{\left(Z+\frac{1}{2}\right)(Z+2)} \qquad \therefore \quad H(Z) = \frac{3Z^2+\frac{15}{2}Z+12}{\left(Z+\frac{1}{2}\right)(Z+2)}$$

$$\therefore \quad \frac{H(Z)}{Z} = \frac{3Z^2+\frac{15}{2}Z+12}{Z\left(Z+\frac{1}{2}\right)(Z+2)}$$

In the partial fraction expansion form we can write,

$$\frac{H(Z)}{Z} = \frac{A_1}{Z} + \frac{A_2}{Z+\frac{1}{2}} + \frac{A_3}{Z+2} \qquad \text{Here} \quad A_1 = \left.\frac{Z\left(3Z^2+\frac{15}{2}Z+12\right)}{Z\left(Z+\frac{1}{2}\right)(Z+2)}\right|_{Z=0}$$

$$\therefore \quad A_1 = \frac{12}{\frac{1}{2}(2)} = \mathbf{1}$$

$$A_2 = \left.\frac{\left(Z+\frac{1}{2}\right)\cdot\left(3Z^2+\frac{15}{2}Z+12\right)}{Z\left(Z+\frac{1}{2}\right)(Z+2)}\right|_{Z=-\frac{1}{2}}$$

$$\therefore \quad A_2 = \frac{3\left(\frac{1}{4}\right)+\left(-\frac{15}{4}\right)+12}{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}+2\right)} = \frac{\frac{3}{3}-\frac{15}{4}+12}{\left(-\frac{1}{2}\right)\left(\frac{3}{2}\right)}$$

$$\therefore \quad A_2 = \frac{-\frac{12+48}{4}}{-3/4} = \frac{36}{-3} = -12$$

$$\text{and} \quad A_3 = \frac{(Z+2)\left(3Z^2+\frac{15}{2}Z+12\right)}{Z\left(Z+\frac{1}{2}\right)(Z+2)}\Bigg|_{Z=-2}$$

$$\therefore \quad A_3 = \frac{(12-15+12)}{(-2)\left(-2+\frac{1}{2}\right)} = \frac{9}{(-2)\left(-\frac{3}{2}\right)}$$

$$\therefore \quad A_3 = \frac{9}{3} = 3$$

$$\therefore \quad \frac{H(Z)}{Z} = \frac{12}{Z} - \frac{12}{Z+\frac{1}{2}} + \frac{3}{Z+2} \qquad \therefore \quad H(Z) = 12 - \frac{12Z}{Z+\frac{1}{2}} + \frac{3Z}{Z+2}$$

$$\therefore \quad H(Z) = 12 - \frac{12}{1+\frac{1}{2}Z^{-1}} + \frac{3}{1+2Z^{-1}}$$

In the general form we can write,

$$H(Z) = H_1(Z) + H_2(Z) + H_3(Z)$$

Here $H_1(Z) = 12$ $\qquad H_2(Z) = -\frac{12}{1+\frac{1}{2}Z^{-1}}$

and $H_3(Z) = \frac{3}{1+\frac{1}{2}Z^{-1}}$

The general structure of parallel form realization is shown in Fig. I-26(a).

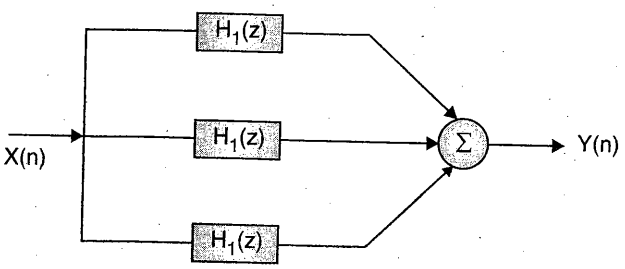


Fig. I-26(a)

Using short cut method, it can be realised as shown in Fig. I-26(b).

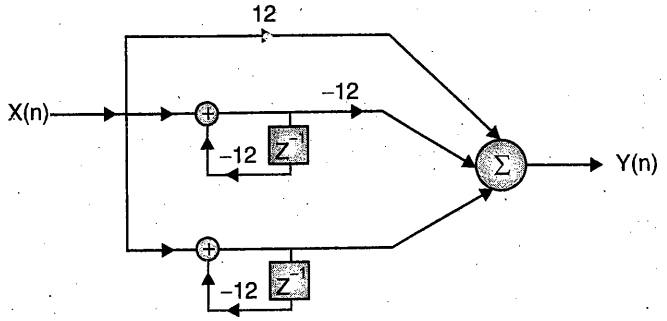


Fig. I-26(b)

(ii) $$H(Z) = \frac{3Z(5Z-2)}{\left(Z+\frac{1}{2}\right)(3Z-1)} \qquad \therefore \quad \frac{H(Z)}{Z} = \frac{3(5Z-2)}{\left(Z+\frac{1}{2}\right)(3Z-1)}$$

$$\therefore \quad \frac{H(Z)}{Z} = \frac{15Z-6}{\left(Z+\frac{1}{2}\right)(3Z-1)}$$

Multiplying numerator and denominator by $\frac{1}{3}$ we get,

$$\frac{H(Z)}{Z} = \frac{5Z-2}{\left(Z+\frac{1}{2}\right)\left(Z-\frac{1}{3}\right)}$$

In partial fraction expansion from we can write,

$$\frac{H(Z)}{Z} = \frac{A_1}{Z+\frac{1}{2}} + \frac{A_2}{Z-\frac{1}{3}}$$

Here $$A_1 = \left(Z+\frac{1}{2}\right) \cdot \frac{(5Z-2)}{\left(Z+\frac{1}{2}\right)\left(Z-\frac{1}{3}\right)}\Bigg|_{Z=-1/2}$$

$$\therefore \quad A_1 = \frac{-5/2-2}{-\frac{1}{2}-\frac{1}{3}} = \frac{-\frac{9}{2}}{-\frac{5}{6}} = \frac{27}{5}$$

and $$A_2 = \left(Z-\frac{1}{3}\right) \frac{(5Z-2)}{\left(Z+\frac{1}{2}\right)\left(Z-\frac{1}{3}\right)}\Bigg|_{Z=1/3}$$

$$\therefore \quad A_2 = \frac{5/3-2}{\frac{1}{3}+\frac{1}{2}} \qquad \therefore \quad A_2 = \frac{-\frac{1}{3}}{5/6} = -2/5$$

$$\therefore \quad \frac{H(Z)}{Z} = \frac{27/5}{Z+\frac{1}{2}} - \frac{2/5}{Z-\frac{1}{3}}$$

$$\therefore \quad H(Z) = \frac{(27/5)Z}{Z+\frac{1}{2}} - \frac{(2/5)Z}{Z-\frac{1}{3}} \qquad \therefore \quad H(Z) = \frac{27/5}{1+\frac{1}{2}Z^{-1}} - \frac{2/5}{1-\frac{1}{3}Z^{-1}}$$

In the general form we can write,

$$H(Z) = H_1(Z) + H_2(Z)$$

Using short cut method it can be realized as shown in Fig. I-26(c).

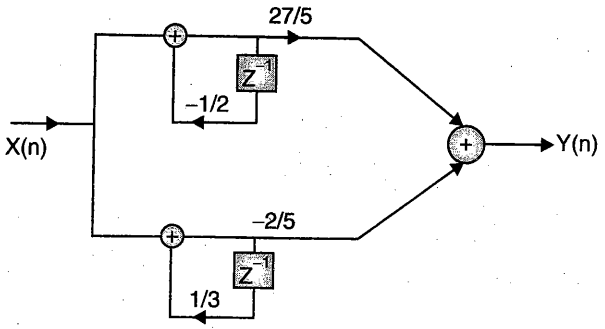


Fig. I-26(c)

1.8 Lattice Structure for IIR Filters :

We have studied lattice structure for FIR filters. For the reference, recall system transfer function of M^{th} order FIR filter. It is,

$$H_M(Z) = 1 + \sum_{K=1}^{M} \beta_M(K) Z^{-K}, \; M \geq 1 \qquad \ldots(1)$$

For this transfer function, we have obtained single stage lattice structure as shown in Fig. I-27.

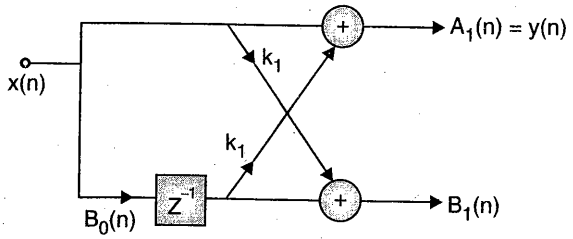


Fig. I-27

With the help of this structure shown in Fig. I-27, we will discuss how to draw lattice structure for IIR filters.

Consider all pole IIR system. For this system the transfer function can be expressed as,

$$H(Z) = \frac{1}{1 + \sum_{K=1}^{N} \beta_N(K) Z^{-K}} \quad ...(2)$$

We know that H (Z), is a transfer function, which can be written as, $H(Z) = \frac{Y(Z)}{X(Z)}$.

$$\therefore \quad \frac{Y(Z)}{X(Z)} = \frac{1}{1 + \sum_{K=1}^{N} \beta_N(K) Z^{-K}}$$

$$\therefore \quad Y(Z) + Y(Z)\left[\sum_{K=1}^{N} \beta_N(K) Z^{-K}\right] = X(Z)$$

Here Y (Z) is independent of summation. So it can be transferred inside the summation.

$$\therefore \quad Y(Z) + \sum_{K=1}^{N} \beta_N(K) Y(Z) Z^{-K} = X(Z)$$

$$\therefore Y(Z) = -\sum_{K=1}^{N} \beta_N(K) Y(Z) \cdot Z^{-K} + X(Z) \quad ...(3)$$

Taking inverse z of both sides we get,

$$y(n) = -\sum_{K=1}^{N} \beta_N(K) y(n-K) + x(n) \quad ...(4)$$

Now we will interchange the role of x (n) and y (n).

$$\therefore \quad x(n) = -\sum_{K=1}^{N} \beta_N(K) \cdot x(n-K) + y(n)$$

Rearranging the terms we get,

$$y(n) = x(n) + \sum_{K=1}^{N} \beta_N(K) x(n-K) \quad ...(5)$$

Taking Z transform of both sides we get,

$$Y(Z) = X(Z) + \sum_{K=1}^{N} \beta_N(K) \cdot Z^{-K} X(Z)$$

$$\therefore \quad Y(Z) = X(Z)\left[1 + \sum_{K=1}^{N} \beta_N(K) Z^{-K}\right]$$

$$\therefore \quad H(Z) = \frac{Y(Z)}{X(Z)} = \left[1 + \sum_{K=1}^{N} \beta_N(K) Z^{-K} \right] \qquad ...(6)$$

Observe Equations (6) and (1). Equation (1) is for all zero FIR system and Equation (6) is for all pole IIR system. Both equations are same. That means we can obtain lattice structure of IIR system by interchanging the role of input and output.

Thus interchanging the role of input and output; from Fig. I-27, we can draw single stage lattice structure of IIR system as shown in Fig. I-28.

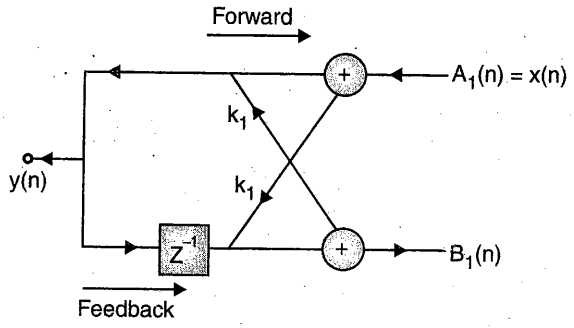


Fig. I-28

Note : IIR systems always require feedback which is shown in Fig. I-28.

Here we have interchanged x (n) and y (n); thus the directions of arrow [of x (n) and y (n)] are also reversed. This same structure is drawn properly as shown in Fig. I-29.

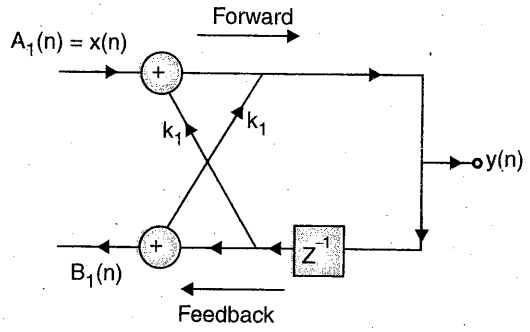


Fig. I-29

To obtain two stage lattice structure we will put N = 2 in Equation (4)

$$\therefore \quad y(n) = - \sum_{K=1}^{2} \beta_N(K)\, y(n-K) + x(n) \qquad ...(7)$$

This is the difference equation of two stage lattice structure. We know that two stage lattice structure is obtained by cascading two single stage lattice structures. It is shown in Fig. I-30.

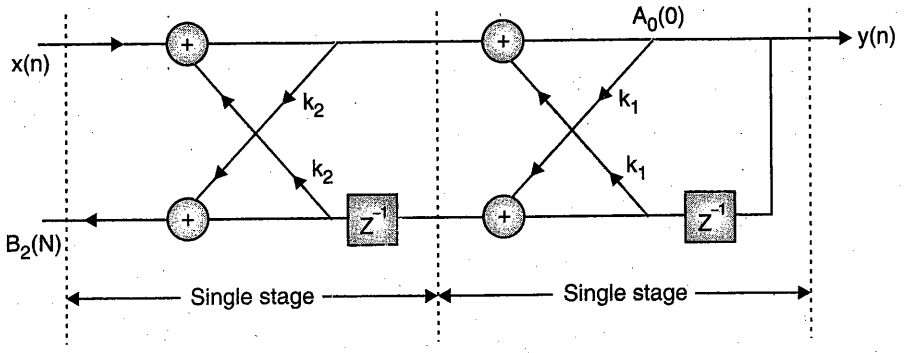


Fig. I-30

Similarly we can draw n^{th} order lattice structure as shown in Fig. I-31.

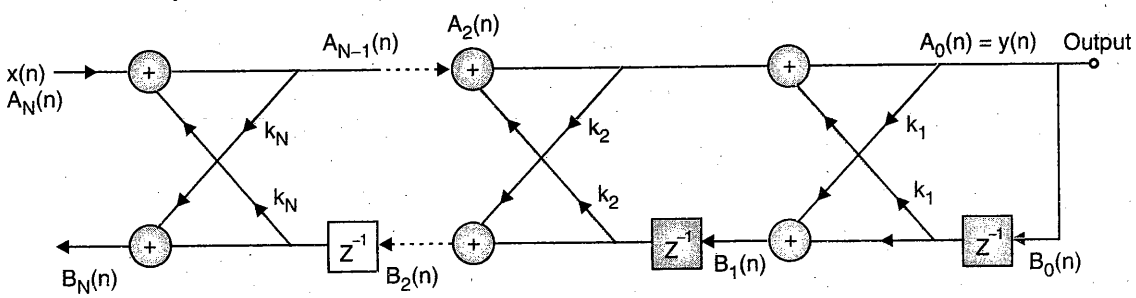


Fig. I-31

1.9 Lattice-ladder Structure of IIR Filter :

Consider all pole IIR system. For this system the transfer function can be expressed as,

$$H(Z) = \frac{1}{1 + \sum_{K=1}^{N} \beta_N(K) Z^{-1}} \quad \ldots(1)$$

The difference equation for this IIR system is

$$y(n) = x(n) - \sum_{K=1}^{N} \beta_N(K)\, y(n-K)$$

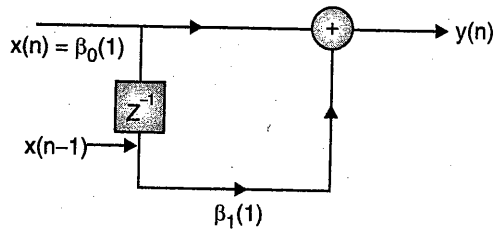
This equation is realized in lattice form as shown in Fig. I-31.

Consider an IIR filter with system function,

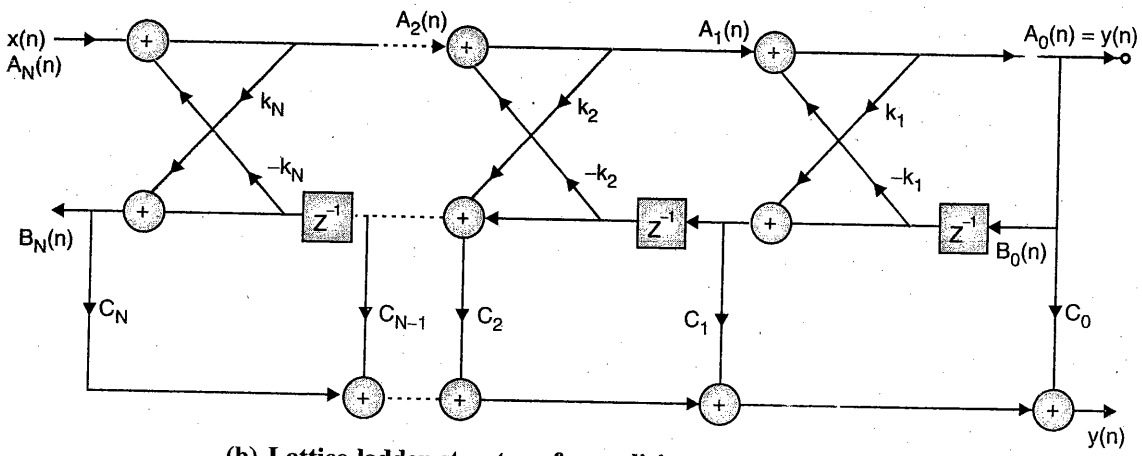
$$H(Z) = \frac{\sum_{K=0}^{M} A_M(K) Z^{-K}}{1 + \sum_{K=1}^{N} \beta_N(K) Z^{-K}}, \quad N > M \quad \ldots(2)$$

For numerator of Equation (2) we have constructed a ladder structure as shown in Fig. I-32(a).

Thus, a lattice structure for Equation (2) can be constructed by first realizing an all-pole lattice coefficients for the denominator and then adding a ladder part as shown in Fig. 1.32(b).



(a)



(b) Lattice-ladder structure for realizing a pole-zero IIR filter

Fig. I-32

The output of the ladder part can be expressed as a weighted linear combination of $B_m(n)$.

Thus, output is given as,

$$y(n) = \sum_{m=0}^{N} C_m B_m(n)$$

where C_m are called the ladder coefficients.

1.9.1 Conversion from Lattice Structure to Direct Form :

The difference equation for an IIR filter is given as,

$$y(n) = x(n) - \sum_{K=1}^{N} \beta_N(K)\, y(n-K) \qquad \ldots(1)$$

For N = 3, Equation (1) becomes,

$$y(n) = x(n) - \beta_3(1)\, y(n-1) - \beta_3(2)\, y(n-2) - \beta_3(3)\, y(n-3) \qquad \ldots(2)$$

The general equations from the lattice structure shown in Fig. I-31 are given as,

$$y(n) = A_0(n) = B_0(n) \quad ...(3)$$

$$x(n) = A_N(n) \quad ...(4)$$

$$A_m(n) = A_{m-1}(n) + K_m B_{m-1}(n-1) \quad ...(5)$$

$$m = N, N-1, \ldots, 1$$

$$B_m(n) = K_m A_{m-1}(n) + B_{m-1}(n-1) \quad ...(6)$$

$$m = N, N-1, \ldots, 1$$

For N = 3, Equations will become,

$$y(n) = A_0(n) = B_0(n) \quad ...(7)$$

$$x(n) = A_3(n) \quad ...(8)$$

$$A_3(n) = A_2(n) + K_3 B_2(n-1) \quad ...(9)$$

$$A_2(n) = A_1(n) + K_2 B_1(n-1) \quad ...(10)$$

$$A_1(n) = A_0(n) + K_1 B_0(n-1) \quad ...(11)$$

$$A_1(n-1) = A_0(n-1) + K_1 B_0(n-2) \quad ...(12)$$

$$B_3(n) = K_3 A_2(n) + B_2(n-1) \quad ...(13)$$

$$B_2(n-1) = K_2 A_1(n-1) + B_1(n-2) \quad ...(14)$$

$$B_1(n-1) = K_1 A_0(n-1) + B_0(n-2) \quad ...(15)$$

$$B_1(n-2) = K_1 A_0(n-2) + B_0(n-3) \quad ...(16)$$

From Equation (8),

$$x(n) = A_3(n)$$

$$\therefore \quad x(n) = A_2(n) + K_3 B_2(n-1)$$

$$= A_1(n) + K_2 B_1(n-1) + K_3 \left[K_2 A_1(n-1) + B_1(n-2) \right]$$

$$= A_0(n) + K_1 B_0(n-1) + K_2 \left[K_1 A_0(n-1) + B_0(n-2) \right]$$

$$+ K_3 \left[K_2 \left[A_0(n-1) + K_1 B_0(n-2) \right] + K_1 A_0(n-2) + B_0(n-3) \right]$$

$$= A_0(n) + K_1 B_0(n-1) + K_1 K_2 A_0(n-1) + K_2 B_0(n-2) + K_2 K_3 A_0(n-1)$$

$$+ K_1 K_2 K_3 B_0(n-2) + K_1 K_3 A_0(n-2) + K_3 B_0(n-3)$$

$$= y(n) + \left[K_1 + K_1 K_2 + K_2 K_3 \right] y(n-1)$$

$$+ \left[K_2 + K_1 K_3 + K_1 K_2 K_3 \right] y(n-2) + K_3 y(n-3)$$

$$= y(n)+\left[K_1(1+K_2)+K_2K_3\right]y(n-1)+\left[K_2+K_1K_3(1+K_2)\right]y(n-2)$$
$$+K_3y(n-3)$$

We have, $\beta_2(1) = K_1(1+K_2)$

$$\therefore \quad x(n) = y(n)+\left[\beta_2(1)+\beta_2(2)\beta_3(3)\right]y(n-1)$$
$$+\left[\beta_2(2)+\beta_2(1)\beta_3(3)\right]y(n-2)+\beta_3(3)y(n-3)$$
$$\therefore y(n) = x(n)-\left[\beta_2(1)\beta_2(2)\beta_3(3)\right]y(n-1)$$
$$-\left[\beta_2(2)+\beta_2(1)\beta_3(3)\right]y(n-2)-\beta_3(3)y(n-3) \quad ...(17)$$

Comparing Equations (2) and (17) we get,

$$\beta_3(0) = 1$$

$$\beta_3(1) = \beta_2(1)+\beta_2(2)\beta_3(3)$$

$$\beta_3(2) = \beta_2(2)+\beta_2(1)\beta_3(3)$$

$$\beta_3(3) = \beta_3(3) = K_3$$

Thus, above equations in general form can be written as,

$$\beta_m(0) = 1$$

$$\beta_m(K) = \beta_{m-1}(K)+\beta_{m-1}(K-1)\beta_m(m)$$

$$\beta_m(m) = K_m$$

Using these equations, we can convert lattice structure to direct form.

1.9.2 Conversion from Direct Form to Lattice Structure :

For N = 2, Equation (1) becomes,

$$y(n) = x(n)-\beta_2(1)y(n-1)-\beta_2(2)y(n-2) \quad ...(18)$$

From equations (8), (9) and (13) we can write

$$x(n) = A_3(n) = A_2(n)+K_3B_2(n-1) \quad ...(19)$$

and $B_3(n) = K_3A_2(n)+B_2(n-1)$

$$\therefore \quad B_2(n-1) = B_3(n)-K_3A_2(n)$$

Putting in Equation (19), we get,

$$x(n) = A_3(n) = A_2(n)+K_3B_3(n)-K_3^2A_2(n)$$

$$\therefore A_3(n) = A_2(n)\left[1-K_3^2\right]+K_3B_3(n)$$

$$\therefore A_2(n) = \frac{A_3(n)-K_3B_3(n)}{1-K_3^2} \quad ...(20)$$

Now let us first find value of $A_3(n) - K_3 B_3(n)$

$$\therefore A_3(n) - K_3 B_2(n) = A_2(n) + K_3 B_2(n-1) - K_3\left[K_2 A_1(n) + B_1(n-1)\right]$$

$$= A_1(n) + K_2 B_1(n-1) + K_3\left[K_2 A_1(n-1) + B_1(n-2)\right]$$

$$- K_3\left\{K_2\left[A_0(n) + K_1 B_0(n-1)\right] + K_1 A_0(n-1) + B_0(n-2)\right\}$$

$$= A_0(n) + K_1 B_0(n-1) + K_2\left[K_1 A_0(n-2) + B_0(n-3)\right]$$

$$+ K_3 K_2\left[A_0(n-1) + K_1 B_0(n-2)\right] + K_3\left[K_1 A_0(n-2) + B_0(n-3)\right]$$
$$- K_2 K_3 A_0(n) - K_1 K_2 K_3 B_0(n-1) - K_1 K_3 A_0(n-1) - K_3 B_0(n-2)$$

$$= A_0(n) + K_1 B_0(n-1) + K_1 K_2 A_0(n-2) + K_2 B_0(n-3)$$
$$+ K_2 K_3 A_0(n-1) + K_1 K_2 K_3 B_0(n-2) + K_3 B_0(n-3)$$
$$+ K_1 K_3 A_0(n-2) - K_2 K_3 A_0(n) - K_1 K_2 K_3 B_0(n-1)$$
$$- K_1 K_3 A_0(n-1) - K_3 B_0(n-2)$$

$$= y(n) + \beta_3(1) y(n-1) + \beta_3(2) y(n-2) + \beta_3(3) y(n-3)$$
$$- K_3 \beta_3(3) y(n) - K_3 \beta_3(2) y(n-2) - K_3 \beta_3(1) y(n-1)$$
$$- \beta_3(3) y(n-3)$$

$$= \left[1 - K_3^2\right] y(n) + \left[\beta_3(1) - \beta_3(3)\beta_3(2)\right] y(n-1)$$
$$+ \left[\beta_3(2) - \beta_3(2)\beta_3(3)\right] y(n-2)$$

Putting in Equation (20), we get,

$$A_2(n) = \frac{\left(1 - K_3^2\right) y(n) + \left[\beta_3(1) - \beta_3(3)\beta_3(2)\right] y(n-1) + \left[\beta_3(2) - \beta_3(2)\beta_3(3)\right] y(n-2)}{\left(1 - K_3^2\right)}$$

$$\therefore A_2(n) = y(n) + \frac{\beta_3(1) - \beta_3(2)\beta_3(3)}{1 - K_3^2} y(n-1) + \frac{\beta_3(2) - \beta_3(2)\beta_3(3)}{1 - K_3^2} y(n-2) \quad \ldots(21)$$

Comparing Equations (18) and (21), we get,

$$\beta_2(0) = 1$$

$$\beta_2(1) = \frac{\beta_3(1) - \beta_3(3)\beta_3(2)}{1 - \beta_3^2(3)}$$

$$\beta_2(2) = \frac{\beta_3(2) - \beta_3(2)\beta_3(3)}{1 - \beta_3^2(3)}$$

Thus, in general these equations can be written as,

$$\beta_m(m) = K_m \qquad \beta_{m-1}(0) = 1$$

$$\beta_m(K) = \frac{\beta_m(K) - \beta_m(m)\beta_m(m-1)}{1 - \beta_m^2(m)}$$

Using these equations, we can convert direct form to lattice structure.

1.10 Ideal Filters and Approximations :

Two important characteristics of ideal filters are :

(1) Ideal filters have constant gain in the passband and zero gain in the stop band.

(2) Ideal filter has linear phase response.

In order to realize (design) digital filter, an important condition is that; the response of filter should be causal. We have studied a causal LTI system. Impulse response of causal LTI system is denoted by $h(n)$. Now system is causal when $h(n)$ has some value for positive values of 'n' and $h(n)$ is zero for negative values of 'n'. Thus the condition of causality can be summarized as,

$$h(n) = 0 \quad \text{for } n < 0 \qquad \ldots(1)$$

As an example, let us consider the magnitude response of ideal low pass filter (L.P.F.). It is shown in Fig. I-33(a).

Here, ω_c = Cut-off frequency

and $|H(\omega)|$ = Magnitude of filter

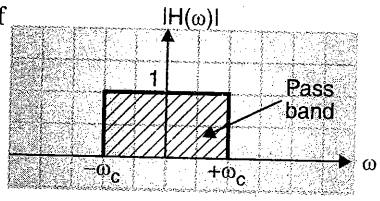


Fig. I-33(a) : Magnitude response of ideal LPF

As shown in Fig. I-33(a), the magnitude is '1' in the frequency range $-\omega_c$ to $+\omega_c$. That means,

$$H(\omega) = \begin{cases} 1 & \text{for } -\omega_c \le \omega \le \omega_c \\ 0 & \text{other wise} \end{cases} \qquad \ldots(2)$$

Now the value of $h(n)$ is obtained by taking inverse fourier of $H(\omega)$. According to the definition of inverse fourier we have,

$$h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\omega) e^{j\omega n} d\omega \qquad \ldots(3)$$

Using Equation (2) we get,

$$h(n) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} 1 \cdot e^{j\omega n} d\omega \qquad \ldots(4)$$

Now we will consider two conditions as follows :

Condition (i) : When n = 0 :

Putting n = 0 in Equation (4) we get,

$$h(n) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} 1 \cdot e^0 \, d\omega$$

$$\therefore \quad h(n) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} d\omega \qquad \ldots(\because e^0 = 1)$$

$$\therefore \quad h(n) = \frac{1}{2\pi} [\omega]_{-\omega_c}^{\omega_c}$$

$$\therefore \quad h(n) = \frac{\omega_c}{\pi} \qquad \ldots\text{when } n = 0 \qquad \ldots(5)$$

Condition (ii) : When n ≠ 0 :

Taking integration of Equation (4) we get,

$$h(n) = \frac{1}{2\pi} \left[\frac{e^{j\omega n}}{jn} \right]_{-\omega_c}^{\omega_c} = \frac{1}{2\pi jn} \left[e^{j\omega_c n} - e^{-j\omega_c n} \right]$$

$$\therefore \quad h(n) = \frac{1}{\pi n} \left[\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j} \right] \qquad \ldots(6)$$

But according to Euler's identity,

$$\frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin\theta$$

Thus Equation (6) becomes,

$$h(n) = \frac{\sin \omega_c n}{\pi n} \qquad \ldots\text{when } n \neq 0 \qquad \ldots(7)$$

Thus combining two conditions we can write,

$$h(n) = \begin{cases} \dfrac{\sin \omega_c n}{\pi n} & \text{For } n \neq 0 \\ \dfrac{\omega_c}{\pi} & \text{For } n = 0 \end{cases} \qquad \ldots(8)$$

Now we will sketch h(n) for different values of n by taking $\omega_c = \frac{\pi}{4}$. It is shown in Fig. I-33(b).

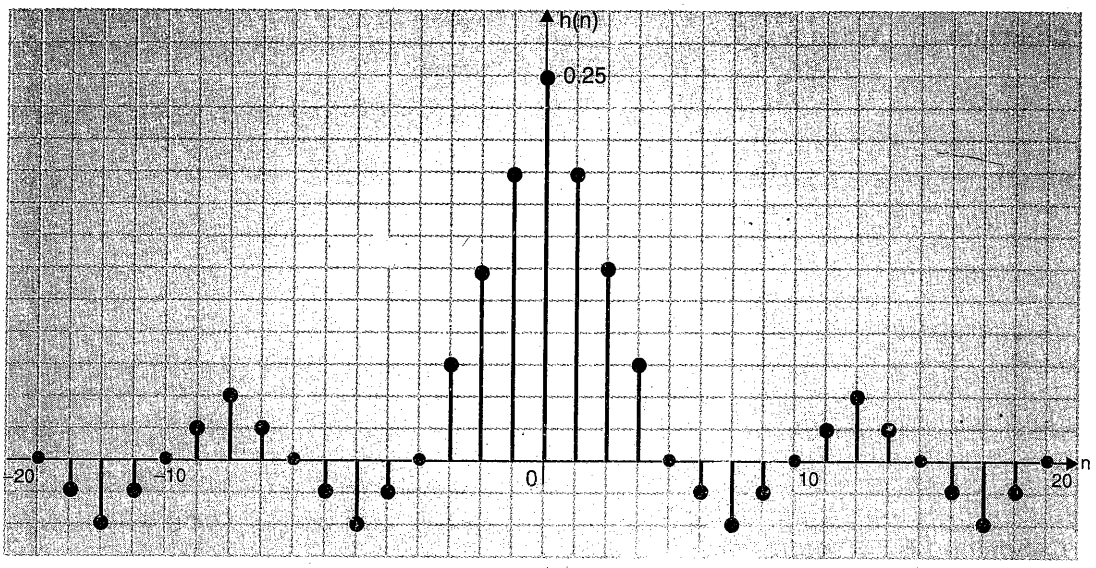


Fig. I-33(b) : Response h (n) of ideal LPF

From Fig. I-33(b), we can easily conclude that h (n) is present for negative values of 'n'. That means it is noncausal. We had discussed that if the frequency response of filter is causal then only it can be realized. Thus ideal filters are practically not realizable.

We have studied that, in practical cases it is not possible to realize idle filter. An important reason for this is, idle filters are non-causal.

Theory Questions

Q. 1 Write note on : Parallel realization of DT system.

Q. 2 Explain direct form-I and direct form-II IIR filter structures and compare w.r.t. memory requirements and computational complexity.

Q. 3 Obtain direct form II filter structure for IIR system whose transfer function is given by :

$$H(z) = \frac{\sum_{k=0}^{M} b_k Z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

Compare Direct form II with form I.

Q. 4 List the various forms of realizations of IIR system.

Q. 5 Mention advantages of direct form II and cascade structures.

Unsolved Numericals

Q. 1 Obtain direct form-I and form-II filter structure for a transfer function

$$H(Z) = \frac{Z^{-1} - 3Z^{-2}}{(10 - Z^{-1})(1 + 0.5Z^{-1} + 0.5Z^{-2})}$$

Ans. :

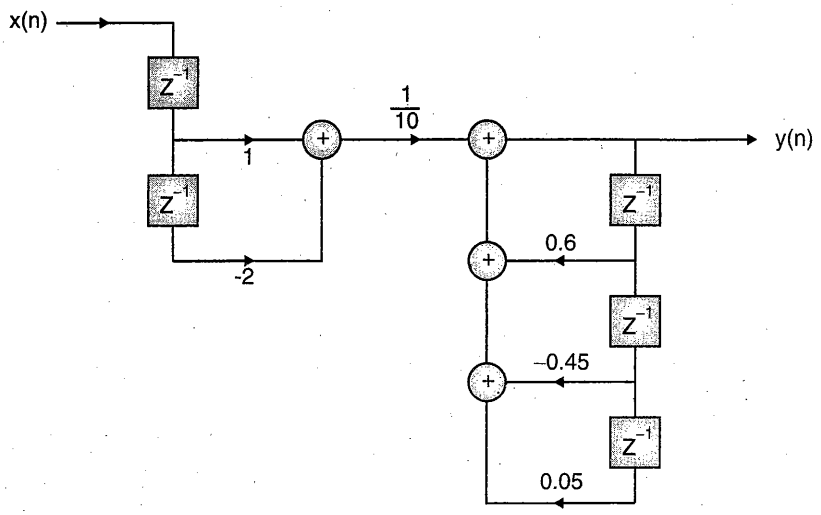


Fig. I-34

Q. 2 Realize the following transfer function using direct form-II structure

$$H(Z) = \frac{1 + 0.2\,Z^{-1} - 0.2\,Z^{-2}}{1 - 0.2\,Z^{-1} + 0.3\,Z^{-2} + Z^{-3}}$$

Ans. :

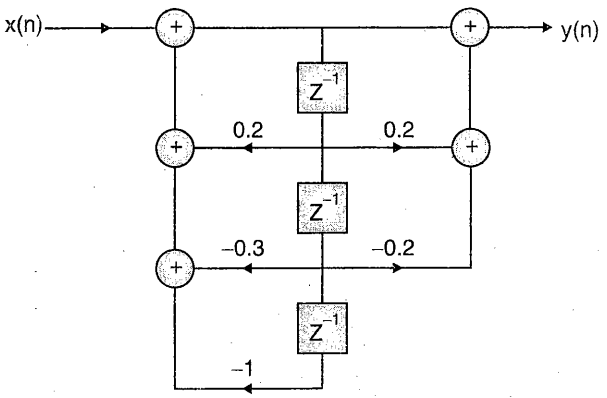


Fig. I-35

Q. 3 Obtain cascade realization of discrete time linear causal system defined by difference equation

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n) + \frac{1}{3}x(n-1)$$

Ans. :

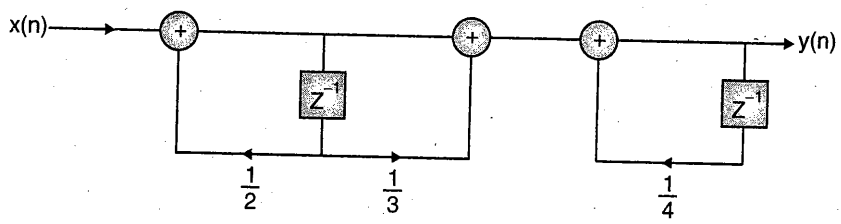


Fig. I-36

Q. 4 A system is described by transfer function H(Z) is given by

$$H(Z) = 3 + \frac{4Z}{Z-\frac{1}{2}} - \frac{2}{Z-\frac{1}{4}}$$

(i) Does this H(z) represent FIR or IIR filter, why ?

(ii) Give difference equation realization of this system using direct form-I and draw the structure.

(iii) Draw the block diagram for direct form-II canonic realization and give the governing equations for implementation.

Ans. :

(i) The given system has poles as well as zeros. So it represents IIR filter ; because FIR filter is always all zero system.

(ii)

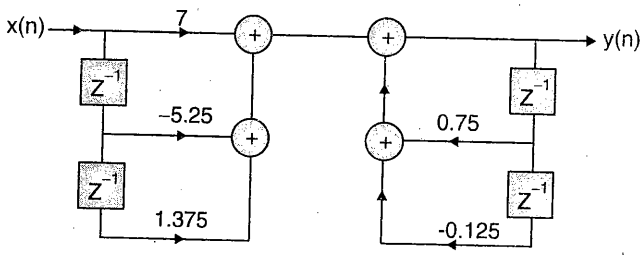


Fig. I-37(a)

(iii)

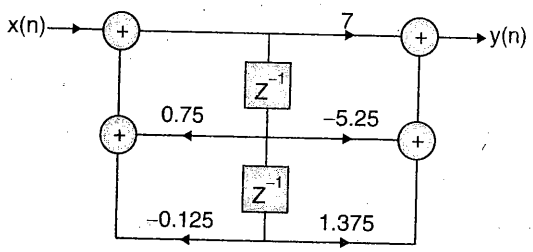


Fig. I-37(b)

Q. 5 A system function is specified by its transfer function,

$$H(Z) = \frac{(Z-1)(Z-2)(Z+1)(Z)}{\left[Z-\left(\frac{1}{2}+j\frac{1}{2}\right)\right]\left[Z-\left(\frac{1}{2}-j\frac{1}{2}\right)\right]\left[\left(Z-\frac{j}{4}\right)(Z+j/4)\right]}$$

Realize the system function in following forms

(i) Direct form-I

(ii) Direct form-II

(iii) Cascade of two biquadratic sections.

(iv) Parallel realization in constant, linear and biquadratic sections.

Ans. :

(i)

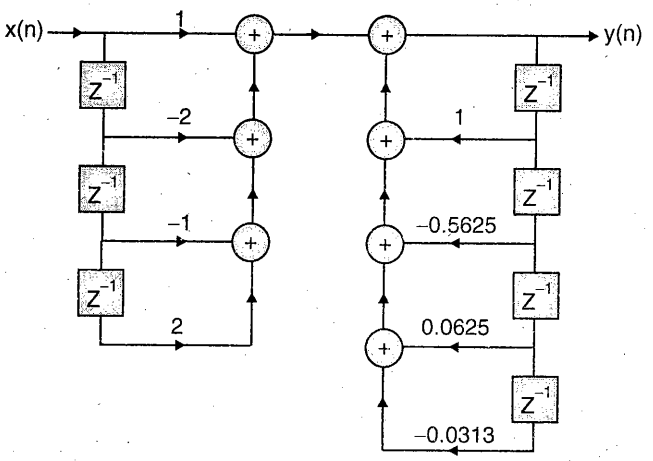


Fig. I-38(a)

(ii)

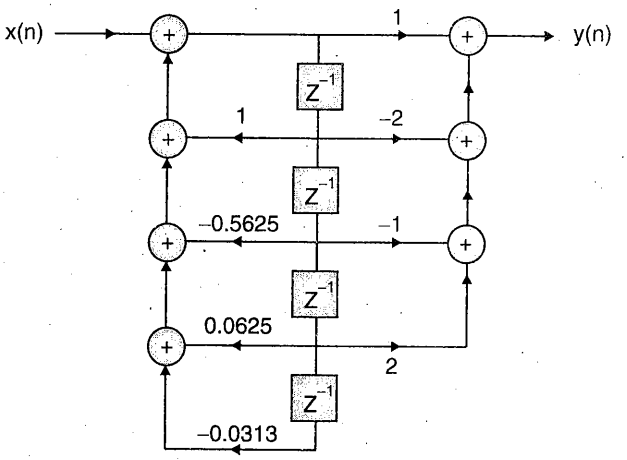


Fig. I-38(b)

(iii)

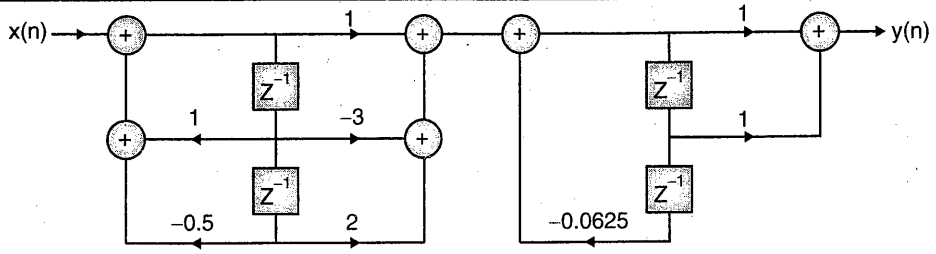


Fig. I-38(c)

(iv)

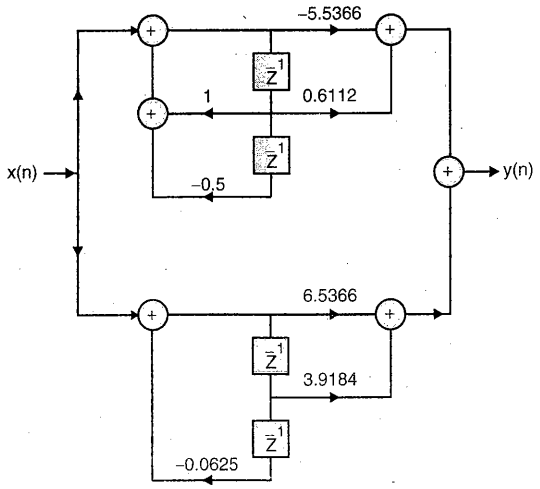


Fig. I-38(d)

Q. 6 Obtain parallel realization of a system described by,

$$H(Z) = \frac{(1 - 0.5\,Z^{-1})(1 - Z^{-1})}{(1 - 0.2\,Z^{-1})(1 + 0.8\,Z^{-1})(1 - 0.8\,Z^{-1})}$$

Ans. :

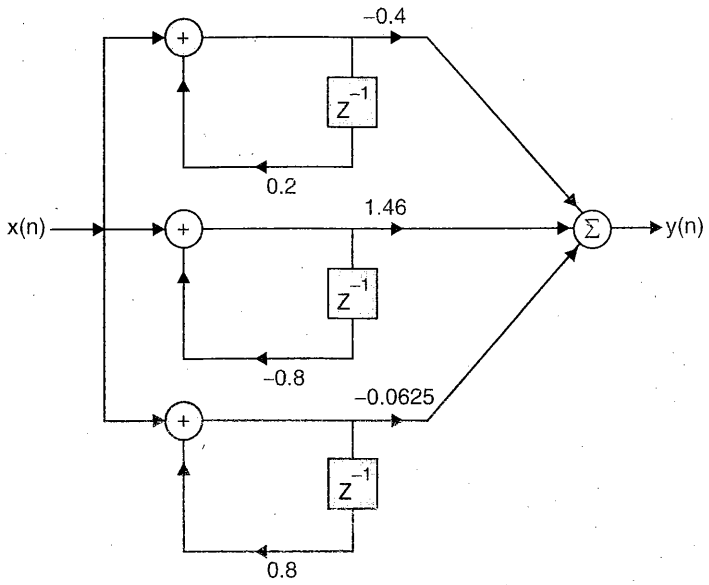


Fig. I-39

Q. 7 A system is cascade combination of two subsystems

$$H_1(Z) = \frac{Z}{Z^2 + 2.5Z + 0.5} \text{ and } H_2(Z) = \frac{Z}{Z+1}$$

(i) Realize it using parallel form (ii) Draw its poles and zeros.

Ans. :

(i)

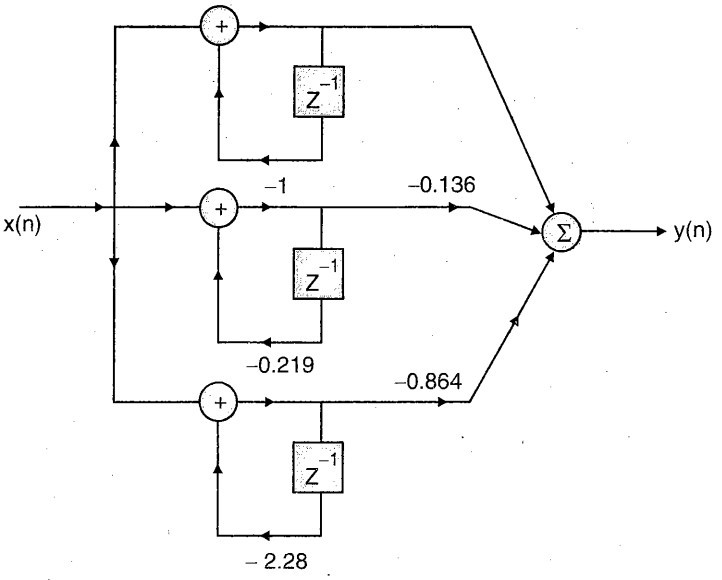


Fig. I-40(a)

(ii)

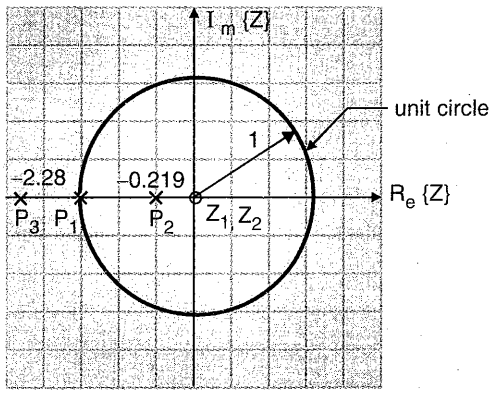


Fig. I-40(b)

Q. 8 A system is described by the following difference equation

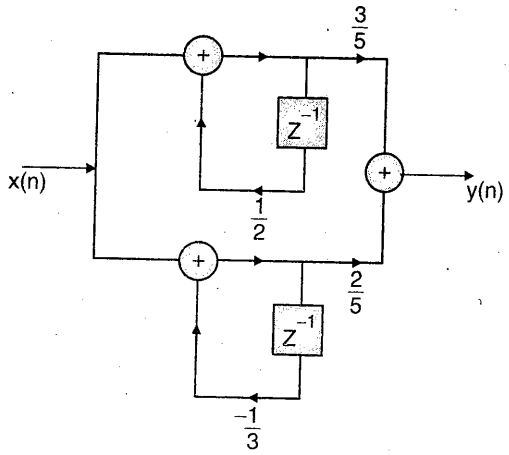
$$y(n)-\frac{1}{6}y(n-1)-\frac{1}{6}y(n-2)=x(n)$$

(i) Find step response of system.

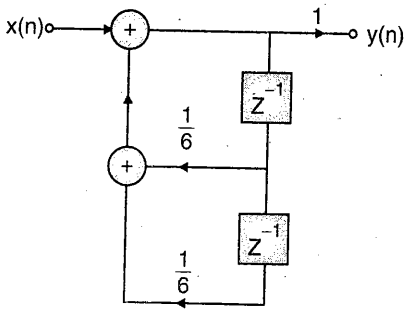
(ii) Realize the system using parallel realization. Compare the number of elements used for parallel and for direct form-II.

(iii) Draw its poles and zeros.

Ans. : (i) $y(n) = -\frac{3}{5}\left(\frac{1}{2}\right)^n u(n)+\frac{1}{10}\left(-\frac{1}{3}\right)^n u(n)+\frac{3}{2}(1)^n u(n)$



(a) Parallel form



(b) Direct form-II

Fig. I-41

(ii) Comparison between parallel and direct form-II :

Number of elements	Parallel	Direct Form-II
Address	3	2
Delays	2	2
Constant Multipliess	3	2

(iii)

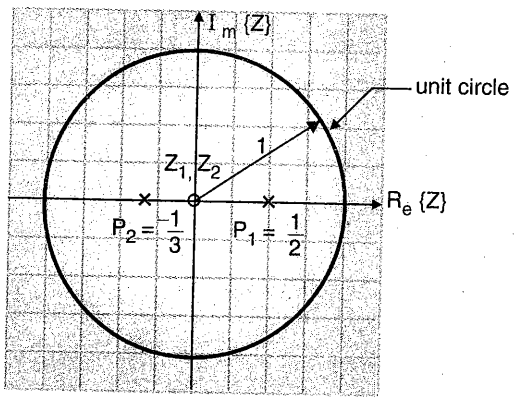


Fig. I-41(c)

Q. 9 System function of initially relaxed causal system is given by,

$$H(Z) = \frac{0.7\,(Z+0.6)\,(Z-0.6)}{Z^2 - 0.1\,Z - 0.72}$$

(i) Find the corresponding difference equation of system.

(ii) Realize the filter using parallel form.

(iii) Find impulse response of the filter.

Ans. :

(i) $$y(n) = 0.1\,y(n-1) + 0.72\,y(n-2) + 0.7\,x(n) - 0.252\,x(n-2)$$

(ii)

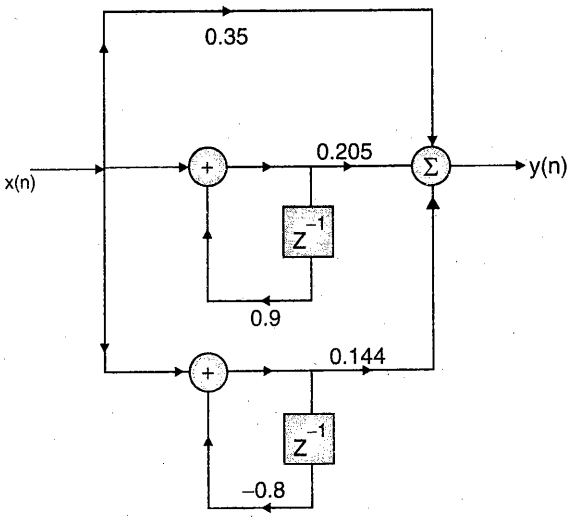


Fig. I-42

(iii)

$$h(n) = 0.35 + 0.205\,(0.9)^n\,u(n) + 0.144\,(-0.8)^n\,u(n)$$

Q. 10 A causal LTI system has a transfer function,

$H(Z) = H_1(Z) \cdot H_2(Z)$ where,

$$H_1(Z) = \frac{1 - 0.2\,Z^{-1}}{1 + 0.5\,Z^{-1}}$$

$$H_2(Z) = \frac{1}{1 + 0.3\,Z^{-1}}$$

(i) If the system is stable, give its ROC condition.

(ii) Show cascade and parallel realization.

(iii) Find impulse response of the system.

(iv) Find the system response if $X(Z) = \dfrac{1}{1 + 0.2\,Z^{-1}}$

(v) Draw the pole-zero diagram.

Ans. :

(i) ROC : $|Z| > 0.5$

(ii)

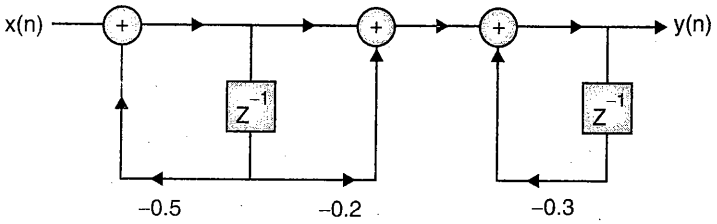


Fig. I-43(a) : Cascade realization

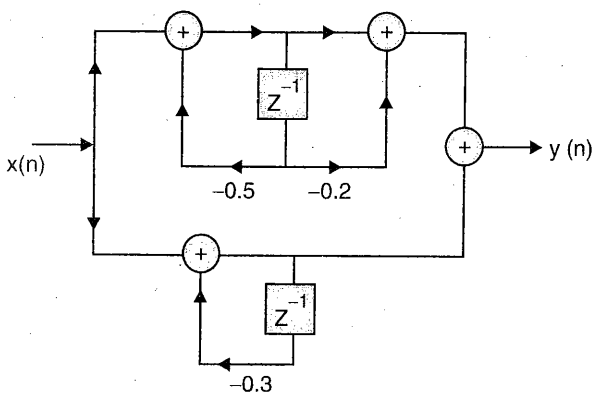


Fig. I-43(b) : Parallel realization

(iii) $h(n) = 3.5\,(-0.5)^n\,u(n) - 2.5\,(-0.3)^n\,u(n)$

(iv) $y(n) = 5.83\,(-0.5)^n\,u(n) - 7.5\,(-0.3)^n\,u(n) + 2.67\,(-0.2)^n\,u(n)$

(v)

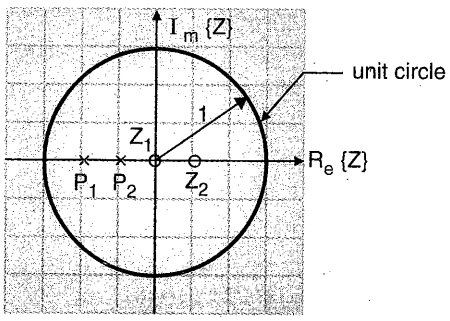


Fig. I-43(c)

Q. 11 Fig. I-44(a) shows direct form-II realization of IIR filter.

(a) Obtain direct form-I structure. (b) Find transfer function H(Z) of filter.
(c) Realize the filter using cascade form using first order modules.
(d) Find impulse response of filter. (e) Show pole-zero diagram of filter.

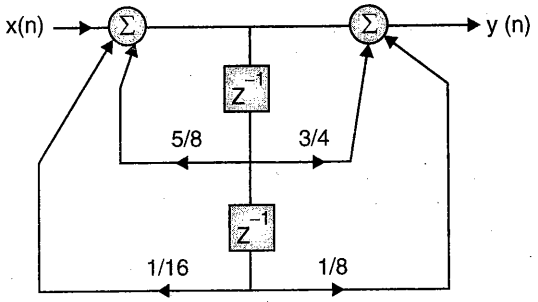


Fig. I-44(a)

Ans. :

(a)

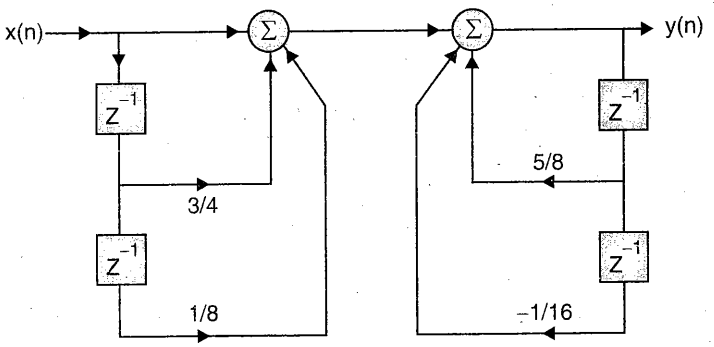


Fig. I-44(b)

(b)
$$H(Z) = \frac{1+\frac{3}{4}Z^{-1}+\frac{1}{8}Z^{-2}}{1-\frac{5}{8}Z^{-1}+\frac{1}{16}Z^{-2}}$$

(c)

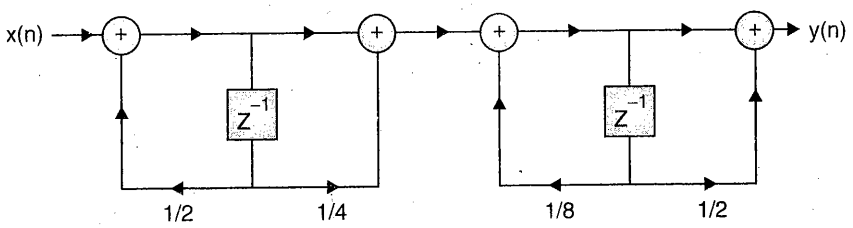


Fig. I-44(c)

(d) $$h(n) = 2\delta(n) + 4\left(\frac{1}{2}\right)^n u(n) - 5\left(\frac{1}{8}\right)^n u(n)$$

(e)

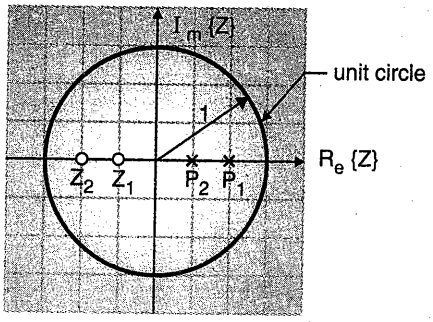


Fig. I-44(d)

❑❑❑

FIR Filter Design

Section No.	Name of the Topic
1.1	Introduction
1.2	Symmetric and Antisymmetric FIR Filters
1.3	Design of FIR Filters using Windows
1.4	Design of Linear Phase FIR Filters using Frequency Sampling Method
1.5	Alternation Theorem in Equiripple Linear Phase FIR Filters
1.6	FIR Differentiators

• Code : DSP8

1.1 Introduction :

FIR stands for Finite Impulse Response. FIR filters are called as non-recursive filters because they do not use the feedback. Before studying the design of FIR filters; we will discuss one important characteristic of FIR filter.

1.1.1 FIR Filters are Inherently Stable :

We know that LSI system is said to be stable if bounded input produces bounded output (BIBO). We have the difference equation of FIR filter,

$$y(n) = \sum_{k=0}^{M-1} b_k x(n-k) \qquad \ldots(1)$$

Taking Z transform of Equation (1) we get,

$$Y(Z) = \sum_{k=0}^{M-1} b_k Z^{-k} X(Z) \qquad \ldots(2)$$

Now transfer function, $H(Z) = \dfrac{Y(Z)}{X(Z)}$

Thus from Equation (2) we get,

$$H(Z) = \frac{Y(Z)}{X(Z)} = \sum_{k=0}^{M-1} b_k Z^{-k} \qquad \ldots(3)$$

Taking IZT of Equation (3) we get

$$h(n) = \begin{cases} b_n & \text{For } 0 \le n \le M-1 \\ 0 & \text{otherwise} \end{cases} \qquad \ldots(4)$$

Expanding Equation (1) we get,

$$y(n) = b_0 x(n) + b_1 x(n-1) + ... + b_{M-1} x(n-M+1) \qquad \ldots(5)$$

Using Equation (4) we get,

$$y(n) = h(0) x(n) + h(1) x(n-1) + ... + h(M-1) x(h-M+1) \qquad \ldots(6)$$

Here h (0), h (1) ... are constants that means they are bounded. Now from Equation (6); the output will be bounded if we apply bounded input. That means for every bounded input; the output of FIR filter is bounded. Thus FIR filters are inherently stable.

1.2 Symmetric and Antisymmetric FIR Filters :

We will discuss the symmetry and antisymmetry of FIR filters. These conditions are related to their unit sample response h (n).

The unit sample response of FIR filter is symmetric if it satisfies the condition.

$$h(n) = h(M-1-n) \ldots\ldots n = 0, 1 \ldots\ldots M-1 \quad \ldots(1)$$

Here M = Number of samples; so if M = 8 we get,

for $n = 0 \Rightarrow h(0) = h(8-1-0) = h(7)$

For $n = 1 \Rightarrow h(1) = h(8-1-1) = h(6)$ etc.

If h (n) is symmetric then, the filter is symmetric.

Now unit sample response of FIR filter is antisymmetric if it satisfies the condition,

$$h(n) = -h(M-1-n), \quad n = 0, 1 \ldots\ldots M-1 \quad \ldots(2)$$

If this condition is satisfied then the filter is antisymmetric

Now the phase of FIR filter is given by,

$$\angle H(\omega) = \begin{cases} -\omega\left(\frac{M-1}{2}\right) & \text{for } |H(\omega)| > 0 \\ -\omega\left(\frac{M-1}{2}\right)+\pi & \text{for } |H(\omega)| < 0 \end{cases} \quad \ldots(3)$$

This equation shows that the phase of FIR filter is piecewise linear. Thus for the symmetric and antisymmetric FIR filters; the condition for linear phase is,

$$h(n) = \pm h(M-1-n)$$

FIR filler can be characterized by,

$$H(Z) = \sum_{k=0}^{M-1} h(k) Z^{-k} \quad \ldots(4)$$

Here M is the length of filter and M – 1 is the order of filter. The frequency response for different conditions is as follows :

(1) If M is odd and symmetric then,

$$H(e^{j\omega}) = e^{\frac{-j\omega(M-1)}{2}} \left[h\left(\frac{M-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n) \cos\omega \left(\frac{M-1}{2} - n\right) \right]$$

(2) If M is odd and antisymmetric then,

$$H(e^{j\omega}) = h\left(\frac{M-1}{2}\right) + 2 \sum_{n=0}^{\frac{M-3}{2}} h(n) \sin\left(\frac{M-1}{2} - n\right)\omega.$$

(3) If M is even and symmetric then,

$$H(e^{j\omega}) = 2 \sum_{n=0}^{\frac{M}{2}-1} h(n) \cos\left(\frac{M-1}{2} - n\right)\omega$$

(4) If M is even and antisymmetric then,

$$H(e^{j\omega}) = 2 \sum_{n=0}^{\frac{M}{2}-1} h(n) \sin\left(\frac{M-1}{2} - n\right)\omega$$

1.2.1 Magnitude Specifications :

The magnitude specifications of FIR filter are as shown in Fig. J-1.

Here δ_p = Peak passband deviation

δ_s = Stopband deviation

ω_p = passband edge frequency

ω_s = Stopband edge frequency

Δf = Transition width = $\frac{\omega_s - \omega_p}{2\pi}$

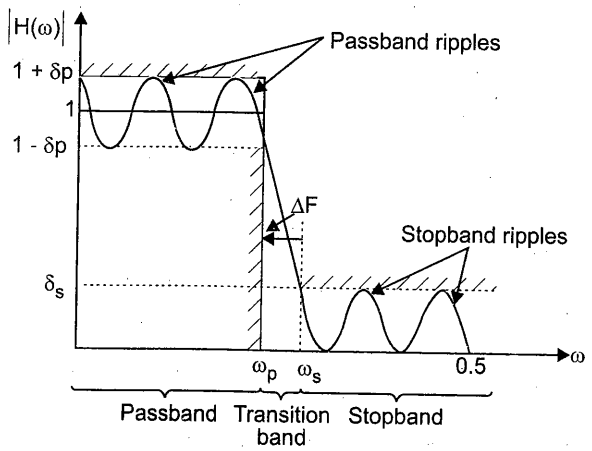


Fig. J -1 : Magnitude specifications

Thus the magnitude specifications of FIR filter can be written as,

$$1 - \delta_p \leq |H(\omega)| \leq 1 + \delta_p \quad \text{for} \quad 0 \leq \omega \leq \omega_p$$

$$\text{and} \quad 0 \leq |H(\omega)| \leq \delta_s \quad \text{for} \quad \omega_s \leq \omega \leq \pi$$

These are the magnitude specifications. For the phase response, we have assumed a linear phase. Therefore while designing the FIR filter only the symmetry of the filter is indicated.

1.2.2 General Filter Coefficient Equation :

The FIR and IIR filters are basically Linear shift Invariant (LSI) systems which are characterized by unit sample response. The FIR system has finite duration unit sample response, as follows,

$$h(n) = 0 \qquad \text{for } n < 0 \text{ and } n \geq M.$$

FIR system is a nonrecursive system i.e. it depends only on past and present input. The difference equation of LSI system is given as,

$$y(n) = -\sum_{K=1}^{N} a_K y(n-K) + \sum_{K=0}^{M} b_k x(n-K) \qquad ...(1)$$

But, the first term represents past outputs and second term represents past and present input. Hence, the difference equation for FIR system is,

$$y(n) = \sum_{K=0}^{M} b_K x(n-K) \qquad ...(2)$$

If we consider 'M' coefficients, then

$$y(n) = \sum_{K=0}^{M-1} b_K x(n-K) \qquad ...(3)$$

Taking 'Z' transform of both sides,

$$Y(Z) = \sum_{K=0}^{M-1} b_K Z^{-K} X(Z)$$

$$\therefore \quad \frac{Y(Z)}{X(Z)} = \sum_{K=0}^{M-1} b_K Z^{-K} \qquad ...(4)$$

$$\text{Let,} \quad \frac{Y(Z)}{X(Z)} = H(Z) \qquad ...(5)$$

which is a system function of FIR system. Hence, from Equation (4) and (5),

$$H(Z) = \sum_{K=0}^{M-1} b_K Z^{-K} \qquad ...(6)$$

Taking inverse 'Z' transform of Equation (6) we get, unit sample response of FIR system as,

$$h(n) = \begin{cases} b_n & \text{for } 0 \leq n \leq M-1 \\ 0 & \text{otherwise} \end{cases}$$

Which gives filter coefficient that means it represents general filter coefficient equation.

Solved Problems :

Prob. 1 : Show that if Z_1 is the zero of linear phase filter then $\frac{1}{Z_1}$ is also zero of the filter.

Soln. : If h (n) is an impulse response then,

$$H(Z) = Z\{h(n)\}$$

For FIR filter having length M we have,

$$H(Z) = \sum_{n=0}^{M-1} h(n) Z^{-n} \quad \ldots(1)$$

If Z_1 is the zero of H (Z) then,

$$H(Z_1) = \sum_{n=0}^{M-1} h(n) Z_1^{-n} = 0 \quad \ldots(2)$$

We have to show that $\frac{1}{Z_1}$ or Z_1^{-1} is also zero of H (Z).

$$\therefore \quad H(Z_1^{-1}) = \sum_{n=0}^{M-1} h(n) (Z_1^{-1})^{-n}$$

$$\therefore \quad H(Z_1^{-1}) = \sum_{n=0}^{M-1} h(n) Z_1^{n} \quad \ldots(3)$$

For the linear phase FIR filter, we have the condition

$$h(n) = h(M-1-n).$$

Putting this value in Equation (3) we get,

$$H(Z_1^{-1}) = \sum_{n=0}^{M-1} h(M-1-n) Z_1^{n} \quad \ldots(4)$$

Substitute $M - 1 - n = p \Rightarrow n = M - 1 - p$

The limits will change as follows :

When $n = 0$, $p = M - 1$

and when $n = M - 1$, $p = 0$

$$\therefore \quad H(Z_1^{-1}) = \sum_{p=0}^{M-1} h(p) Z_1^{M-1-p}$$

$$\therefore \quad H(Z_1^{-1}) = \sum_{p=0}^{M-1} h(p) Z_1^{M-1} Z_1^{-p}$$

$$\therefore \quad H(Z_1^{-1}) = Z_1^{M-1} \sum_{p=0}^{M-1} h(p) Z_1^{-p} \qquad ...(5)$$

Now changing the variable in summation; p by n we get

$$H(Z_1^{-1}) = Z_1^{M-1} \sum_{n=0}^{M-1} h(n) Z_1^{-n} \qquad ...(6)$$

According to Equation (2); the summation term is $H(Z_1)$ and it is zero.

$$\therefore \quad H(Z_1^{-1}) = Z_1^{M-1} \cdot 0 = 0 \qquad ...(7)$$

Thus from Equation (7) we can conclude that Z_1^{-1} or $\frac{1}{Z_1}$ is also zero for linear phase FIR filter.

Prob. 2 : A fourth order FIR filter has following two pairs of complex conjugate zeros.

$Z_1, Z_2 = e^{\pm j\frac{\pi}{3}}$ and $Z_3, Z_4 = 2\, e^{\pm\frac{j\pi}{3}}$

State whether the filter has linear phase property.

Soln. :

The order of FIR filter is given by N – 1; in this case order is 4. Thus N – 1 = 4. That means these are four poles and four zeros. Given zeros are :

$Z_1 = e^{j\frac{\pi}{3}}$, $\quad Z_2 = e^{-\frac{j\pi}{3}}$

$Z_3 = 2\, e^{\frac{j\pi}{3}}$, $\quad Z_4 = 2\, e^{-j\pi/3}$

For FIR filter zeros occur in reciprocal order. That means if Z_1 is the zero of system then $\frac{1}{Z_1}$ is also zero of the system. But for Z_3 and Z_4 this condition is not satisfied. Thus this filter will not have linear phase property.

Prob. 3 : Frequency response of a filter is given by,

$H(e^{j\omega}) = e^{j3\omega}[2 + 1.8\cos 3\omega + 1.2\cos 2\omega + 0.5\cos\omega]$

Find impulse response h (n) of a filter.

Soln. : Given frequency response is,

$$H(e^{j\omega}) = e^{-j3\omega}[2 + 1.8\cos 3\omega + 1.2\cos 2\omega + 0.5\cos\omega] \qquad ...(1)$$

Now the phase is given by,

$$\theta(\omega) = e^{-j\omega\frac{(M-1)}{2}} \qquad \therefore \quad \frac{M-1}{2} = 3, \qquad \therefore M = 7$$

That means length of filter is 7 and order of filter is 6. Since length is odd we have the expression for frequency response as :

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{M-1}{2}\right)}\left[h\left(\frac{M-1}{2}\right)+\sum_{n=0}^{\left(\frac{M-3}{2}\right)} 2h(n)\cos\left\{\omega\left(\frac{M-1}{2}-n\right)\right\}\right]$$

Here M = 7

$$\therefore \quad H(e^{j\omega}) = e^{-j\omega 3}\left[h(3)+\sum_{n=0}^{2} 2h(n)\cos\{\omega(n-3)\}\right]$$

$$\therefore \quad H(e^{j\omega}) = e^{-j\omega 3}\left[h(3)+2h(0)\cos 3\omega+2h(1)\cos 2\omega+2h(2)\cos\omega\right] \qquad ...(2)$$

Comparing Equations (1) and (2) we get,

$h(3) = 2, \quad 2h(0) = 1.8 \Rightarrow h(0) = 0.9$

$2h(1) = 1.2 \Rightarrow h(1) = 0.6$

and $2h(2) = 0.5 \Rightarrow h(2) = 0.25$

Since length of filter is M = 7; there are 7 samples in the impulse response from h (0) to h (6).

Now for linear phase filter we have the condition,

$h(n) = h(M-1-n) \qquad n = 0, 1 \ldots M-1.$

For $n = 0 \Rightarrow h(0) = h(7-1-0) = h(6) \qquad \therefore h(6) = h(0) = 0.9$

For $n = 1 \Rightarrow h(1) = h(7-1-1) = h(5) \qquad \therefore h(5) = h(1) = 0.6$

For $n = 2 \Rightarrow h(2) = h(7-1-2) = h(4) \qquad \therefore h(4) = h(2) = 0.25$

Thus impulse response h (n) is,

$h(n) = \{ h(0), h(1), h(2), h(3), h(4), h(5), h(6) \}$

$\therefore \quad h(n) = \{ 0.9, 0.6, 0.25, 2, 0.25, 0.6, 0.9 \}$

Prob. 4 : A linear phase FIR filter has response given by $e^{-j2\omega}$. What is the order of filter ? Justify your answer.

Soln. : Assume filter is symmetric.

When M is odd and symmetric then the frequency response is given by,

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{M-1}{2}\right)}\left[h\left(\frac{M-1}{2}\right)+\sum_{n=0}^{\frac{M-3}{2}} 2h(n)\cos\omega\left(\frac{M-1}{2}-n\right)\right]$$

Here $e^{-j\omega\left(\frac{M-1}{2}\right)}$ represents phase $\theta(\omega)$.

given phase is $\theta(\omega) = e^{-j2\omega}$

Comparing these two equations,

$$\frac{M-1}{2} = 2$$

$$\therefore \quad M-1 = 4$$

Here M represents length of filter = 5 and M – 1 represents order of filter. Thus order of filter is 4.

Prob. 5 : Prove that a linear phase FIR filter having antisymmetric coefficients cannot be a high-pass filter.

OR

To design a high pass linear phase filter, which type of linear phase filter can be used ?

Soln. :

Let 'M' be the order of filter.

When M is even and antisymmetric we have,

$$\left| H(e^{j\omega}) \right| = 2 \sum_{n=0}^{\frac{M}{2}-1} h(n) \sin\left(\frac{M-1}{2} - n\right)\omega \qquad ...(1)$$

And when M is odd and antisymmetric,

$$\left| H(e^{j\omega}) \right| = h\left(\frac{M-1}{2}\right) + 2 \sum_{n=0}^{\frac{M-3}{2}} h(n) \sin\left(\frac{M-1}{2} - n\right)\omega \qquad ...(2)$$

In Equation (1), at $\omega = 0$, $|H(e^{j\omega})| = 0$ and in Equation (2) for $h\left(\frac{M-1}{2}\right) = 0$; at $\omega = 0$, $|H(e^{j\omega})| = 0$. Thus low pass filter cannot be designed using antisymmetric coefficients. Now when M is even then at the frequency $\omega = \pi$, the response $|H(e^{j\omega})|$ is zero. Thus it is not possible to design a high pass filter. For the design of high pass filter Hilbert transforms can be used.

1.3 Design of FIR Filters using Windows :

Different types of windows are used to design FIR filter. First we will discuss the design of FIR filter using rectangular window. The rectangular window is as shown in Fig. J-2

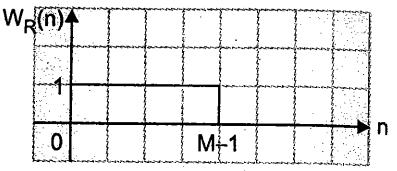


Fig. J-2 : Rectangular window

It is denoted by $W_R(n)$. Its magnitude is 1 for the range, $n = 0$ to $M-1$. Now let $h_d(n)$ be the impulse response having infinite duration. If $h_d(n)$ is multiplied by $W_R(n)$ then a finite impulse response is obtained as shown in Fig. J-3. That means we will get only limited pulse of $h_d(n)$; not all (∞) pulse. Since we are truncating the input sequence by using a window, this process is called as truncation process. Since the shape of window function is rectangular; it is called as rectangular window.

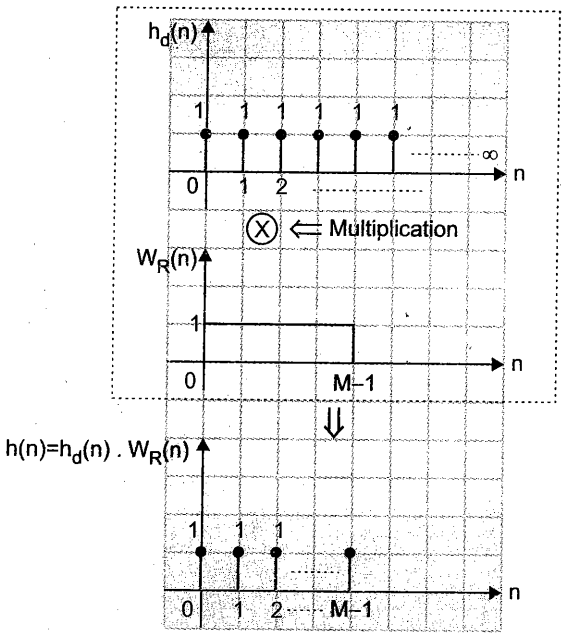


Fig. J-3 : Truncation process

Magnitude response of rectangular window :

The rectangular window is defined as,

$$W_R(n) = \begin{cases} 1 & \text{for } n = 0, 1, 2 \ldots M-1 \\ 0 & \text{otherwise} \end{cases} \quad \ldots(1)$$

Let $h_d(n)$ be infinite duration impulse response. We know that the finite duration impulse response $h(n)$ is obtained by multiplying $h_d(n)$ by $W_R(n)$.

$$\therefore \quad h(n) = h_d(n) \cdot W_R(n) \qquad ...(2)$$

We will denote fourier transform of $W_R(n)$ by $W_R(\omega)$. Thus using the definition of fourier transform we can write,

$$W_R(\omega) = \sum_{n=0}^{M-1} W_R(n) e^{-j\omega n} \qquad ...(3)$$

But the value of $W_R(n)$ is 1 for the range $n = 0$ to $M - 1$.

$$\therefore \quad W_R(\omega) = \sum_{n=0}^{M-1} 1 \cdot e^{-j\omega n} \qquad ...(4)$$

We can represent the window sequence as,

$$W_R(n) = u(n) - u(n-M)$$

here $u(n)$ is unit step having duration $n = 0$ to $n = \infty$ and $u(n-M)$ is delay unit step.

Thus Equation (4) becomes,

$$W_R(\omega) = \sum_{n=0}^{\infty} [u(n) - u(n-M)] e^{-j\omega n}$$

$$\therefore \quad W_R(\omega) = \sum_{n=0}^{\infty} u(n) e^{-j\omega n} - \sum_{n=0}^{\infty} u(n-M) e^{-j\omega n} \qquad ...(5)$$

Consider the first term at R.H.S. It represents fourier transform (FT) of unit step. And we have,

$$\text{F.T. of } u(n) = \sum_{n=0}^{\infty} 1 \cdot e^{-j\omega n} = \sum_{n=0}^{\infty} \left(e^{-j\omega}\right)^n = \frac{1}{1 - e^{-j\omega}}$$

Now consider the second term. It represents the fourier tranform of delayed unit step.

$$\therefore \quad \text{F.T. of } u(n-M) \longleftrightarrow e^{-j\omega M} F\{u(n)\}$$

$$= e^{-j\omega M} \frac{1}{1 - e^{-j\omega}} = \frac{e^{-j\omega M}}{1 - e^{-j\omega}}$$

Thus Equation (5) becomes,

$$W_R(\omega) = \frac{1}{1-e^{-j\omega}} - \frac{e^{-j\omega M}}{1-e^{-j\omega}}$$

$$\therefore \quad W_R(\omega) = \frac{1-e^{-j\omega M}}{1-e^{-j\omega}} \qquad ...(6)$$

We will rearrange Equation (6) as follows,

$$W_R(\omega) = \frac{e^{-j\omega\frac{M}{2}} \cdot e^{j\omega\frac{M}{2}} - e^{-j\omega\frac{M}{2}} \cdot e^{-j\omega\frac{M}{2}}}{e^{-j\frac{\omega}{2}} \cdot e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}} \cdot e^{-j\frac{\omega}{2}}}$$

$$\therefore \quad W_R(\omega) = \frac{e^{-j\omega\frac{M}{2}}\left(e^{j\omega\frac{M}{2}} - e^{-j\omega\frac{M}{2}}\right)}{e^{-j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right)} \qquad ...(7)$$

We have the trignometric identity,

$$e^{j\theta} - e^{-j\theta} = 2\sin\theta$$

Thus Equation (7) becomes,

$$W_R(\omega) = \frac{e^{-j\omega\frac{M}{2}}\left[2\sin\left(\frac{\omega M}{2}\right)\right]}{e^{-j\frac{\omega}{2}}\left[2\sin\left(\frac{\omega}{2}\right)\right]}$$

$$\therefore \quad W_R(\omega) = e^{-j\omega\frac{M}{2}} \cdot e^{j\frac{\omega}{2}} \cdot \frac{\sin\left(\frac{\omega M}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

$$\therefore \quad W_R(\omega) = \frac{\sin\left(\frac{\omega M}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} \cdot e^{-j\omega\left(\frac{M-1}{2}\right)} \qquad ...(8)$$

Now $W_R(\omega)$ can be expressed in terms of magnitude and angle as;

$$W_R(\omega) = |W_R(\omega)| \cdot e^{j\angle W_R(\omega)} \qquad ...(9)$$

Comparing Equations (8) and (9) we can write,

$$|W_R(\omega)| = \frac{\left|\sin\left(\frac{\omega M}{2}\right)\right|}{\left|\sin\left(\frac{\omega}{2}\right)\right|} \qquad ...(10)$$

1.3.1 Properties of Commenly used Windows :

Some other types of window functions are as follows :

1) Hamming window
2) Hanning window
3) Triangular (Bartlett) window
4) Blackman window
5) Kaiser window

We will consider that the range of each window is from – Q to Q. Here Q is positive integer number.

(1) Hamming window :

Hamming window function is given by,

$$W_{H_m}[n] = \begin{cases} 0.54 + 0.46 \cos \dfrac{2\pi n}{M-1} \\ 0 \end{cases} \quad \text{elesewhere}$$

Fig. J-4 shows shape of Hamming window.

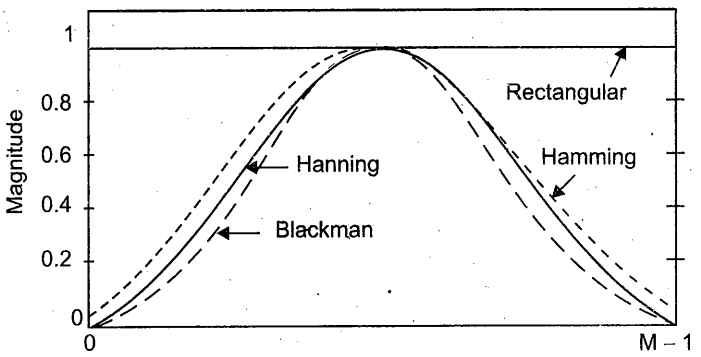


Fig. J-4

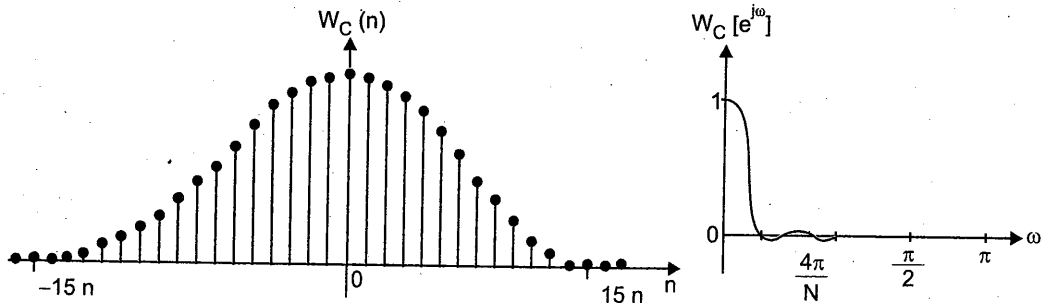
$W_{H_m}[n]$ is a bell-shaped sequence that is symmetric about n = 0.

(2) Hanning window function :

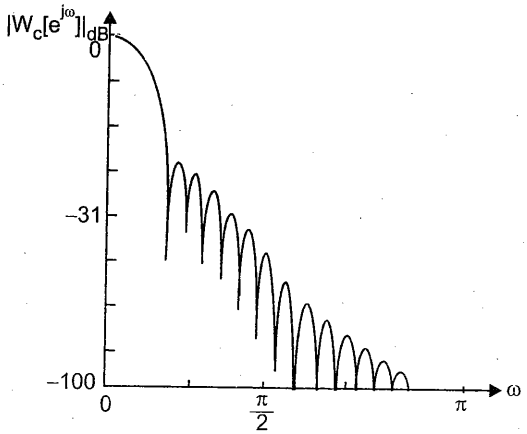
Hanning window function is also called as *Raised-cosine window*. The function is denoted by,

$$W_{hn} = \frac{1}{2}\left[1 - \frac{\cos 2\pi n}{M-1}\right]$$

Refer Figs. J-5(a), (b) and (c) to have a look to the spectrum.



(a) Hanning window sequence (N = 31) **(b) Spectrum**



(c) Log-magnitude spectrum

Fig. J-5

Magnitude of first side-lobe level is − 31 dB relative to maximum value.

(3) Triangular window function OR Bartlett window :

Triangular function is like ***tapering the rectangular window sequence linearly from the middle to the ends***. Triangular window function can be given by

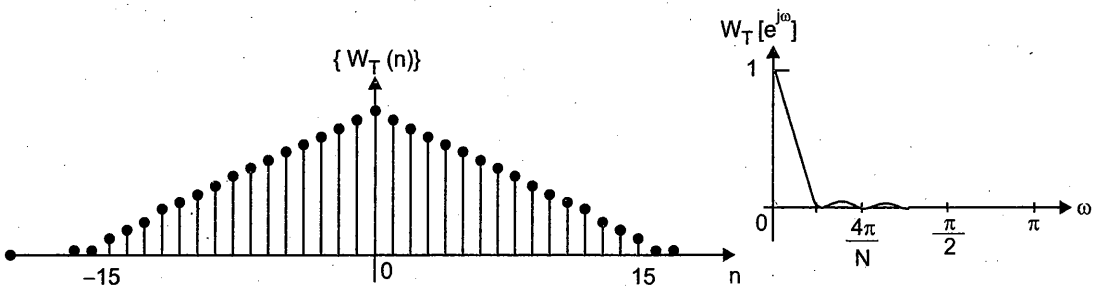
$$W_T [n] = 1 - \frac{2|n|}{M-1} \text{ for } |n| \leq M-1$$

Window and its spectrum is shown in Fig. J-6.

Sidelobe level is smaller than that of rectangular window.

Triangular window produces smoother magnitude response than that of rectangular window function.

The transition from passband to stopband is not as steep as that for the rectangular window. In the stopband, the response is smoother, but attenuation is less than that produced by rectangular window, therefore because of this characteristic, triangular window is not usually good choice.

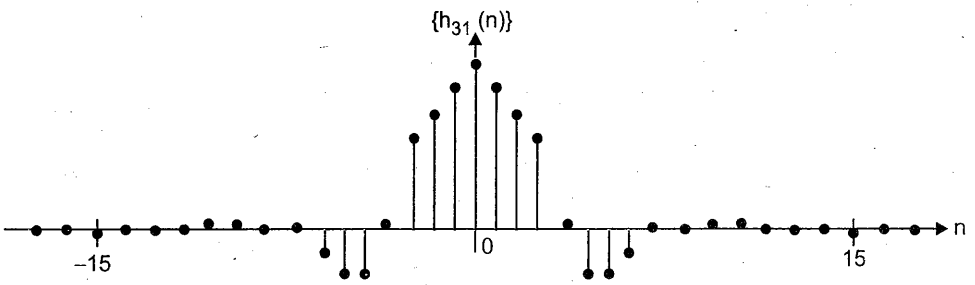


(a) (b)

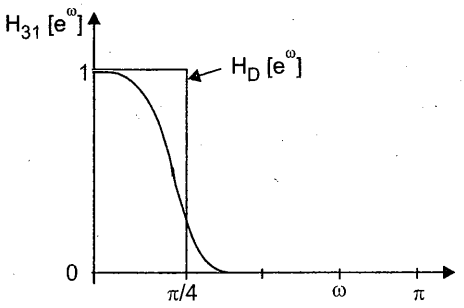
Fig. J-6

If we compare hanning window with triangular then hanning window function is smoother at the ends. Smoother ends reduces sidelobe level, while broaden middle section.

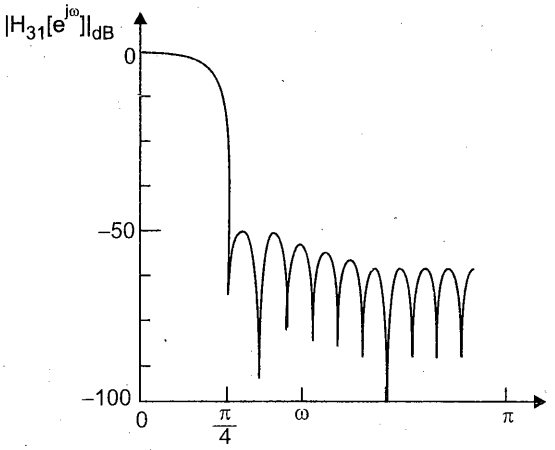
(4) Blackman window function :



(a) : Window sequence (N = 31)



(b) Seectrum



(c) Log-magnitude

Fig. J-7 : Blackman window function

In blackman window function, we will find one more additional term in comparision with hamming and hanning window. Because of additional cosine term, sidelobs are reduced further.

Window function for blackman can be given by ,

$$W_B[n] = 0.42 + 0.5\cos\frac{2\pi n}{M-1} + 0.08\cos\frac{4\pi n}{M-1}$$

Window and its spectrum are shown in Fig. J-7.

(5) Kaiser window function :

Kaiser window function can be defined as,

$$W_k[n] = \begin{cases} \dfrac{I_o(\beta)}{I_o(\alpha)} & |n| \le Q \\ 0 & \text{elsewhere} \end{cases}$$

Where $I_o(x)$ is modified bessel function of the first kind and zero order. The tradeoff between main lobe width and side lobe level can be adjusted by varying parameter α.

Here α is independent variable.

β can be expressed as

$$\beta = \alpha\sqrt{1-(n/Q)^2}$$

$$W_k[n] = \begin{cases} I_0\left(2\sqrt{1-(n/Q)^2}\right)/I_0(\alpha) & |n| \le Q \\ 0 & \text{elsewhere} \end{cases}$$

The spectrum of Kaiser window is as shown in Fig. J-8.

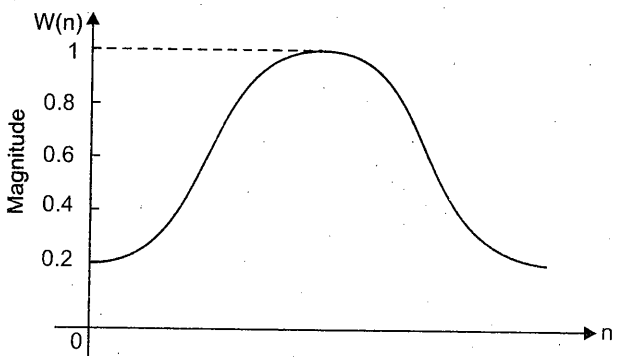


Fig. J-8 : Kaiser window function

Table J-1. shows the summary of window functions.

Table J-1

Sr. No.	Window	Function
1.	Triangular Bartlett	$W_T[n] = 1 - \frac{2\lvert n\rvert}{M-1}$ for $\lvert n\rvert \le M-1$
2.	Hanning	$W_{hn} = \frac{1}{2}\left[1 - \frac{\cos 2\pi n}{M-1}\right]$
3.	Hamming	$W_{Hm}[n] = \begin{cases} 0.54 + 0.46\frac{\cos 2\pi n}{M-1} & \lvert n\rvert \le Q \\ 0 & \text{elsewhere} \end{cases}$
4.	Blackman	$W_B[n] = 0.42 + 0.5\cos\frac{2\pi n}{M-1} + 0.08\cos\frac{4\pi n}{(M-1)}$
5.	Kaiser	$W_k[n] = \begin{cases} \frac{I_o(\beta)}{I_o(\alpha)} & \lvert n\rvert \le Q \\ 0 & \text{elsewhere} \end{cases}$ OR
		$W_k[n] = \begin{cases} \frac{I_0\left(\alpha\sqrt{1-(n/Q)^2}\right)}{I_0[\alpha]} & \lvert n\rvert \le Q \\ 0 & \text{elsewhere} \end{cases}$

1.3.2 Gibb's Phenomenon :

The impulse response of FIR filter in terms of rectangular window is given by,

$$h(n) = h_d(n) \cdot W_R(n) \qquad \ldots(1)$$

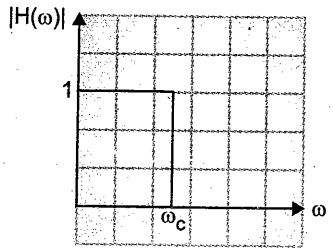
The frequency response of filter is obtained by taking fourier transform of Equation (1).

$$\therefore \quad H(\omega) = FT\left\{h_d(n) \cdot W_R(n)\right\}$$

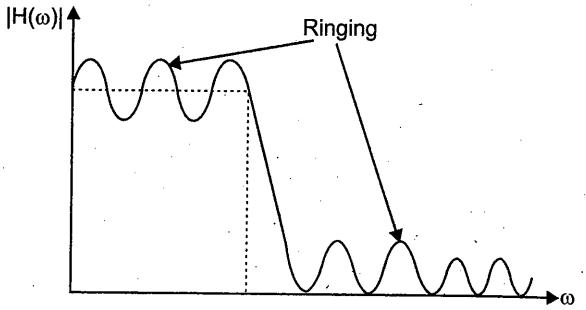
$$\therefore \quad H(\omega) = H_d(\omega) * W_R(\omega) \qquad \ldots(2)$$

This shows that the frequency response of FIR filter is equal to the convolution of desired frequency response, $H_d(\omega)$ and the fourier transform of window function.

Now the desired frequency response of low pass FIR is shown in Fig. J-9(a); while the frequency response of FIR filter obtained because of windowing is shown in Fig. J-9(b).



(a) Desired frequency response



(b) Frequency response obtained because of windowing

Fig. J-9

The sidelobes are present in the frequency response of window function. Because of these sidelobes; the ringing is observed in the frequency response of FIR filter. This ringing is predominantly present near the bandedge and it is known as Gibb's phenomenon.

Now the question arises, **why the side lobes are present in the frequency response of window function ?** This is because of the sudden discontinuities in the window function. Observe the magnitude response of rectangular window. In this case, the discontinuity is very abrupt. Therefore the sidelobes are of larger amplitude. Thus the ringing effect is maximum in case of rectangular window.

Because of this reason; other window functions are developed which will not have the abrupt discontinuities. That means the window function will change more gradually in the time domain.

1.3.3 Advantages and Disadvantages of Window Method :

Advantages :

1. The windowing method requires minimum amount of computational effort; so window method is simple to implement.
2. For the given window; the maximum amplitude of ripple in the filter response is fixed. Thus the stopband attenuation is fixed in the given window.

Disadvantages :

1. The designing of FIR filters using windows is not flexible.
2. The frequency response of FIR filter shows the convolution of spectrum of window function and desired frequency response. Because of this; the passband and stopband edge frequencies cannot be precisely specified.
3. In many applications the expression for the desired filter response will be too complicated.

Design steps for FIR filter :

1. Get the desired frequency response, $H_d(\omega)$.
2. Take inverse fourier transform (IFT) of $H_d(\omega)$ to obtain $h_d(n)$.
3. Decide the length of FIR filter.
4. Multiply $h_d(n)$ by selected window function to get $h(n)$.
5. From $h(n)$ obtain $H(Z)$ and then realize it, if asked.

Solved Problems :

Prob. 1 : Design a linear phase FIR low pass filter of order seven with cut-off frequency 1 rad/sec using rectangular window.

Soln. :

Step I : The desired frequency response $H_d(\omega)$ for the low pass FIR filter is given by,

$$H_d(\omega) = \begin{cases} e^{-j\omega\left(\frac{M-1}{2}\right)} & \text{for } |\omega| < |\omega_c| \\ 0 & \text{otherwise} \end{cases} \quad ...(1)$$

Here M = length of filter = 7 (given)

$$\therefore \quad H_d(\omega) = \begin{cases} e^{-3j\omega} & \text{for } |\omega| < |\omega_c| \\ 0 & \text{otherwise} \end{cases} \quad ...(2)$$

Step II : We will obtain $h_d(n)$ by taking IFT of Equation (2). Using the definition of IFT we get,

$$h_d(n) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} H_d(\omega) \cdot e^{j\omega n} d\omega \quad ..(3)$$

Here given cut-off frequency = $\omega_c = 1$. Now for the symmetric filter we can write the range of integration (ω_c), from $\omega_c = -1$ to $\omega_c = 1$.

$$\therefore \quad h_d(n) = \frac{1}{2\pi} \int_{-1}^{1} e^{-3j\omega} \cdot e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-1}^{1} e^{j\omega(n-3)} d\omega \quad ...(4)$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\omega(n-3)}}{j(n-3)} \right]_{-1}^{1}$$

$$\therefore \quad h_d(n) = \frac{1}{2\pi} \left[\frac{e^{j(n-3)} - e^{-j(n-3)}}{j(n-3)} \right]$$

Now we have the trignometric identity,

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\therefore \quad h_d(n) = \frac{\sin(n-3)}{\pi(n-3)} \quad ...\text{for } n \neq 3 \quad ...(5)$$

Now if n = 3 then Equation (4) becomes,

$$h_d(n) = \frac{1}{2\pi}\int_{-1}^{1} e^{0}\cdot d\omega = \frac{1}{2\pi}\int_{-1}^{1} d\omega = \frac{1}{2\pi}[1-(-1)] = \frac{1}{\pi} \quad ...(6)$$

Thus we can write,
$$h_d(n) = \begin{cases} \dfrac{\sin(n-3)}{\pi(n-3)} & \text{for } n \neq 3 \\ \dfrac{1}{\pi} & \text{for } n = 3 \end{cases} \quad ...(7)$$

Step III : Here we have to make use of rectangular window of the order 7. We have for rectangular window,

$$\therefore \quad W_R(n) = \begin{cases} 1 & \text{for } n = 0 \text{ to } 6 \\ 0 & \text{otherwise} \end{cases} \qquad \text{(for } n = 0 \text{ to } M-1\text{)}$$

Now h (n) is, $h(n) = h_d(n)\cdot W_R(n)$

Thus using Equation (7) we get,

$$h(n) = \begin{cases} h_d(n) & \text{for } n = 0 \text{ to } 6 \\ 0 & \text{otherwise} \end{cases} \quad ...(8)$$

Equation (8) gives unit impulse response of FIR filter. Making use of Equation (7), we can obtain the values of $h_d(n)$ and h (n) as shown in the Table J-2.

Table J-2

Value of n	Value of $h_d(n) = h(n)$
0	$h_d(0) = h(0) = \dfrac{\sin(-3)}{-3\pi} = 0.015$
1	$h_d(1) = h(1) = \dfrac{\sin(-2)}{-2\pi} = 0.145$
2	$h_d(2) = h(2) = \dfrac{\sin(-1)}{-\pi} = 0.268$
3	$h_d(3) = h(3) = \dfrac{1}{\pi} = 0.318$
4	$h_d(4) = h(4) = \dfrac{\sin(1)}{\pi} = 0.268$
5	$h_d(5) = h(5) = \dfrac{\sin(2)}{2\pi} = 0.145$
6	$h_d(6) = h(6) = \dfrac{\sin(3)}{3\pi} = 0.0149$

Prob. 2 : Find the magnitude and phase response function of seventh order low pass linear phase FIR filter with cut-off frequency 1 rad/sec. using hanning window.

Soln. :

This is similar to problem (1); only difference is that here, we have to use hanning window.

We have obtained the equation of $h_d(n)$. It is,

$$h_d(n) = \begin{cases} \dfrac{\sin(n-3)}{\pi(n-3)} & \text{for } n \neq 3 \\ \dfrac{1}{\pi} & \text{for } n = 3 \end{cases} \quad \ldots(1)$$

Now hanning window is defined as,

$$W_h(h) = \frac{1}{2}\left[1 - \cos\frac{2\pi n}{M-1}\right]$$

But M = 7 (given)

$$\therefore \quad W_h(n) = \frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{6}\right)\right] \quad \ldots(2)$$

Now the coefficients of FIR filter that means h (n) is obtained by using the equation,

$$h(n) = h_d(n) \cdot W_h(n) \quad \ldots(3)$$

Making use of these equations we can obtain the values of h (n) as shown in Table J-3.

Table J-3

Value of n	$h_d(n) = \begin{cases} \frac{\sin(n-3)}{\pi(n-3)} & \text{for } n \neq 3 \\ \frac{1}{\pi} & \text{for } n = 3 \end{cases}$	$W_h(n) = \frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{6}\right)\right]$	$h(n) = h_d(n) \cdot W_h(n)$
0	0.015	0	0
1	0.145	0.25	0.03625
2	0.268	0.75	0.201
3	0.318	1	0.318
4	0.268	0.75	0.201
5	0.145	0.25	0.03625
6	0.149	0	0

Magnitude and phase :

$$\text{We have,} \quad H_d(\omega) = \begin{cases} 1 \cdot e^{-j\left(\frac{M-1}{2}\right)} & \text{for } |\omega| \leq |\omega_c| \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore \quad |H_d(\omega)| = \begin{cases} 1 & \text{for } |\omega| \leq |\omega_c| \\ 0 & \text{otherwise} \end{cases}$$

$$\text{and} \quad \angle H_d(\omega) = -\omega\left(\frac{M-1}{2}\right) = -\omega\left(\frac{7-1}{2}\right) = -3\omega$$

Note : Windowing method is also called as Inverse Fourier Method.

Prob. 3 : Determine the filter coefficients $h_d(n)$ for the desired frequency response of a low pass filter given by,

$$H_d(e^{j\omega}) = \begin{cases} e^{-j2\omega}, & -\frac{\pi}{4} \leq \omega \leq \frac{\pi}{4} \\ 0, & \frac{\pi}{4} \leq \omega \leq \pi \end{cases}$$

If we define new filter coefficients by,

$h(n) = h_d(n) \cdot W(n)$

where $W(n) = \begin{cases} 1 & \text{for } 0 \leq n \leq 4 \\ 0 & \text{otherwise} \end{cases}$

Then determine $h(n)$.

Soln. :

Given :

$$H_d(e^{j\omega}) = \begin{cases} e^{-j2\omega}, & -\frac{\pi}{4} \le \omega \le \frac{\pi}{4} \\ \quad , & -\frac{\pi}{4} \le \omega \le \pi \end{cases} \qquad ...(1)$$

Calculation of $h_d(n)$:

$h_d(n)$ is given by,

$$h_d(n) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} H_d(e^{j\omega}) e^{j\omega n} d\omega$$

$$\therefore \quad h_d(n) = \frac{1}{2\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{-j2\omega} \cdot e^{j\omega n} \cdot d\omega$$

$$\therefore \quad h_d(n) = \frac{1}{2\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{j(n-2)\omega} d\omega \qquad ...(2)$$

$$\therefore \quad h_d(n) = \frac{1}{2\pi} \left[\frac{e^{j(n-2)\omega}}{j(n-2)} \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{1}{2\pi} \frac{e^{j(n-2)\frac{\pi}{4}} - e^{-j(n-2)\frac{\pi}{4}}}{j(n-2)}$$

$$\therefore \quad h_d(n) = \frac{\sin(n-2)\frac{\pi}{4}}{\pi(n-2)} \qquad \text{for } n \neq 2 \qquad ...(3)$$

To get the value at n = 2, put n = 2 in Equation (2),

$$\therefore \quad h_d(2) = \frac{1}{2\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{j0}\, d\omega = \frac{1}{2\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 1\, d\omega$$

$$\therefore \quad h_d(2) = \frac{1}{2\pi} \left[\frac{\pi}{4} - \left(-\frac{\pi}{4} \right) \right] = \frac{1}{2\pi} \cdot \frac{\pi}{2} = 0.25$$

$$\therefore \quad h_d(n) = \begin{cases} \dfrac{\sin(n-2)\frac{\pi}{4}}{\pi(n-2)} & \text{for } n \neq 2 \\ 0.25 & \text{for } n = 2 \end{cases} \qquad \ldots(4)$$

This is the required expression of $h_d(n)$.

Calculation of h (n) :

We have, $h(n) = h_d(n) \cdot W(n)$

and $W(n) = \begin{cases} 1 & \text{for } 0 \le n \le 4 \\ 0 & \text{otherwise} \end{cases}$

That means we have to calculate h (n) for n = 0, 1, 2, 3, 4

$$\text{For } n = 0 \Rightarrow h(0) = h_d(0) = \frac{\sin(-2)\frac{\pi}{4}}{\pi(-2)} = 0.16$$

$$\text{For } n = 1 \Rightarrow h(1) = h_d(1) = \frac{\sin(1-2)\frac{\pi}{4}}{\pi(1-2)} = 0.225$$

$$\text{For } n = 2 \Rightarrow h(2) = h_d(2) = 0.25$$

$$\text{For } n = 3 \Rightarrow h(3) = h_d(3) = \frac{\sin(3-2)\frac{\pi}{4}}{\pi(3-2)} = 0.225$$

$$\text{For } n = 4 \Rightarrow h(4) = h_d(4) = \frac{\sin(4-2)\frac{\pi}{4}}{\pi(4-2)} = 0.1($$

Prob. 4 : Design FIR filter (low pass) using rectangular window with passband gain of unity, cutoff frequency of 200 Hz, sampling frequency of 1 kHz. Assume the length of impulse response as 7.

Soln. :

Given :

Cut-off frequency $= f_c = 200$ Hz

Sampling frequency $= f_s = 1$ kHz $= 1000$ Hz

Length $= M = 7$

Step I : From the given specifications, the desired response of ideal low pass filter can be expressed as,

$$H_d(\omega) = \begin{cases} 1, & 0 \le f \le 200 \text{ Hz} \\ 0, & f > 200 \text{ Hz} \end{cases} \qquad \ldots(1)$$

Now the normalized cut-off frequency (ω_c), is given by,

$$\omega_c = \frac{2\pi f_c}{f_s} = \frac{2\pi \times 200}{1000} = 1.257 \text{ rad/sec}$$

Thus the desired frequency response can be expressed in terms of ω_c as,

$$H_d(\omega) = \begin{cases} 1, & 0 \le \omega \le 1.257 \\ 0, & 1.257 < \omega \le \pi \end{cases} \quad \ldots(2)$$

Step II : The filter coefficients are given by,

$$h_d(n) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} H_d(\omega) e^{j\omega n} d\omega$$

$$\therefore h_d(n) = \frac{1}{2\pi} \int_{-1.257}^{1.257} 1 \cdot e^{j\omega n} d\omega$$

$$\therefore h_d(n) = \frac{1}{2\pi} \left[\frac{e^{j\omega n}}{jn} \right]_{-1.257}^{1.257}$$

$$\therefore h_d(n) = \frac{1}{2\pi} \frac{e^{j1.257n} - e^{-j1.257n}}{jn}$$

Using trignometric identity,

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \text{ we get,}$$

$$h_d(n) = \frac{\sin 1.257n}{\pi n}, \quad n \neq 0 \quad \ldots(3)$$

Now for n = 0 we get,

$$h_d(0) = \frac{1}{2\pi} \int_{-1.257}^{1.257} 1 \cdot e^0 \, d\omega = \frac{1}{2\pi} [1.257 - (-1.257)]$$

$$\therefore h_d(0) = 0.4$$

Step III : For rectangular window we have,

$$h(n) = h_d(n) \cdot W_R(n)$$

$$\text{and } W_R(n) = \begin{cases} 1 & \text{for } n = 0 \text{ to } M-1 \text{ that means } 0 \text{ to } 6 \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore h(n) = h_d(n) \text{ for } n = 0 \text{ to } 6$$

Value of n	Value of $h_d(n) = h(n)$
0	$h_d(0) = 0.4$
1	$h_d(1) = h(1) = \frac{\sin 1.257}{\pi} = 0.302$
2	$h(2) = h_d(2) = \frac{\sin 1.257 \times 2}{2\pi} = 0.093$
3	$h(3) = h_d(3) = \frac{\sin 1.257 \times 3}{3\pi} = -0.062$
4	$h(4) = h_d(4) = \frac{\sin 1.257 \times 4}{4\pi} = -0.075$
5	$h(5) = h_d(5) = \frac{\sin 1.257 \times 5}{5\pi} = 1.15 \times 10^{-4}$
6	$h(6) = h_d(6) = \frac{\sin 1.257 \times 6}{6\pi} = 0.05$

1.3.4 FIR Filter Design using Kaiser Window :

By using Kaiser window it is possible to obtain seperate control upon length or order (M) of filter and the transition width of the main lobe.

The magnitude specifications of required FIR filter is as shown in Fig. J-10.

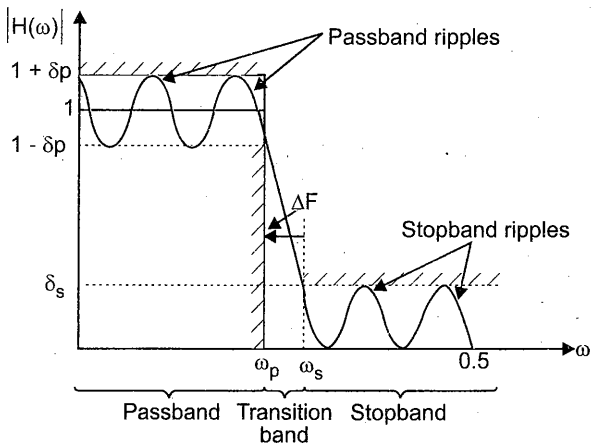


Fig. J-10 : Magnitude specifications

Here δ_p = Peak passband deviation

δ_s = Stopband deviation

ω_p = passband edge frequency

ω_s = Stopband edge frequency

Δf = Transition width = $\frac{\omega_s - \omega_p}{2\pi}$

In Kaiser window there are two main parameters. The length $(M+1)$ of window and the shape parameter β. The Kaiser window is defined as,

$$\omega(n) = \begin{cases} \dfrac{I_0\left\{\beta\left[1-\left(\dfrac{n-\alpha}{\alpha}\right)^2\right]\right\}^{\frac{1}{2}}}{I_0(\beta)} & \text{for } 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

Here $\alpha = \frac{M}{2}$ and $I_0(\ )$ is the 0^{th} order modified bessel function of first kind. This function is obtained by using the equation,

$$I_0(x) = 1 + \frac{0.25\,x^2}{(1!)^2} + \frac{(0.25\,x^2)^2}{(2!)^2} + \frac{(0.25\,x^2)^3}{(3!)^2} + \ldots.$$

Design steps :

Step I : Choose the optimum value of ripple 'δ' using the equation,

$$\delta = \text{minimum of } \delta_P \text{ and } \delta_S$$

Here $\delta_P = 10^{-0.05\,A_s}$

or obtain δ_P by solving the equation,

$$A_P = 20\log_{10}\left(\frac{1+\delta_P}{1-\delta_P}\right)$$

And $\delta_s = \dfrac{10^{0.05\,A_P} - 1}{10^{0.05\,A_P} + 1}$

OR obtain δ_s by solving the equation,

$$A_s = -20\log_{10}(\delta_s)$$

Step II : Calculate attenuation 'A' in dB using the equation,

$$A = -20\log_{10}(\delta)$$

Step III : Calculate the parameter β from empirical equations proposed by Kaiser as follows :

$$\beta = \begin{cases} 0.1102\,(A - 8.7) & \text{for } A > 50 \\ 0.5842\,(A - 21)^{0.4} + 0.07886\,(A - 21) & \text{for } 21 \le A \le 50 \\ 0 & \text{for } A < 21 \end{cases}$$

Step IV : The length of filter is $M + 1$. Then calculate value of M using the equation,

$$M = \frac{A - 8}{2.285\,\Delta\omega}$$

Here $\Delta\omega$ = transition width and is given by,

$$\Delta\omega = \omega_s - \omega_P$$

Where ω_s = Stopband edge frequency

and ω_P = Passband edge frequency.

Step V : Select the desired impulse response depending on type of filter e.g. low pass, high pass etc.

Step VI : Calculate the FIR filter coefficients using the relation $h(n) = h_d(n) \cdot W(n)$.

Solved problems :

Prob. 1 : Design an FIR linear phase filter using Kaiser window to meet the following specifications

$0.99 \le |H(e^{j\omega})| \le 1.01, \quad 0 \le \omega \le 0.19\,\pi$

$|H(e^{j\omega})| \le 0.01, \quad 0.21\,\pi \le \omega \le \pi.$

Soln. : The magnitude specifications of FIR filter are given by,

$$1 - \delta_P \le |H(\omega)| \le 1 + \delta_P \quad , 0 \le \omega \le \omega_P$$

$$\text{and} \quad 0 \le |H(\omega)| \le \delta_s \quad , \omega_s \le \omega \le \pi \qquad \ldots(1)$$

Comparing Equation (1) with given specifications we get,

$$1 - \delta_P = 0.99 \Rightarrow \delta_P = 0.01$$

$$\delta_s = 0.01,\ \omega_P = 0.19\,\pi \text{ and } \omega_s = 0.21\,\pi$$

Step I : Optimum value of ripple 'δ' is,

$$\delta = \text{Minimum of } \delta_P \text{ and } \delta_s$$

$$\therefore \quad \delta = 0.01$$

Step II : Attenuation 'A' is given by,

$$A = -20 \log_{10}(\delta)\ \text{dB}$$

$$\therefore \quad A = -20 \log_{10} (0.01)\ dB$$

$$\therefore \quad A = 40\ dB$$

Step III : Here the value of A is 40 dB, that means in the range $21 \leq A \leq 50$. Thus we will calculate β using the equation,

$$\beta = 0.5842 (A - 21)^{0.4} + 0.07886 (A - 21)$$

$$\therefore \quad \beta = 0.5842 (40 - 21)^{0.4} + 0.07886 (40 - 21)$$

$$\therefore \quad \beta = 3.395$$

Step IV : The length of filter is M + 1. The value of M is calculated using the equation.

$$M = \frac{A - 8}{2.285\ \Delta\omega}$$

$$\text{Here} \quad \Delta\omega = \omega_s - \omega_P = 0.21\ \pi - 0.19\ \pi = 0.02\ \pi$$

$$\therefore \quad M = \frac{40 - 8}{2.285\ (0.02\ \pi)} = 222.87$$

$$\therefore \quad M \approx 223$$

Step V : We have, the desired frequency response for the low pass filter.

$$H_d(\omega) = \begin{cases} e^{-j\omega\left(\frac{M-1}{2}\right)} & \text{for } |\omega| < |\omega_c| \\ 0 & \text{otherwise} \end{cases} \qquad ...(2)$$

Remember that Equation (2) is for the filter length M.

For this filter using Kaiser window, the length is M + 1.

Thus putting M = M + 1 in Equation (2) we get,

$$H_d(\omega) = \begin{cases} e^{-j\omega\frac{M}{2}} & \text{for } |\omega| < |\omega_c| \\ 0 & \text{otherwise} \end{cases} \qquad ...(3)$$

Now $h_d(n)$ is calculated by taking inverse fourier of Equation (3).

$$\therefore \quad h_d(n) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} H_d(\omega)\, e^{j\omega n}\, d\omega \qquad ...(4)$$

Here ω_c = Cutoff frequency. It is obtained by using the equation,

$$\omega_c = \frac{\omega_P + \omega_s}{2}$$

$$\therefore \quad \omega_c = \frac{0.19\pi + 0.21\pi}{2} = 0.2\pi$$

$$\therefore \quad h_d(n) = \frac{1}{2\pi} \int_{-0.2\pi}^{0.2\pi} e^{-j\omega\left(\frac{223}{2}\right)} e^{j\omega n} \, d\omega$$

$$\therefore \quad h_d(n) = \frac{1}{2\pi} \int_{-0.2\pi}^{0.2\pi} e^{-j\omega 111.5} \cdot e^{j\omega n} \, d\omega$$

$$\therefore \quad h_d(n) = \frac{1}{2\pi} \int_{-0.2\pi}^{0.2\pi} e^{+j\omega(n-111.5)} \, d\omega \qquad \ldots(5)$$

$$\therefore \quad h_d(n) = \frac{1}{2\pi} \left[\frac{e^{j\omega(n-111.5)}}{j(n-111.5)} \right]_{-0.2\pi}^{0.2\pi}$$

$$\therefore \quad h_d(n) = \frac{1}{2\pi j(n-111.5)} \left[e^{j0.2\pi(n-111.5)} - e^{-j0.2\pi(n-111.5)} \right]$$

We have the trignometric identity.

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\therefore \quad h_d(n) = \frac{\sin[0.2\pi(n-111.5)]}{\pi(n-111.5)} \quad \text{... for } n \neq 111.5 \qquad \ldots(6)$$

Now for n = 111.5, using Equation (5) we get,

$$h_d(n) = \frac{1}{2\pi} \int_{-0.2\pi}^{0.2\pi} e^{j\omega(111.5-111.5)} \, d\omega$$

$$\therefore \quad h_d(n) = \frac{1}{2\pi} \int_{-0.2\pi}^{0.2\pi} e^0 \, d\omega = \frac{1}{2\pi} \int_{-0.2\pi}^{0.2\pi} 1 \, d\omega$$

$$\therefore \quad h_d(n) = \frac{1}{2\pi} [0.2\pi - (-0.2\pi)]$$

$$\therefore \quad h_d(n) = 0.2 \qquad \text{for } n = 111.5 \qquad \ldots(7)$$

Now combining Equations (6) and (7) we can write,

$$h_d(n) = \begin{cases} \dfrac{\sin[0.2\pi(n-111.5)]}{\pi(n-111.5)} & \text{for } n \neq 111.5 \\ 0.2 & \text{for } n = 111.5 \end{cases} \qquad \ldots(8)$$

Step VI : The filter coefficients are calculated using the equation

$$h(n) = h_d(n) \cdot W(n) \qquad ...(9)$$

We have, $$\omega(n) = \frac{I_0\left\{\beta\left[1-\left(\frac{n-\alpha}{\alpha}\right)^2\right]^{1/2}\right\}}{I_0(\beta)}$$

We have $$\alpha = \frac{M}{2} = \frac{223}{2} = 111.5$$

$$\therefore \omega(n) = \frac{I_0\left\{3.395\left[1-\left(\frac{n-111.5}{111.5}\right)^2\right]^{1/2}\right\}}{I_0(3.395)} \qquad ...(10)$$

Thus using Equations (8) and (10), we can calculate the required filter coefficients.

1.4 Design of Linear Phase FIR Filters using Frequency Sampling Method :

(1) The desired frequency response is denoted by $H_d(\omega)$.

(2) This frequency response is sampled uniformly at M points.

(3) The frequency samples are given by,

$$\omega_k = \frac{2\pi}{M}k, \qquad k = 0, 1 \ldots\ldots M-1$$

(4) A set of samples determined from $H_d(\omega)$ are identified as discrete fourier transform (DFT) coefficients. It is denoted by $H(k)$.

There are two types of design as follows :

(A) Type I :

Here frequency samples are :

$$\omega_k = \frac{2\pi k}{M}, \qquad k = 0, 1 \ldots\ldots M-1 \qquad ...(1)$$

The sampled desired frequency response is denoted by $H(k)$.

$$\text{and } H(k) = H_d(\omega)\Big|_{\omega = \omega_k} \qquad k = 0, 1 \ldots\ldots M-1$$

$$\therefore H(k) = H_d\left(\frac{2\pi k}{M}\right) \qquad k = 0, 1 \ldots\ldots M-1 \qquad ...(2)$$

The filter coefficient h(n) can be obtained by taking IDFT of H(k).

$$\therefore\ h(n) = \frac{1}{M}\sum_{k=0}^{M-1} H(k)\, e^{j2\pi kn/M}, \qquad n = 0, 1 \ldots\ldots M-1 \qquad \ldots(3)$$

Now according to symmetry property of DFT we can write,

$$H(M-k) = H^*(k) \qquad \ldots(4)$$

Then the filter coefficients can be determined using the equation,

$$h(n) = \frac{1}{M}\left\{ H(0) + 2\sum_{k=1}^{(M-1)/2} \mathrm{Re}\left[H(k)\, e^{j2\pi kn/M} \right] \right\} \text{ .. when M is odd} \qquad \ldots(5)$$

$$\text{and } h(n) = \frac{1}{M}\left\{ H(0) + 2\sum_{k=1}^{\frac{M}{2}-1} \mathrm{Re}\left[H(k)\, e^{j2\pi\, kn/M} \right] \right\} \text{ ... when M is even} \qquad \ldots(6)$$

(B) Type II :

In this method frequency samples are taken at,

$$\omega_k = \frac{2\pi(k+1)}{2M}, \qquad k = 0, 1 \ldots\ldots M-1 \qquad \ldots(7)$$

The DFT coefficients are obtained by using,

$$H(k) = H_d\left[\frac{\pi(2k+1)}{M}\right], \qquad k = 0, 1 \ldots\ldots M-1 \qquad \ldots(8)$$

The filter coefficients are obtained by using the equation,

$$h(n) = \frac{2}{M}\sum_{k=0}^{(M-3)/2} \mathrm{Re}\left\{ H(k)\, e^{j\pi n(2k+1)/M} \right\} \text{ ... M is odd} \qquad \ldots(9)$$

$$\text{and } h(n) = \frac{2}{M}\sum_{k=0}^{\frac{M}{2}-1} \mathrm{Re}\left\{ H(k)\, e^{j\,\pi\, n(2k+1)/M} \right\} \text{ ...M is even} \qquad \ldots(10)$$

Solved Problems :

Prob. 1 : Design a low-pass digital filter with cut-off frequency $\omega_c = \frac{\pi}{2}$ using frequency sampling technique for N = 17.

Soln. :

We will use type I design. The value of H(k) is given by,

$$H(k) = H_d\left(\frac{2\pi k}{M}\right)$$

Note that in the given problem notation N is used instead of M.

$$\therefore \quad H(k) = H_d\left(\frac{2\pi k}{N}\right) \qquad \ldots(1)$$

$H_d(\omega)$ is the desired frequency response and for low-pass filter it is given by,

$$H_d(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)}, \qquad 0 \le \omega \le \omega_c \qquad \ldots(2)$$

Now we have,

$$H(k) = H_d(\omega)\Big|_{\omega = \frac{2\pi k}{N}}$$

$$\therefore \quad H(k) = e^{-j\frac{2\pi k}{N}\left(\frac{N-1}{2}\right)}$$

Now N = 17.

$$\therefore \quad H(k) = e^{-j\frac{16\pi k}{17}} \qquad \ldots(3)$$

Here N is odd so we will calculate the filter coefficients by using the equation,

$$h(n) = \frac{1}{N}\left\{H(0) + 2\sum_{k=1}^{\left(\frac{N-1}{2}\right)} \mathrm{Re}\left[H(k)\,e^{j2\pi kn/N}\right]\right\}$$

$$\therefore \quad h(n) = \frac{1}{17}\left\{H(0) + 2\sum_{k=1}^{8} \mathrm{Re}\left[H(k)\,e^{j2\pi kn/17}\right]\right\} \qquad \ldots(4)$$

Substituting the value of H(k) from Equation (3),

$$\therefore \quad h(n) = \frac{1}{17}\left\{H(0) + 2\sum_{k=1}^{8} \mathrm{Re}\left[e^{-j16\pi k/17} \cdot e^{j2\pi kn/17}\right]\right\}$$

$$\therefore \quad h(n) = \frac{1}{17}\left\{H(0) + 2\sum_{k=1}^{8} \mathrm{Re}\left[e^{\frac{-j2\pi k}{17}(8-n)}\right]\right\} \qquad \ldots(5)$$

Putting k = 0 in Equation (3) we can obtain the value of H(0).

$$\therefore \quad H(0) = e^{\frac{-j16\pi(0)}{17}} = e^0 = 1$$

$$\therefore \quad h(n) = \frac{1}{17}\left\{1 + 2\sum_{k=1}^{8} \mathrm{Re}\left[e^{\frac{-j2\pi k}{17}(8-n)}\right]\right\} \qquad \ldots(6)$$

Now using Euler's identity we can write,

$$e^{\frac{-j2\pi k(8-n)}{17}} = \cos\left[\frac{2\pi k(8-n)}{17}\right] - j\sin\left[\frac{2\pi k(8-n)}{17}\right]$$

$$\therefore \quad \text{Re}\left[e^{\frac{-j2\pi k(8-n)}{17}}\right] = \cos\left[\frac{2\pi k(8-n)}{17}\right] \qquad ...(7)$$

$$\therefore \quad h(n) = \frac{1}{17}\left\{1 + 2\sum_{k=1}^{8}\cos\frac{2\pi k(8-n)}{17}\right\} \qquad ...(8)$$

Now we will obtain different filter coefficients by putting n = 0 to N – 1 that means for the values of n from n = 0 to 16.

$$\text{For } n = 0 \Rightarrow h(0) = \frac{1}{17}\left[1 + 2\sum_{k=1}^{8}\frac{\cos 2\pi k(8-0)}{17}\right]$$

$$\therefore h(0) = \frac{1}{17}\left\{1 + 2\left[\cos\frac{(2\pi\times1\times8)}{17} + \cos\frac{(2\pi\times2\times8)}{17} + \cos\frac{(2\pi\times3\times8)}{17} + \cos\frac{(2\pi\times4\times8)}{17} + \cos\frac{(2\pi\times5\times8)}{17} + \cos\frac{(2\pi\times6\times8)}{17} + \cos\frac{(2\pi\times7\times8)}{17} + \cos\frac{(2\pi\times8\times8)}{17}\right]\right\}$$

$$\therefore \quad h(0) = \frac{1}{17}\{1 + 2[-0.98 + 0.93 - 0.85 + 0.73 - 0.6 + 0.44 - 0.27 + 0.092]\}$$

$$\therefore \quad h(0) = -9.4\times10^{-4}$$

Note : As you further go on solving, you will observe that $h(1) = h(15)$, $h(2) = h(14)$, $h(3) = h(13)$ .... $h(7) = h(9)$ $h(8)$ has to be calculated separately. Thus, all 16 values need not be calculated, only 9 values need to be calculated.

1.5 Alternation Theorem in Equiripple Linear Phase FIR Filters :

Alternation theorem is basically used to determine the filter parameters that minimize the maximum absolute value of error $E(\omega)$ over the frequency bands in which approximation is to be performed. It is a theorem applicable in the theory of Chebyshev approximations.

The real-valued frequency response characteristic of filter is given as,

$$H_r(\omega) = Q(\omega)P(\omega) \quad ...(1)$$

where,

$$Q(\omega) = \begin{cases} 1 & \text{case 1} \\ \cos\frac{\omega}{2} & \text{case 2} \\ \sin\omega & \text{case 3} \\ \sin\frac{\omega}{2} & \text{case 4} \end{cases}$$

$$\text{and } P(\omega) = \sum_{k=0}^{L} a(k)\cos\omega k$$

Here, $L = (M-1)/2$ for case 1

$L = (M-3)/2$ for case 3

$L = M/2-1$ for case 2 and 4.

A necessary and sufficient condition for $P(\omega)$ to be unique, best weighted Chebyshev approximation to $H_{dr}(\omega)$ in compact subset of interval $(0, \pi)$, (S) is that the error function $E(\omega)$ exhibit at least $(L+2)$ external frequencies in S. That is, there must exist at least $(L+2)$ frequencies $\{\omega_k\}$ in S such that $\omega_1 < \omega_2 < ..... < \omega_{L+2}$

$$E(\omega_k) = -E(\omega_{k+1})$$

and $$|E(\omega_k)| = \max|E(\omega)|$$

where, $k = 1, 2, ..., L+2$

Thus, it can be seen that the error function $E(\omega)$ alternates in sign between two successive external frequencies. Hence the theorem is called alternation theorem.

Let us consider the design of a low pass filter with passband $0 \le \omega \le \omega_p$ and stopband $\omega_s \le \omega \le \pi$. Since the desired frequency response $H_{dr}(\omega)$ and the weighting function $W(\omega)$ are piecewise constant, we have

$$\frac{dE(\omega)}{d\omega} = \frac{d}{d\omega}\{W(\omega)[H_{dr}(\omega) - H_r(\omega)]\}$$

$$= -\frac{dH_r(\omega)}{d(\omega)} = 0$$

Consequently, the frequencies $\{\omega_i\}$ corresponding to the peaks of $E(\omega)$ also correspond to peaks at which $H_r(\omega)$ meets the error tolerance. Since $H_r(\omega)$ is a trignometric polynomial of degree L, for case 1, for example,

$$H_r(\omega) = \sum_{k=0}^{L} \alpha(k) \cos \omega k$$

$$= \sum_{k=0}^{L} \alpha(k) \left[\sum_{n=0}^{k} \beta nk (\cos \omega)^n \right]$$

$$= \sum_{k=0}^{L} \alpha'(k) (\cos \omega)^k$$

It follows that $H_r(\omega)$ can have at most $L-1$ local maxima and minima in the open interval $0 < \omega < \pi$. In addition, $\omega = 0$ and $\omega = \pi$ are usually extrema of $H_r(\omega)$ and also, of $E(\omega)$. Therefore, $H_r(\omega)$ has at most $L+1$ extremal frequencies. Furthermore, the band-edge frequencies ω_p and ω_s are also extrema of $E(\omega)$, since $|E(\omega)|$ is maximum at $\omega = \omega_p$ and $\omega = \omega_s$. As a consequence, there are at most $L+3$ external frequencies in $E(\omega)$ for the unique, best approximation of the ideal lowpass filter. On the other hand, the alternation theorem states that there alternation theorem states that there are at least $L+2$ extreme frequencies in $E(\omega)$. Thus the error function for the lowpass filter design has either $L+3$ or $L+2$ extrema. In general, filter designs that contain more than $L+2$ alternations or ripples are called extra ripple filters. When the filter design contains the maximum number of alternations, it is called a maximal ripple filter.

The alternation theorem guarantees a unique solution for the Chebyshev optimization problem.

1.6 FIR Differentiators :

Basically the differentiators are used to take the derivative of input signal. For an analog ideal differentiator the frequency response is linearly proportional to the frequency.

While the digital differentiator is the device which has frequency response,

$$H_d(\omega) = j\omega, \quad -\pi \le \omega \le \pi \qquad ..(1)$$

Now unit impulse response, $h(n)$ is obtained by taking inverse fourier transform of Equation (1).

$$\therefore \; h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) \cdot e^{j\omega n} \, d\omega \qquad ...(2)$$

Putting Equation (1) in Equation (2) we get,

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} j\omega e^{j\omega n} d\omega$$

$$\therefore \; h_d(n) = \frac{\cos \pi n}{n}, \; -\infty < n < \infty \text{ and } n \neq 0$$

Now we will use chebyshev approximation to design linear phase FIR differentiators. An ideal differentiator has an antisymmertic unit sample response. That means,

$$h(n) = -h(M-1-n) \qquad ...(3)$$

An ideal differentiator should satisfy the condition of providing zero response at zero frequency. In practical cases the desired frequency response characteristic need to be linear in the limited frequency range $0 \leq \omega \leq 2\pi f_p$. Here f_p is called as the bandwidth of the differentiator.

The weighting function $W(\omega)$ is given by,

$$W(\omega) = \frac{1}{\omega} \qquad 0 \leq \omega \leq 2\pi f_p \qquad ...(4)$$

This is required to keep the ripple in the passband at the constant level. Now we know that the weighted approximation error is given by,

$$E(\omega) = W(\omega)[H_{dr}(\omega) - H_r(\omega)]$$

Now from Equation (1) we have,

$$H_{dr}(\omega) = \omega$$

$$\therefore \; E(\omega) = \frac{1}{\omega}\left[\omega - H_r(\omega)\right]$$

Theory Questions

Q. 1 Write note on : Gibbs phenomenon.

Q. 2 Discuss in detail 'Rectangular window' in FIR filter design.

Q. 3 Write note on : Window functions.

Q. 4 Why FIR filters are called linear phase filters ?

Q. 5 What are the advantages of windowing method in FIR filter design ?

Q. 6 Why FIR filter are called as linear phase filter ?

Q. 7 How will you select a window in the design of FIR filter ?

Unsolved Numericals

Q. 1 Using rectangular window technique design a low pass filter with pass band gain of unity, cut-off frequency of 1000 Hz and working at a sampling frequency of 5 kHz. The length of impulse response is 7.

Ans. :

$$h(0) = -0.062,\ h(1) = 0.093,\quad h(2) = 0.302,\quad h(3) = 0.4$$
$$h(4) = 0.302,\ h(5) = 0.093,\quad h(6) = -0.062$$

Q. 2 Design an ideal bandpass filter with frequency response $H_d(e^{j\omega}) = 1$ for $\frac{\pi}{4} \le |\omega| \le \frac{3\pi}{4}$

Use a rectangular window with N = 11 in your design.

Hint : The desired unit impulse response of rectangular window is,

$$h_{d(n)} = \frac{\sin \omega_{c2}(n-T) - \sin \omega_{c1}(n-T)}{\pi(n-T)} \quad \text{for } n \neq T$$

$$h_{d(n)} = \frac{\omega_{c2} - \omega_{c1}}{\pi} \quad \text{for } n = T$$

Here $T = \frac{M-1}{2}$

Ans. :

$h(0) = 0,\ h(1) = 0,\ h(2) = 0,\ h(3) = -0.3183,\ h(4) = 0,\ h(5) = 0.2,\ h(6) = 0,$
$h(7) = -0.3183,\ h(8) = 0,\ h(9) = 0,\ h(10) = 0$

Q. 3 Describe low pass filter with cut-off frequency $\omega_c = \frac{\pi}{2}$ Use frequency sampling method. Assume M = 7.

Ans. :

$$h(n) = \frac{1}{17}\left[H(0) + 2\sum_{k=1}^{4} \frac{\cos 2\pi k(8-n)}{17}\right]$$

Q. 4 Design a low pass filter with cut-off frequency $\omega_c = \frac{\pi}{4}$, a transition width $\Delta\omega = 0.02\pi$ and stop band ripple $\delta s = 0.01$, use keiser window in your design.

Ans. :

$$P = 3.395,\ M = 222.88$$

$$h(n) = \frac{\sin\left[0.2\pi\left(\frac{n-223}{2}\right)\right]}{\pi\left(n-\frac{223}{2}\right)},\ \frac{I_O\left\{3.395\left[1-\left(\frac{n-111.5}{111.5}\right)^2\right]^{\frac{1}{2}}\right\}}{I_o(3.395)}$$

Q. 5 Describe a low pass digital filter with cut-off frequency $\omega_c = \frac{\pi}{2}$ using frequency sampling technique for M = 17.

Ans. :
$$h(n) = \frac{1}{17}\left[1 + 2\sum_{k=1}^{\infty} \cos\left(\frac{2\pi k(8-n)}{17}\right)\right]$$

Q. 6 Design a low pass filter with cut-off frequency 1 kHz, length of unit sample response is 9 and hanning window smoothening. Use sampling frequency of 10 kHz for your design.

Ans. :

$h(0) = h(8) = 0$, $h(1) = h(7) = 0.0146$, $h(2) = h(6) = 0.0756$,
$h(3) = h(5) = 0.159$, $h(4) = 0.2$

❑❑❑

Note

IIR Filter Design

Section No.	Name of the Topic
1.1	Introduction
1.2	Filter Design Methods
1.3	Butterworth Filter Approximation
1.4	Chebyshev Filter Approximation
1.5	Frequency Transformations
1.6	Elliptic Filters

• Code : DSP9

1.1 Introduction :

Analog filters are designed using analog components like resistors (R), inductors (L) and capacitors (C). While digital filters are implemented using difference equation.

The digital filters described by differential equations can be implemented using software like 'C' or assembly language. We can easily change the algorithm; so we can easily change the filter characteristics according to our requirement.

Basically there are two types of filters as follows :

1. FIR (Finite impulse response) filter.
2. IIR (Infinite impulse response) filter.

We will study each type in detail; later in this chapter. Presently we will compare analog and digital filters by studying advantages and disadvantages of digital filters.

1.1.1 Advantages of Digital Filters :

1. Many input signals can be filtered by one digital filter without replacing the hardware.
2. Digital filters have characteristic like linear phase response. Such characteristic is not possible to obtain in case of analog filters.
3. The performance of digital filters, does not vary with environmental parameters. But the environmental parameters like temperature, humidity etc., change the values of components in case of analog filters. So it is required to calibrate analog filters periodically.
4. In case of digital filters; since the filtering is done with the help of digital computer, both filtered and unfiltered data can be saved for further use.
5. Unlike analog filters; the digital filters are portable.
6. From unit to unit the performance of digital filters is repeatable.
7. The digital filters are highly flexible.
8. Using VLSI technology; the hardware of digital filters can be reduced. Similarly the power consumption can be reduced.
9. Digital filters can be used at very low frequencies, for example in Biomedical applications.
10. In case of analog filters; maintenance is frequently required. But for digital filters it is not required.

1.1.2 Disadvantages of Digital Filters :

1. Speed limitation :

In case of digital filters, ADC and DAC are used. So the speed of digital filter depends on the conversion time of ADC and the settling time of DAC. Similarly the speed of operation of digital filter depends on the speed of digital processor. Thus the bandwidth of input signal processed is limited by ADC and DAC. In real time applications, the bandwidth of digital filter is much lower than analog filters.

2. Finite wordlength effect :

The accuracy of digital filter depends on the wordlength used to encode them in binary form. Wordlength should be long enough to obtain the required accuracy.

The digital filters are also affected by the ADC noise, resulting from the quantization of continuous signals. Similarly the accuracy of digital filters is also affected by the roundoff noise occurred during computation.

3. Long design and development time :

An initial design and development time for digital hardware is more than analog filters.

1.2 Filter Design Methods :

In order to design the digital IIR filter; analog IIR filter is designed first. Then analog filter is converted into the digital filter. Here you may ask a question, why to design digital filter from analog filter ?

The reasons are as follows :

(1) The procedure to design analog filter is readily available and it is highly advanced.

(2) When we design digital filter using analog filter then the implementation becomes simple.

The different methods use to design IIR filter are as follows :

(1) Approximation of derivatives. (2) Impulse invariance. (3) Bilinear transformation.

(4) Matched Z transform. (5) Least square filter design.

1.2.1 Approximation of Derivatives :

Consider an analog differentiator with transfer function $H_a(s)$. The function of analog differentiator is to take the derivative of analog input signal. Let $x(t)$ be the input signal applied to analog differentiator. Then its output can be written as,

$$y(t) = \frac{d}{dt} x(t) \qquad ...(1)$$

Consider analog signal as shown in Fig. K-1(a).

Since the output is the differentiation of input, then from Fig. K-1(a) we can write,

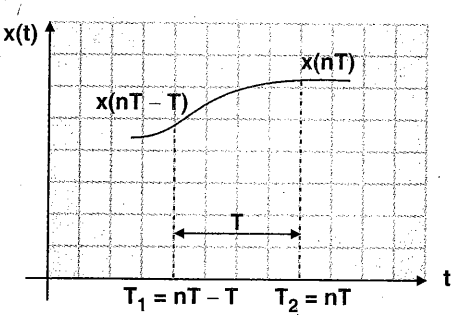


Fig. K-1(a) : Analog signal

$$y(nT) = \frac{x(nT) - x(nT-T)}{T}$$

$$\text{Thus } y(n) = \frac{x(n) - x(n-1)}{T} \quad \ldots(2)$$

Taking Z transform of both sides we get,

$$Y(Z) = \frac{X(Z) - Z^{-1}X(Z)}{T}$$

$$\therefore Y(Z) = X(Z)\left[\frac{1 - Z^{-1}}{T}\right]$$

$$\therefore \frac{Y(Z)}{X(Z)} = H(Z) = \left[\frac{1 - Z^{-1}}{T}\right] \quad \ldots(3)$$

Equation (3) gives the transfer function of digital filter.

Now we will obtain the transfer function of analog filter.

We have,

$$y(t) = \frac{d}{dt}x(t)$$

Taking Laplace transform we get,

$$Y(s) = sX(s)$$

$$\therefore \frac{Y(s)}{X(s)} = H(s) = s \quad \ldots(4)$$

Equation (4) gives the transfer function of analog filter.

Comparing Equations (3) and (4) we get,

$$s = \frac{1 - Z^{-1}}{T} \quad \ldots(5)$$

Thus transfer function of digital filter is obtained by putting $s = \frac{1 - Z^{-1}}{T}$ in the equation of the transfer function of digital filter.

That means,

$$H(Z) = H_a(s)\Big|_{s = \frac{1 - Z^{-1}}{T}} \quad \ldots(6)$$

Mapping between s and Z plane :

We have,

$$s = \frac{1 - Z^{-1}}{T}$$

Multiplying numerator and denominator by Z we get,

$$s = \frac{Z - 1}{ZT}$$

$$\therefore \quad sZT = Z - 1$$

$$\therefore \quad sZT - Z = -1$$

$$\therefore \quad -sZT + Z = 1$$

$$\therefore \quad Z(1 - sT) = 1$$

$$\therefore \quad Z = \frac{1}{1 - sT} \qquad ...(7)$$

We know that 's' is the laplace operator and it is expressed as,

$$s = \sigma + j\Omega$$

Putting this value in Equation (7) we get,

$$Z = \frac{1}{1 - T(\sigma + j\Omega)}$$

$$\therefore \quad Z = \frac{1}{1 - \sigma T - j\Omega T}$$

$$\therefore \quad Z = \frac{1}{1 - \sigma T - j\Omega T} \times \frac{1 - \sigma T + j\Omega T}{1 - \sigma T + j\Omega T}$$

$$\therefore \quad Z = \frac{1 - \sigma T + j\Omega T}{(1 - \sigma T)^2 + (\Omega T)^2}$$

$$\therefore \quad Z = \frac{1 - \sigma T}{(1 - \sigma T)^2 + (\Omega T)^2} + j\frac{\Omega T}{(1 - \sigma T)^2 + (\Omega T)^2} \qquad ...(8)$$

Putting $\sigma = 0$ in Equation (8) we get,

$$Z = \frac{1}{1 + (\Omega T)^2} + j\frac{\Omega T}{1 + (\Omega T)^2} \qquad ...(9)$$

Now as Ω varies from $-\infty$ to $+\infty$, the corresponding locus of points in Z plane is a circle of radius $\frac{1}{2}$ and its centre at $Z = \frac{1}{2}$. This mapping is shown in Fig. K-1(b).

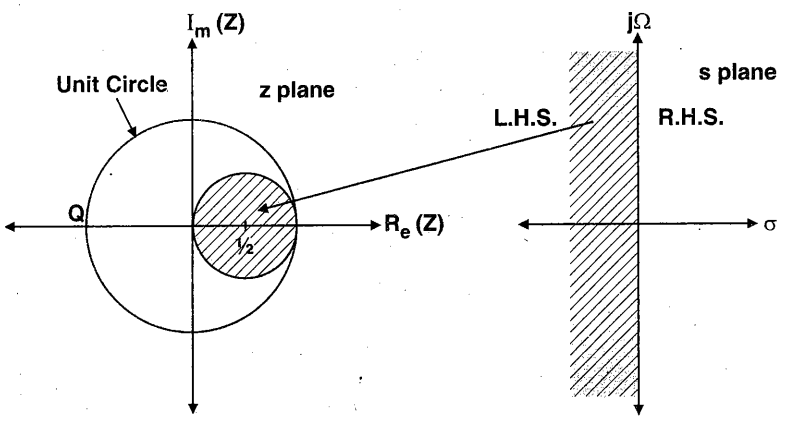


Fig. K-1(b) : Mapping between s and Z plane

This mapping shows that,

i) L.H.S. of s plane is mapped onto the points inside the circle in Z plane having radius $= \frac{1}{2}$ and centre at $Z = \frac{1}{2}$.

ii) R.H.S. of s plane is mapped onto the points outside the circle in Z plane.

iii) The stable analog filter is converted into stable digital filter.

Limitation of approximation of derivatives method :

This method is suitable only for designing of low pass and bandpass IIR digital filters with relatively small resonant frequencies.

1.2.2 Impulse Invariance Method :

In this method, the design starts from the specifications of analog filter. Here we have to replace analog filter by digital filter. This is achieved if impulse response of digital filter resembles the sampled version of impulse response of analog filter. If impulse response of both, analog and digital filter matches then, both filters perform in a similar manner.

Before studying this method we will list out the different notations, we are going to use.

$h(t)$ = Impulse response in time domain

$H_a(s)$ = Transfer function of analog filter; here 's' is laplace operator

$h(nT_s)$ = Sampled version of $h(t)$, obtained by replacing t by nT_s.

$H(Z)$ = Z transform of $h(nT_s)$. This is response of digital filter.

Ω = Analog frequency

ω = Digital frequency

Transformation of analog system function $H_a(s)$ to digital system function $H(Z)$:

Now let the system transfer function of analog filter be $H_a(s)$. We can express $H_a(s)$ in terms of partial fraction expansion. That means,

$$H_a(s) = \frac{A_1}{s-P_1} + \frac{A_2}{s-P_2} + \frac{A_3}{s-P_3} \ldots.$$

$$\therefore \quad H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s-P_k} \qquad \ldots(1)$$

Here A_k = $A_1, A_2 \ldots A_N$ are the coefficients of partial fraction expansion.

and P_k = $P_1, P_2 \ldots P_N$ are the poles.

Here 's' is the laplace operator. So we can obtain impulse response of analog filter, $h(t)$ from $H_a(s)$ by taking inverse laplace of $H_a(s)$. So using standard relation of inverse laplace we get,

$$h(t) = \sum_{k=1}^{N} A_k e^{P_k t} \quad \ldots(2)$$

Now unit impulse response for discrete structure is obtained by sampling $h(t)$. That means, $h(n)$ can be obtained from $h(t)$ by replacing 't' by $n T_s$ in Equation (2).

$$\therefore \quad h(n) = \sum_{k=1}^{N} A_k e^{P_k n T_s} \quad \ldots(3)$$

Here T_s is the sampling time.

The system transfer function of digital filter is denoted by $H(Z)$. It is obtained by taking Z-transform of $h(n)$. According to the definition of Z-transform for causal system,

$$H(Z) = \sum_{n=0}^{\infty} h(n) Z^{-n} \quad \ldots(4)$$

Putting Equation (3) in Equation (4) we get,

$$H(Z) = \sum_{n=0}^{\infty} \left[\sum_{k=1}^{N} A_k e^{P_k n T_s} \right] \cdot Z^{-n}$$

$$\therefore \quad H(Z) = \sum_{k=1}^{N} A_k \sum_{n=0}^{\infty} e^{P_k T_s n} \cdot Z^{-n}$$

$$\therefore \quad H(Z) = \sum_{k=1}^{N} A_k \sum_{n=0}^{\infty} \left(e^{P_k T_s} \cdot Z^{-1} \right)^n \quad \ldots(5)$$

Using the standard summation formula,

$$\sum_{n=0}^{\infty} a^k = \frac{1}{1-a};$$

Equation (5) becomes,

$$H(Z) = \sum_{k=1}^{N} A_k \cdot \frac{1}{1 - e^{P_k T_s} \cdot Z^{-1}} \quad \ldots(6)$$

This is the required transfer function of digital filter.

Thus comparing Equation (1) and Equation (6), we can say that the transfer function of digital filter is obtained from the transfer function of analog filter by doing the transformation.

$$\frac{1}{s - P_k} \longrightarrow \frac{1}{1 - e^{P_k T_s} \cdot Z^{-1}} \qquad ...(7)$$

Equation (7) shows, how the poles from analog domain are transferred into the digital domain. This transformation of poles is called as mapping of poles.

Relationship of s-plane to Z plane :

We know that the poles of analog filters are located at $s = P_k$. Now from Equation (7) we can say that the poles of digital filter, H (Z) are located at,

$$Z = e^{P_k T_s} \qquad ...(8)$$

This equation indicates that the poles of analog filter at $s = P_k$ are transformed into the poles of digital filter at $Z = e^{P_k T_s}$. Thus the relationship between laplace ('s' domain) and Z domain is given by,

$$Z = e^{sT_s} \qquad ...(9)$$

Here $s = P_k$ and T_s is the sampling time.

Now 's' is the laplace operator and it is expressed as,

$$s = \sigma + j\Omega \qquad ...(10)$$

Here σ = Attenuation factor

and Ω = Analog frequency

We know the 'Z' can be expressed in polar form as,

$$Z = re^{j\omega} \qquad ...(11)$$

Here 'r' is magnitude and 'ω' is the digital frequency.

Putting Equations (10) and (11) in Equation (9) we get,

$$re^{j\omega} = e^{(\sigma + j\Omega)T_s}$$

$$\therefore \quad re^{j\omega} = e^{\sigma T_s} \cdot e^{j\Omega T_s} \qquad ...(12)$$

Separating real and imaginary parts of Equation (12) we get,

$$r = e^{\sigma T_s} \qquad ...(13)$$

$$\text{and} \quad e^{j\omega} = e^{j\Omega T_s}$$

$$\therefore \quad \omega = \Omega T_s \qquad ...(14)$$

Now we will find the relationship between s plane and Z plane. Basically plot in 's'-domain means, σ is plotted on X-axis and $j\Omega$ is plotted on Y- axis. And Z-domain representation means real Z is plotted on X- axis and imaginary Z is plotted on Y-axis.

Now consider Equation (13), it is

$$r = e^{\sigma T_s}$$

We will discuss the following conditions :

(i) If $\sigma < 0$, then r is equal to reciprocal of 'e' raise to some constant. Thus range of r will be 0 to 1.

$$\therefore \sigma < 0 \Rightarrow 0 < r < 1 \qquad ...(15)$$

Now $\sigma < 0$ means negative values of σ. That is L.H.S. of s plane. We know that 'r' is the radius of circle is Z plane.

So '$0 < r < 1$' indicates interior part of unit circle. Thus we can conclude that,

L.H.S. of 's' plane is mapped inside the unit circle.

(ii) If $\sigma = 0$ then $r = e^0 = 1$

$$\therefore \sigma = 0 \Rightarrow r = 1$$

Now $\sigma = 0$ indicates $j\Omega$ axis and $r = 1$ indicates unit circle. Thus,

$j\Omega$ axis in 's' plane is mapped on the unit circle.

(iii) If $\sigma > 0$ then, r is equal to 'e' raise to some constant. That means $r > 1$.

$$\therefore \sigma > 0 \Rightarrow r > 1$$

Now $\sigma > 0$ indicates R.H.S. of 's' plane and 'r' > 1 indicates exterior part of unit circle. Thus,

R.H.S. of 's' plane is mapped outside the unit circle.

Combining all conditions; this mapping is shown in Fig. K-2.

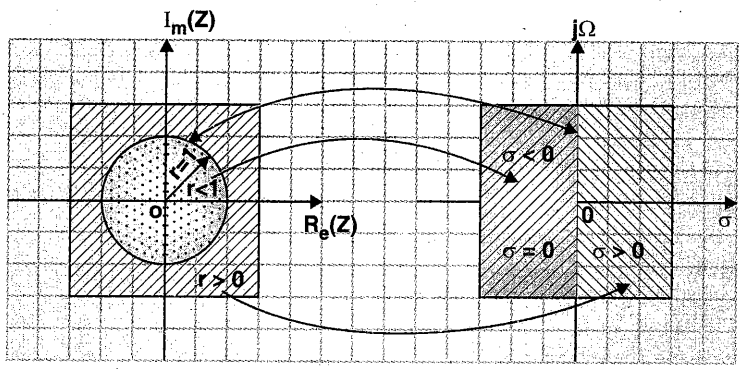


Fig. K-2 : Relationship of s plane to Z-plane

Disadvantages of impulse invariance method :

(1) We know that 'Ω' is analog frequency and its range is from $\frac{\pi}{T_s}$ to $-\frac{\pi}{T_s}$. While the digital frequency 'ω' varies from $-\pi$ to π. That means from $\frac{\pi}{T_s}$ to $-\frac{\pi}{T_s}$ 'ω' maps from $-\pi$ to π. Let k be any integer. Then, we can write the general range of Ω as $(k-1)\frac{\pi}{T_s}$ to $(k+1)\frac{\pi}{T_s}$; but

for this range also; 'ω' maps from $-\pi$ to π. Thus mapping from analog frequency 'Ω' to digital frequency 'ω' is many to one. This mapping is not one to one.

(2) Analog filters are not band limited so there will be aliasing due to the sampling process. Because of this aliasing, the frequency response of resulting digital filter will not be identical to the original frequency response of analog filter.

(3) The change in the value of sampling time (T_s) has no effect on the amount of aliasing.

Some standard formulae for transformation in impulse invariance method are as follows :

(i) $$\frac{1}{s-P_k} \longrightarrow \frac{1}{1-e^{P_k T_s} \cdot Z^{-1}}$$

(ii) $$\frac{s+a}{(s+a)^2+b^2} \longrightarrow \frac{1-e^{-aT_s}[\cos bT_s]Z^{-1}}{1-2e^{-aT_s}[\cos bT_s]Z^{-1}+e^{-2aT_s}\cdot Z^{-2}}$$

(iii) $$\frac{b}{(s+a)^2+b^2} \longrightarrow \frac{e^{-aT_s}[\sin bT_s]Z^{-1}}{1-2e^{-aT_s}[\cos bT_s]Z^{-1}+e^{-2aT_s}\cdot Z^{-2}}$$

Design steps for impulse invariance method :

Step I : Analog frequency transfer function H (s) will be given. If it is not given then, obtain expression of H (s) from the given specifications.

Step II : If required expand H (s) by using partial fraction expansion (PFE).

Step III : Obtain Z transform of each PFE term using impulse invariance transformation equation.

Step IV : Obtain H (Z), this is required digital IIR filter.

Solved Problems :

Prob. 1 : Find out H (Z) using impulse invariance method at 5 Hz sampling frequency from H (s) as given below :

$$H(s) = \frac{2}{(s+1)(s+2)}$$

Soln. :

Step I : Given analog transfer function is,

$$H(s) = \frac{2}{(s+1)(s+2)} \qquad ...(1)$$

Step II : We will expand H (s) using partial fraction expansion as :

$$\therefore \quad H(s) = \frac{A_1}{(s+1)} + \frac{A_2}{(s+2)} \qquad ...(2)$$

Thus poles are at $P_1 = -1$ and $P_2 = -2$.

Now values of A_1 and A_2 are calculated as follows :

$$A_1 = (s-P_1)\,H(s)\Big|_{s=P_1}$$

$$\therefore \quad A_1 = (s+1)\cdot\frac{2}{(s+1)(s+2)}\Big|_{s=-1}$$

$$\therefore \quad A_1 = \frac{2}{-1+2} = 2$$

$$\text{and} \quad A_2 = (s-P_2)H(s)\Big|_{s=P_2} = (s+2)\cdot\frac{2}{(s+1)(s+2)}\Big|_{s=-2}$$

$$\therefore \quad A_2 = \frac{2}{-2+1} = -2$$

Putting values of A_1 and A_2 in Equation (2) we get,

$$H(s) = \frac{2}{(s+1)} - \frac{2}{(s+2)} \qquad ...(3)$$

Step III : Now we will obtain the Z-transform using impulse invariance transformation equation. It is,

$$\frac{1}{s-P_k} \longrightarrow \frac{1}{1-e^{P_k T_s}\cdot Z^{-1}} \qquad ...(4)$$

Here T_s = Sampling time. Now given sampling frequency is F_s = 5 Hz.

$$\therefore \quad T_s = \frac{1}{F_s} = \frac{1}{5} = 0.2 \text{ sec.}$$

we have poles at $P_1 = -1$ and $P_2 = -2$

So using Equation (4) we get,

$$\frac{1}{s+1} \longrightarrow \frac{1}{1-e^{-1(0.2)}\cdot Z^{-1}} = \frac{1}{1-e^{-0.2}\cdot Z^{-1}} \qquad ...(5)$$

$$\text{and} \quad \frac{1}{s+2} \longrightarrow \frac{1}{1-e^{-2(0.2)}\cdot Z^{-1}} = \frac{1}{1-e^{-0.4}Z^{-1}} \qquad ...(6)$$

Step IV : The transfer function of digital filter is given by,

$$H(Z) = \sum_{k=1}^{N} \frac{A_k}{1-e^{P_k T_s}\cdot Z^{-1}}$$

In this case we get,

$$H(Z) = \frac{A_1}{1-e^{P_1 T_s}\cdot Z^{-1}} + \frac{A_2}{1-e^{P_2 T_s}\cdot Z^{-1}} \qquad ...(7)$$

Using Equations (5) and (6) we get,

$$H(Z) = \frac{2}{1-e^{-0.2}\cdot Z^{-1}} - \frac{2}{1-e^{-0.4}\cdot Z^{-1}}$$

$$\therefore \quad H(Z) = \frac{2}{1-0.818\,Z^{-1}} - \frac{2}{1-0.67\,Z^{-1}}$$

To convert each term into positive powers of Z; multiplying numerator and denominator of each term by Z we get,

$$H(Z) = \frac{2Z}{Z-0.818} - \frac{2Z}{Z-0.67}$$

$$\therefore \quad H(Z) = \frac{2Z(Z-0.67) - 2Z(Z-0.818)}{(Z-0.818)(Z-0.67)}$$

$$\therefore \quad H(Z) = \frac{2Z^2 - 1.34Z - 2Z^2 + 1.636Z}{Z^2 - 0.67Z - 0.818Z + 0.54}$$

$$\therefore \quad H(Z) = \frac{0.29Z}{Z^2 - 1.488Z + 0.54}$$

$$\therefore \quad H(Z) = \frac{0.29Z^{-1}}{1 - 1.488Z^{-1} + 0.54Z^{-2}}$$

This is the required transfer function for digital IIR filter.

Prob. 2 : Determine H(Z) using impulse invariance method for the system function,

$$H(s) = \frac{1}{(s+0.5)(s^2+0.5s+2)}$$

Soln. :

Step I : The given transfer function is,

$$H(s) = \frac{1}{(s+0.5)(s^2+0.5s+2)} \qquad ...(1)$$

Step II : In the partial fraction expansion form H(s) can be written as,

$$H(s) = \frac{1}{(s+0.5)(s^2+0.5s+2)} = \frac{A}{(s+0.5)} + \frac{Bs+C}{(s^2+0.5s+2)} \qquad ...(2)$$

Let us obtain the values of A, B and C.

$$\frac{1}{(s+0.5)(s^2+0.5s+2)} = \frac{A(s^2+0.5s+2)+(Bs+C)(s+0.5)}{(s+0.5)(s^2+0.5s+2)}$$

$$\therefore \quad A(s^2+0.5s+2)+(Bs+C)(s+0.5) = 1$$

$$\therefore \quad As^2 + A\,0.5\,s + 2A + Bs^2 + B\,0.5\,s + C.s + 0.5\,C = 1$$

$$s^2(A+B) + s(0.5\,A + 0.5\,B + C) + (2A + 0.5\,C) = 1 \qquad ...(3)$$

Now s^2 term is absent in R.H.S.

$$\therefore \quad A + B = 0 \qquad ...(4)$$

Similarly 's' term is absent in R.H.S.

$$\therefore \quad 0.5\,A + 0.5\,B + C = 0 \qquad ...(5)$$

$$\text{And} \quad 2\,A + 0.5\,C = 1 \qquad ...(6)$$

Now we will solve Equations (4), (5) and (6) to obtain the values of A, B and C.

From Equation (4),

$$B = -A$$

Putting this value in Equation (5) we get,

$$0.5A - 0.5A + C = 0$$

$$\therefore \quad C = 0$$

From Equation (6), $2A + 0 = 1$

$$\therefore \quad A = 0.5$$

Since $B = -A$ we get,

$$\therefore \quad B = -0.5$$

Putting these values in Equation (2) we get,

$$H(s) = \frac{0.5}{s+0.5} - \frac{0.5\,s}{s^2+0.5\,s+2} \qquad \ldots(7)$$

To use the standard transformation formulae we will convert the second term on R.H.S. in the form $\frac{s+a}{(s+a)^2+b^2}$ and $\frac{b}{(s+a)^2+b^2}$

Consider the term $s^2+0.5\,s+2$. It can be expressed as,

$$s^2+0.5\,s+2 = (s^2+0.5\,s+0.0625)+(1.9375)$$

$$\therefore \quad s^2+0.5\,s+2 = (s+0.25)^2+(1.39)^2$$

Putting this value in Equation (7) we get,

$$H(s) = \frac{0.5}{s+0.5} - \frac{0.5\,s}{(s+0.25)^2+(1.39)^2}$$

$$\therefore \quad H(s) = \frac{0.5}{s+0.5} - 0.5\left[\frac{s+0.25-0.25}{(s+0.25)^2+(1.39)^2}\right]$$

$$\therefore \quad H(s) = \frac{0.5}{s+0.5} - 0.5\left[\frac{s+0.25}{(s+0.25)^2}\right] - 0.5\left[\frac{-0.25}{(s+0.25)^2+(1.39)^2}\right]$$

Now we want the numerator of third term equal to 1.39. It is arranged as follows,

$$H(s) = \frac{0.5}{s+0.5} - 0.5\left[\frac{s+0.25}{(s+0.25)^2}\right] + \frac{0.5\times 0.25}{1.39}\left[\frac{1.39}{(s+0.25)^2+(1.39)^2}\right]$$

$$\therefore \quad H(s) = \frac{0.5}{s+0.5} - 0.5\left[\frac{s+0.25}{(s+0.25)^2}\right] + 0.089\left[\frac{1.39}{(s+0.25)^2+(1.39)^2}\right] \qquad \ldots(8)$$

Now recall the standard transformation formulae,

(i) $$\frac{1}{s-P_k} \longrightarrow \frac{1}{1-e^{P_k T_s}\cdot Z^{-1}}$$

(ii) $$\frac{s+a}{(s+a)^2+b^2} \longrightarrow \frac{1-e^{-aT_s}[\cos bT_s]Z^{-1}}{1-2e^{-aT_s}[\cos bT_s]Z^{-1}+e^{-2aT_s}\cdot Z^{-2}}$$

(iii) $$\frac{b}{(s+a)^2+b^2} \longrightarrow \frac{e^{-aT_s}[\sin bT_s]Z^{-1}}{1-2e^{-aT_s}[\cos bT_s]Z^{-1}+e^{-2aT_s}\cdot Z^{-2}}$$

Step III : Using these formulae, equation of H (Z) can be obtained from Equation (8) as follows :

$$H(Z) = \frac{0.5}{1-e^{0.5T_s}\cdot Z^{-1}} - 0.5\left[\frac{1-e^{-0.25T_s}[\cos 1.39\,T_s]Z^{-1}}{1-2e^{-0.25T_s}[\cos 1.39\,T_s]Z^{-1}+e^{-0.5T_s}\cdot Z^{-2}}\right]$$

$$+ 0.089\left[\frac{e^{-0.25T_s}(\sin 1.39\,T_s)Z^{-1}}{1-2e^{-0.25T_s}(\cos 1.39\,T_s)Z^{-1}+e^{-0.5T_s}\cdot Z^{-2}}\right]$$

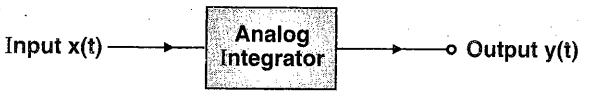
This is the required transfer function. Note that the value of T_s is not given in the problem; so T_s is kept as it is.

1.2.3 Bilinear Transformation Method :

In case of impulse invariance method, we have studied that the mapping is many to one. So this method is not suitable to design high-pass filter and band reject filter.

In case of bilinear transformation; the mapping is one to one from s domain to the Z domain. So there is no aliasing effect. The limitations of impulse invariance method are overcome by using BLT method.

Consider an analog integrator as shown in Fig. K-2(A)



$H_a(s) = \frac{1}{S}$

Fig. K-2(A)

The response of analog integrator is,

$$H_a(s) = \frac{1}{s} \qquad ...(1)$$

Input in laplace domain is X (s) and output in laplace domain is Y (s).

For time period T; the difference in output is given by,

$$Y(t_2) - Y(t_1) = \int_{t_1}^{t} x(nT_s)\,dT \qquad ...(2)$$

Consider input signal as shown in Fig. K-2(B).

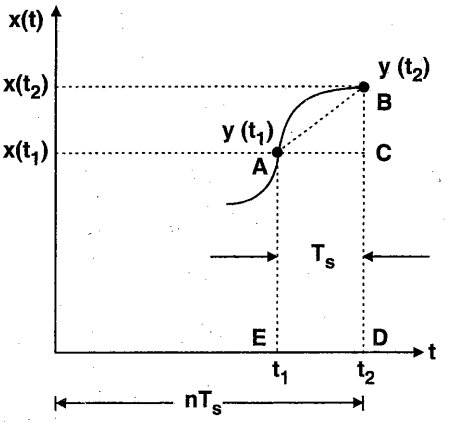


Fig. K-2(B)

Here we have assumed two input positions $x(t_1)$ and $x(t_2)$. Corresponding output is denoted by $Y(t_1)$ and $Y(t_2)$ respectively. Now area under the curve is addition of area of triangle ABC and area of rectangle ACDE.

$$\therefore \quad Y(t_2) - Y(t_1) = \frac{1}{2}(t_2 - t_1)\left[x(t_2) - x(t_1)\right] + x(t_1)(t_2 - t_1) \qquad ...(3)$$

As shown in Fig. K-2(B), time period $t_2 = nT_s$. Thus time period t_1 is $nT_s - T_s$ That means $t_2 - t_1 = T_s$

Putting these values in Equation (3) we get,

$$Y(nT_s) - Y(nT_s - T_s) = \frac{1}{2}T\left[x(nT_s) - x(nT_s - T)\right] + x(nT_s - T)T_s$$

$$Y(nT_s) - Y(nT_s - T_s) = \frac{1}{2}T_s x(nT) - \frac{1}{2}T_s x(nT_s - T_s) + T_s x(nT_s - T_s)$$

$$\therefore \quad Y(nT_s) - Y(nT_S - T_s) = \frac{1}{2}T_s x(nT_s) + \frac{1}{2}T_s x(nT_s - T_s)$$

$$\therefore \quad Y(nT_s) - Y(nT_s) - T_s = \frac{T_s}{2}\left[x(nT_s) + x(nT_s - T_s)\right] \qquad ...(4)$$

Taking Z transform of both sides we get,

$$Y(Z) - Z^{-1}Y(Z) = \frac{T_s}{2}\left[X(Z) + Z^{-1}X(Z)\right]$$

$$\therefore Y(Z)\left[1 - Z^{-1}\right] = \frac{T_s}{2}\left[X(Z)(1 + Z^{-1})\right]$$

$$\therefore \quad \frac{Y(Z)}{X(Z)} = H(Z) = \frac{T_s}{2}\frac{(1+Z^{-1})}{(1-Z^{-1})} \qquad ...(5)$$

Now we have transfer function of analog filter,

$$H_a(s) = \frac{1}{s} \qquad ...(6)$$

Equating (5) and (6) we get,

$$\frac{1}{s} = \frac{T_s}{2}\frac{(1+Z^{-1})}{(1-Z^{-1})} \qquad ...(7)$$

Thus relationship between s plane and Z-plane is given by,

$$s = \frac{2}{T_s}\left(\frac{Z-1}{Z+1}\right) \qquad ..(8)$$

Here T_s is the sampling time.

We know that 's' is the laplace operator and it can be expressed as,

$$s = \sigma + j\Omega \qquad ...(9)$$

Now the equation of Z in polar form is,

$$Z = r e^{j\omega} \qquad ...(10)$$

Putting Equations (9) and (10) in Equation (8) and multiplying numerator and denominator by $(re^{-j\omega}+1)$ we get,

$$\sigma + j\Omega = \frac{2}{T_s}\left[\frac{re^{j\omega}-1}{re^{j\omega}+1}\right]\times\frac{re^{-j\omega}+1}{re^{-j\omega}+1}$$

$$\sigma + j\Omega = \frac{2}{T_s}\left[\frac{r^2\cdot e^{j\omega}\cdot e^{-j\omega}+re^{j\omega}-re^{-j\omega}-1}{r^2 e^{j\omega}\cdot e^{-j\omega}+re^{j\omega}+re^{-j\omega}+1}\right] \qquad ...(11)$$

But $e^{j\omega}\cdot e^{-j\omega} = e^0 = 1$

$$\therefore \quad \sigma + j\Omega = \frac{2}{T_s}\left[\frac{r^2+re^{j\omega}-re^{-j\omega}-1}{r^2+re^{j\omega}+re^{-j\omega}+1}\right] \qquad ...(12)$$

Now we have, $\frac{e^{j\omega}-e^{-j\omega}}{2j} = \sin\omega$ and $\frac{e^{j\omega}+e^{-j\omega}}{2} = \cos\omega$.

We will rearrange Equation (12) as follows :

$$\sigma + j\Omega = \frac{2}{T_s}\left[\frac{r^2-1+j2r\left(\frac{e^{j\omega}-e^{-j\omega}}{2j}\right)}{r^2+2r\left(\frac{e^{j\omega}+e^{-j\omega}}{2}\right)+1}\right]$$

$$\therefore \quad \sigma + j\Omega = \frac{2}{T_s}\left[\frac{r^2-1+j2r\sin\omega}{r^2+2r\cos\omega+1}\right]$$

$$\therefore \quad \sigma + j\Omega = \frac{2}{T_s}\left[\frac{r^2-1}{r^2+2r\cos\omega+1} + j\frac{2r\sin\omega}{r^2+2r\cos\omega+1}\right] \quad ...(13)$$

Equating real and imaginary parts of Equation (13) we get,

$$\sigma = \frac{2}{T_s} \times \frac{r^2-1}{r^2+2r\cos\omega+1} \quad ...(14)$$

$$\text{and} \quad \Omega = \frac{2}{T_s} \times \frac{2r\sin\omega}{r^2+2r\cos\omega+1} \quad ...(15)$$

Now we will discuss the following conditions related to Equation (14).

(i) When $r < 1$ then $\sigma < 0$

Here $r < 1$, means interior part of circle having unit circle and $\sigma < 0$, means σ is negative which is L.H.S. of s-plane. So this condition indicates that L.H.S. of s plane maps inside the unit circle.

(ii) When $r = 1$ then $\sigma = 0$

Now $r = 1$ means unit circle and $\sigma = 0$ means $j\Omega$ axis. Thus this condition indicates that the $j\Omega$ axis maps on the unit circle.

(iii) When $r > 1$ then $\sigma > 0$

Here $r > 1$, means exterior part of unit circle and $\sigma > 0$ indicates that σ is positive means R.H.S. of s-plane. So this condition indicates that R.H.S. of s-plane maps outside the unit circle.

This mapping is similar to the mapping in impulse invariance method, as shown in Fig. K-2. But in impulse invariance method mapping is valid only for poles ; while in bilinear transformation, mapping is valid for poles as well as zeros.

How stable analog filter is converted into stable digital filter ?

Analog filter is stable if the poles lie on the L.H.S. of s-plane. While the digital filter is stable if the poles are inside the unit circle in the Z-domain. Now condition (i) indicates that L.H.S. of s-plane maps inside the unit circle. Thus stable analog filter is converted into stable digital filter.

Frequency warping concept :

Here we will obtain the relationship of $j\Omega$ axis in s-plane to the unit circle in the Z-plane ($r = 1$).

Recall Equation (15).

$$\Omega = \frac{2}{T_s} \times \frac{2r\sin\omega}{r^2+2r\cos\omega+1}$$

For the unit circle, $r = 1$. Thus putting $r = 1$ in the equation of Ω we get,

$$\Omega = \frac{2}{T_s} \times \frac{2\sin\omega}{1+2\cos\omega+1}$$

$$\therefore \quad \Omega = \frac{2}{T_s} \times \frac{2\sin\omega}{2+2\cos\omega}$$

$$\therefore \quad \Omega = \frac{2}{T_s} \times \frac{\sin \omega}{1 + \cos \omega} \qquad ...(16)$$

We have the trignometric identities,

$$\sin \omega = 2 \sin \frac{\omega}{2} \cdot \cos \frac{\omega}{2} \text{ and } 2 \cos^2 \frac{\omega}{2} = 1 + \cos \omega.$$

Thus Equation (16) becomes,

$$\Omega = \frac{2}{T_s} \times \frac{2 \sin \frac{\omega}{2} \cos \frac{\omega}{2}}{2 \cos^2 \frac{\omega}{2}}$$

$$\therefore \quad \Omega = \frac{2}{T_s} \times \frac{2 \sin \frac{\omega}{2}}{2 \cos \omega / 2}$$

$$\therefore \quad \Omega = \frac{2}{T_s} \tan \frac{\omega}{2}$$

$$\therefore \quad \omega = 2 \tan^{-1} \left(\frac{\Omega T_s}{2} \right) \qquad ...(17)$$

Now for different values of ΩT_s; the graph of ΩT_s versus ω is as shown in Fig. K-3.

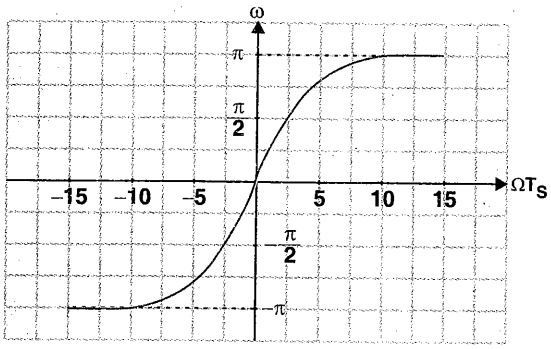


Fig. K-3 : Mapping between ω and Ω

Observations :

1. The entire range in Ω is mapped only once into the range $-\pi \leq \omega \leq \pi$.
2. The mapping is one to one.
3. The mapping is highly non-linear.

Now because of the non-linearity of tangent function $2 \tan^{-1} \left(\frac{\Omega T_s}{2} \right)$; there exists frequency warping or frequency compression.

What is frequency warping ?

Because of the non-linear mapping ; the amplitude response of digital IIR filter is expanded at lower frequencies and compressed at higher frequencies in comparison to the analog filter. This effect is called as frequency warping.

Prewarping procedure :

We have discussed the warping effect. Because of this effect, there is non-linear compression of Ω to ω values. To compensate this effect ; prewarping or prescaling procedure is used. This procedure is as follows :

(i) The 'Ω' scale is changed to prewarpped scale denoted by Ω^* and $\Omega^* = \frac{2}{T_s} \tan\left(\frac{\omega T_s}{2}\right)$

(ii) Then analog filter transfer function $H(s)$ is obtained using values of Ω^*.

(iii) By applying the bilinear transformation, the desired digital frequency response $H(Z)$ is obtained.

Advantages of bilinear transformation method :

1. There is one to one transformation from the s-domain to the Z- domain.
2. The mapping is one to one.
3. There is no aliasing effect.
4. Stable analog filter is transformed into the stable digital filter.

Disadvantage of bilinear transformation method :

The mapping is non-linear and because of this; frequency warping effect takes place.

Comparison between Impulse invariance and Bilinear transformation method :

Sr. No.	Impulse invariance method	Bilinear transformation method
1.	Poles are transferred by using the equation, $\frac{1}{s-P_k} \rightarrow \frac{1}{1-e^{P_k T_s} \cdot Z^{-1}}$	Poles are transferred by using the equation. $s = \frac{2}{T_s}\left[\frac{Z-1}{Z+1}\right]$
2.	Mapping is many to one.	Mapping is one to one.
3.	Aliasing effect is present.	Aliasing effect is not present.
4.	It is not suitable to design high-pass filter and band reject filter.	High pass filter and band reject filter can be designed.
5.	Only poles of the system can be mapped.	Poles as well as zeros can be mapped.
6.	No frequency warping effect.	Frequency warping effect is present.

Solved Problems :

Prob. 1 : An analog filter has the following transfer function $H(S) = \frac{1}{s+1}$. Using bilinear transformation technique, determine the transfer function of digital filter $H(Z)$ and also write the difference equation of digital filter.

Soln. : The given transfer function is,

$$H(S) = \frac{1}{s+1} \quad \ldots(1)$$

In bilinear transformation H (Z) is obtained by putting,

$$s = \frac{2}{T_s}\left[\frac{Z-1}{Z+1}\right]$$

Here T_s is the sampling time; which is not given. So assume T_s = 1 sec.

$$\therefore \quad s = 2\left[\frac{Z-1}{Z+1}\right] \qquad ...(2)$$

Putting this value in Equation (1),

$$H(Z) = \frac{1}{1+2\left(\frac{Z-1}{Z+1}\right)} = \frac{1}{\frac{(Z+1)+2(Z-1)}{Z+1}}$$

$$\therefore \quad H(Z) = \frac{Z+1}{Z+1+2Z-2}$$

$$\therefore \quad H(Z) = \frac{Z+1}{3Z-1} \qquad ...(3)$$

This is the transfer function of digital filter. Now we have,

$$H(Z) = \frac{Y(Z)}{X(Z)} = \frac{Z+1}{3Z-1}$$

Converting into negative power of Z we get,

$$\frac{Y(Z)}{X(Z)} = \frac{1+Z^{-1}}{3-Z^{-1}}$$

$$\therefore \quad Y(Z)(3-Z^{-1}) = X(Z)(1+Z^{-1})$$

$$\therefore \quad 3Y(Z)-Z^{-1}\cdot Y(Z) = X(Z)+Z^{-1}X(Z)$$

Taking IZT of both sides,

$$3y(n)-y(n-1) = x(n)+x(n-1)$$

$$\therefore \quad 3y(n) = x(n)+x(n-1)+y(n-1)$$

$$\therefore \quad y(n) = \frac{1}{3}x(n)+\frac{1}{3}x(n-1)+\frac{1}{3}y(n-1)$$

This is the difference equation of digital filter.

Prob. 2 : The transfer function of analog filter is :

$$H(s) = \frac{3}{(s+2)(s+3)} \text{ with } T_s = 0.1 \text{ sec.}$$

Design the digital IIR filter using BLT.

Soln. : The given transfer function is,

$$H(s) = \frac{3}{(s+2)(s+3)} \qquad ...(1)$$

In bilinear transformation, H (Z) is obtained by putting :

$$s = \frac{2}{T_s}\left[\frac{Z-1}{Z+1}\right] \qquad ...(2)$$

Here T_s = sampling time = 0.1 sec.

$$s = \frac{2}{0.1}\left[\frac{Z-1}{Z+1}\right] = 20\left[\frac{Z-1}{Z+1}\right]$$

Putting this value in Equation (1) we get,

$$H(Z) = \frac{3}{\left[20\left(\frac{Z-1}{Z+1}\right)+2\right]\left[20\left(\frac{Z-1}{Z+1}\right)+3\right]}$$

$$\therefore \quad H(Z) = \frac{3}{\left[\frac{20Z-20}{Z+1}+2\right]\left[\frac{20Z-20}{Z+1}+3\right]}$$

$$\therefore \quad H(Z) = \frac{3(Z+1)(Z+1)}{(20Z-20+2Z+2)(20Z-20+3Z+3)}$$

$$\therefore \quad H(Z) = \frac{3(Z+1)^2}{(22Z-18)(23Z-17)}$$

$$\therefore \quad H(Z) = \frac{3(Z^2+2Z+1)}{506Z^2-374Z-414Z+306}$$

$$\therefore \quad H(Z) = \frac{3(Z^2+2Z+1)}{506Z^2-788Z+306} = \frac{Z^2+2Z+1}{168.67Z^2-262.67Z+102}$$

$$= \frac{1+2Z^{-1}+Z^{-2}}{168.67-262.67Z^{-1}+102Z^{-2}}$$

This is the required transfer function for digital IIR filter.

Prob. 3 : Design a single pole low pass digital filter with a 3 dB bandwidth of 0.2 π by use of bilinear transformation applied to the analog filter.

$H(s) = \frac{\Omega_c}{s+\Omega_c}$, where Ω_c is the 3 dB bandwidth of analog filter.

Soln. : The given transfer function of analog filter is,

$$H(s) = \frac{\Omega_c}{s+\Omega_c} \qquad ...(1)$$

The cut-off frequency of digital filter (ω_c) is given. It is,

$$\omega_c = 0.2\pi \qquad ...(2)$$

We know that in BLT, Ω_c and ω_c are related as,

$$\Omega_c = \frac{2}{T_s}\tan\left[\frac{\omega_c}{2}\right] = \frac{2}{T_s}\tan\left[\frac{0.2\pi}{2}\right] = \frac{2}{T_s}\tan[0.1\pi]$$

$$\therefore \quad \Omega_c = \frac{0.65}{T_s} \qquad \ldots(3)$$

Putting this value in Equation (1) we get,

$$H(s) = \frac{\frac{0.65}{T_s}}{s + \frac{0.65}{T_s}} \qquad \ldots(4)$$

Thus we have obtained the transfer function of analog filter in the appropriate form.

Now to obtain $H(Z)$ we have to put,

$$s = \frac{2}{T_s}\left[\frac{Z-1}{Z+1}\right] \text{ in Equation (4)}$$

$$\therefore \quad H(Z) = \frac{0.65/T_s}{\frac{2}{T_s}\left[\frac{Z-1}{Z+1}\right] + \frac{0.65}{T_s}}$$

Multiplying numerator and denominator by T_s we get,

$$H(Z) = \frac{0.65}{2\left[\frac{Z-1}{Z+1}\right] + 0.65} = \frac{0.65}{\frac{2Z-2}{Z+1} + 0.65}$$

$$\therefore \quad H(Z) = \frac{0.65(Z+1)}{2Z-2+0.65Z+0.65} = \frac{0.65(Z+1)}{2.65Z-1.35}$$

Dividing numerator and denominator by 2.65 we get,

$$H(Z) = \frac{0.245(Z+1)}{Z-0.509} \quad \therefore \quad H(Z) = \frac{0.245[1+Z^{-1}]}{1-0.509Z^{-1}} \qquad \ldots(5)$$

This is the required transfer function for digital filter. Note that there is a single pole at $P_1 = 0.509$.

Now in the problem frequency of digital filter is given which is $\omega_c = 0.2\pi$ and bandwidth is 3 dB. We will check the frequency response. The frequency response of digital filter is obtained by putting $Z = e^{j\omega}$ in the equation of $H(Z)$. Thus we get,

$$H(\omega) = H(Z)\Big|_{Z=e^{j\omega}}$$

$$\therefore \quad H(\omega) = \frac{0.245(e^{j\omega}+1)}{e^{j\omega}-0.509} \qquad \ldots(6)$$

If $\omega = 0$ then Equation (6) becomes,

$$H(0) = \frac{0.245\,(e^{0}+1)}{e^{0}-0.509} = \frac{0.49}{0.49} = 1$$

Thus at $\omega = 0$; the magnitude $|H(\omega)| = 1$.

Now at $\omega = 0.2\pi$ we get,

$$H(\omega) = \frac{0.245\,(e^{j\,0.2\pi}+1)}{e^{j\,0.2\pi}-0.509} = \frac{0.245\,[\cos(0.2\pi)+j\sin(0.2\pi)+1]}{\cos(0.2\pi)+j\sin(0.2\pi)-0.509}$$

$$\therefore \quad H(\omega) = \frac{0.245\,[0.809+j\,0.587+1]}{0.809+j\,0.587-0.509}$$

$$\therefore \quad H(\omega) = \frac{0.245\,(1.809+j\,0.589)}{0.3+j\,0.587}$$

$$\therefore \quad |H(\omega)| = 0.707 = 3 \text{ dB}$$

Thus the desired response is obtained.

Prob. 4 : The system transfer function of analog filter is given by,

$$H(s) = \frac{s+0.1}{(s+0.1)^2+16}$$

Obtain the system transfer function of digital filter using BLT which is resonant at $\omega_r = \frac{\pi}{2}$

Soln. : The given transfer function is,

$$H(s) = \frac{s+0.1}{(s+0.1)^2+16}$$

$$\therefore \quad H(s) = \frac{s+0.1}{(s+0.1)^2+(4)^2} \qquad ..(1)$$

From Equation (1) we can say that $\Omega = 4$.

The value of ω_r is given as $\omega_r = \frac{\pi}{2}$

Now we will find out the value of sampling time (T_s) using the relation.

$$\Omega = \frac{2}{T_s}\tan\left(\frac{\omega}{2}\right)$$

$$\therefore \quad 4 = \frac{2}{T_s}\tan\left(\frac{\pi}{4}\right)$$

$$\therefore \quad T_s = \frac{2}{4}\tan\left(\frac{\pi}{4}\right)$$

$$\therefore \quad T_s = 0.5 \text{ sec.} \qquad ...(2)$$

Using bilinear transformation H(Z) can be obtained by putting, $s = \frac{2}{T_s}\left(\frac{Z-1}{Z+1}\right)$ in the equation of H(s).

$$\therefore \quad H(Z) = \frac{\frac{2}{0.5}\left(\frac{Z-1}{Z+1}\right)+0.1}{\left[\frac{2}{0.5}\left(\frac{Z-1}{Z+1}\right)+0.1\right]^2+16}$$

$$\therefore \quad H(Z) = \frac{4\left(\frac{Z-1}{Z+1}\right)+0.1}{\left[4\left(\frac{Z-1}{Z+1}\right)+0.1\right]^2+16} = \frac{\frac{4Z-4}{Z+1}+0.1}{\left[\frac{4Z-4}{Z+1}+0.1\right]^2+16}$$

$$\therefore \quad H(Z) = \frac{\frac{4Z-4}{Z+1}+0.1}{\left[\frac{4Z-4+0.1Z+0.1}{Z+1}\right]^2+16} = \frac{\frac{4Z-4+0.1Z+0.1}{Z+1}}{\left[\frac{4.1Z-3.9}{Z+1}\right]^2+16}$$

$$\therefore \quad H(Z) = \frac{\frac{4.1Z-3.9}{Z+1}}{\frac{(16.81Z^2-31.98Z+15.21)}{(Z+1)^2}+16}$$

$$\therefore \quad H(Z) = \frac{\frac{(4.1Z-3.9)}{Z+1}\times(Z+1)^2}{16.81Z^2-31.98Z+15.21+16(Z+1)^2}$$

$$\therefore \quad H(Z) = \frac{(4.1Z-3.9)(Z+1)}{16.81Z^2-31.98Z+15.21+16Z^2+32Z+16}$$

$$\therefore \quad H(Z) = \frac{4.1Z^2+4.1Z-3.9Z-3.9}{32.81Z^2+0.02Z+31.21}$$

$$\therefore \quad H(Z) = \frac{4.1Z^2+0.2Z-3.9}{32.81Z^2+0.02Z+31.21}$$

$$\therefore \quad H(Z) = \frac{4.1+0.2Z^{-1}-3.9Z^{-2}}{32.81+0.02Z^{-1}+31.2/Z^{-2}}$$

This is the required transfer function for digital IIR filter.

Prob. 5 : The analog transfer function of low pass filter is, $H(s) = \frac{1}{s+2}$ and its bandwidth is 1 rad/sec.

Design the digital filter using BLT method whose cut-off frequency is 20π and sampling time is 0.0167 sec.; by considering the warping effect.

Soln. : Given analog transfer function is,

$$H(s) = \frac{1}{s+2} \qquad ...(1)$$

For the prewarping procedure we have,

$$\Omega_p^* = \frac{2}{T_s}\tan\left(\frac{\omega T_s}{2}\right)$$

$$\therefore \quad \Omega_p^* = \frac{2}{0.0167}\tan\left(\frac{20\pi \times 0.0167}{2}\right) \qquad ...\text{Here } \omega = 20\pi \text{ (given)}$$

$$\therefore \quad \Omega_p^* = 69.31$$

Now the value of $H^*(s)$ is obtained by putting $s = \frac{s}{\Omega_p^*}$ in Equation (1).

$$\therefore \quad H^*(S) = \frac{1}{\frac{s}{\Omega_p^*}+2}$$

$$\therefore \quad H^*(S) = \frac{1}{\frac{s}{69.31}+2}$$

$$\therefore \quad H^*(S) = \frac{69.31}{s+138.62} \qquad ...(2)$$

Now $H(Z)$ is obtained by putting $s = \frac{2}{T_s}\left(\frac{Z-1}{Z+1}\right)$ in Equation (2),

$$\therefore \quad H(Z) = \frac{69.31}{\frac{2}{0.0167}\left(\frac{Z-1}{Z+1}\right)+138.62} = \frac{69.31}{119.76\left(\frac{Z-1}{Z+1}\right)+138.62}$$

$$\therefore \quad H(Z) = \frac{69.31\,(Z+1)}{119.76\,Z - 119.76 + 138.62\,Z + 138.62}$$

$$\therefore \quad H(Z) = \frac{69.31\,(Z+1)}{258.38\,Z + 18.86}$$

$$\therefore \quad H(Z) = \frac{Z+1}{3.73\,Z+0.27} \qquad \therefore \quad H(Z) = \frac{1+Z^{-1}}{3.73+0.27\,Z^{-1}}$$

This is the required transfer function for digital filter.

1.2.4 Matched Z transform :

In this method, the analog filter is transformed into the digital filter by mapping poles and zeros of H (s) directly into the poles and zeros in Z plane.

Consider that the transfer function of analog filter is given by,

$$H(s) = \frac{\prod_{k=1}^{M} (s - Z_k)}{\prod_{k=1}^{M} (s - P_k)} \quad \ldots(1)$$

Here Z_k represents the zeros of system and P_k represents the poles of system.

In this method; the poles obtained from matched Z transform are identical to poles obtained with impulse invariance method. But the zero positions are different than impulse invariance method.

The transfer function of digital filter can be given by,

$$H(Z) = \frac{\prod_{k=1}^{M} (1 - e^{Z_k T_s} \cdot Z^{-1})}{\prod_{k=1}^{M} (1 - e^{P_k T_s} \cdot Z^{-1})}$$

Here T_s is the sampling time. In this method, to maintain the frequency response characteristics of analog filter normal; the sampling interval (T_s) is kept very low in order to avoid aliasing.

1.2.5 Least squares filter design :

This method is used to design digital IIR filter directly without doing any conversion. Consider that we have to design digital IIR filter which contains only poles. The transfer function of such filter is given by,

$$H(Z) = \frac{b_0}{1 + \sum_{k=1}^{N} a_k Z^{-k}} \quad \ldots(1)$$

The desired transfer function is denoted by $H_d(Z)$. Consider the cascade connection of $H_d(Z)$ with $\frac{1}{H(Z)}$ as shown in Fig. A. Note that $\frac{1}{H(Z)}$ is all zero system.

Consider that input applied to this cascade connection is $\delta(n)$. Actually $y(n) = \delta(n)$; this is because we are cascading a filtering and inverse filtering operation. Here $H(Z)$ is approximation of desired frequency response $H_d(Z)$.

As shown in Fig.K-4 the input applied to inverse filter is $h_d(n)$ and its response is $\frac{1}{H(Z)}$. From Equation (1) we have,

$$\frac{1}{H(Z)} = \frac{1 + \sum_{k=1}^{N} a_k Z^{-k}}{b_0} \qquad ...(2)$$

The actual output is,

$$y(n) = \frac{1}{b_0}\left[h_d(n) + \sum_{k=1}^{N} a_k h_d(n-k) \right] \qquad ...(3)$$

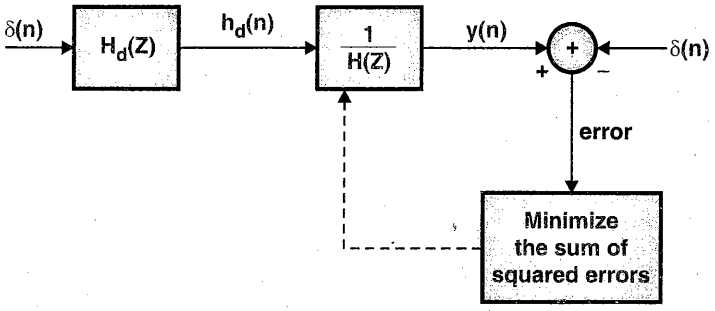


Fig.K-4 : Least square inverse filter design

Here for $n > 0$, $y(n)$ represents the error between the desired output and the actual output. Here the parameter a_k is selected to minimize the sum of squares of error sequence.

$$\in = \sum_{n=1}^{\infty} y^2(n) \qquad ...(4)$$

Putting Equation (3) in Equation (4) we get,

$$\in = \sum_{n=1}^{\infty} \left\{ \frac{1}{b_0}\left[h_d(n) + \sum_{k=1}^{N} a_k h_d(n-k) \right] \right\}^2 \qquad ...(5)$$

By differentiating Equation (5) with respect to a_k results into the set of linear differential equations. By solving these equations the coefficients of desired filter can be obtained.

1.3 Butterworth Filter Approximation :

A typical characteristic of a butterworth low pass filter is as shown in Fig. K-5.

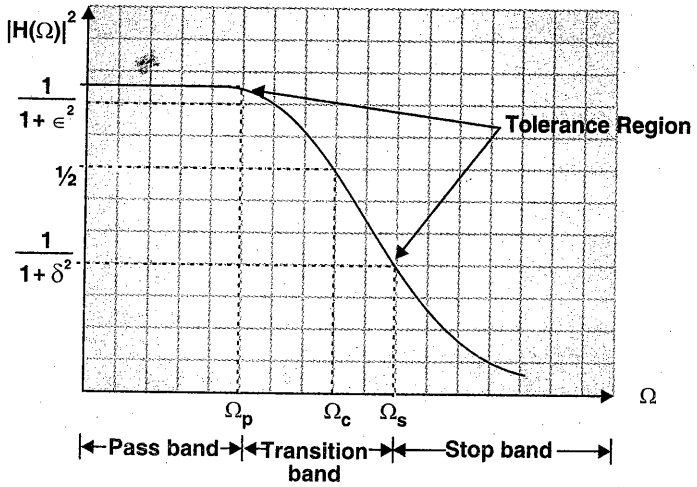


Fig. K-5 : Typical characteristics of analog L.P.F.

This type of response is called as butterworth response because its main characteristic is that the passband is maximally flat. That means there are no variations (ripples) in the passband.

Now the magnitude squared response of low pass butterworth filter is given by,

$$|H(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}} \quad ...(1)$$

This equation is also expressed as,

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}} \quad ..(2)$$

Here $|H(\Omega)|$ = Magnitude of analog low pass filter.

Ω_c = Cut-off frequency (– 3dB frequency).

Ω_p = Pass band edge frequency.

$1+\epsilon^2$ = Pass band edge value.

$1+\delta^2$ = Stop band edge value.

ϵ = Parameter related to ripples in pass band.

δ = Parameter related to ripples in stop band.

N = Order of the filter.

We know that, in case of low pass filter the frequencies will pass upto the value of cut-off frequency (Ω_c). This is called as pass band. After that the frequencies are attenuated. This is called as stop band. Ideal characteristic is shown by dotted line in Fig. K-5. Ideally, at the value of cut-off frequency (Ω_c) the frequencies should be stopped. But in practical cases this is not happening.

Now the order of filter is denoted by 'N'. Roughly we can say order of filter means, the number of stages used in the design of analog filter. As the order of filter 'N' increase's, the response of filter is more close to the ideal response as shown in Fig. K-6.

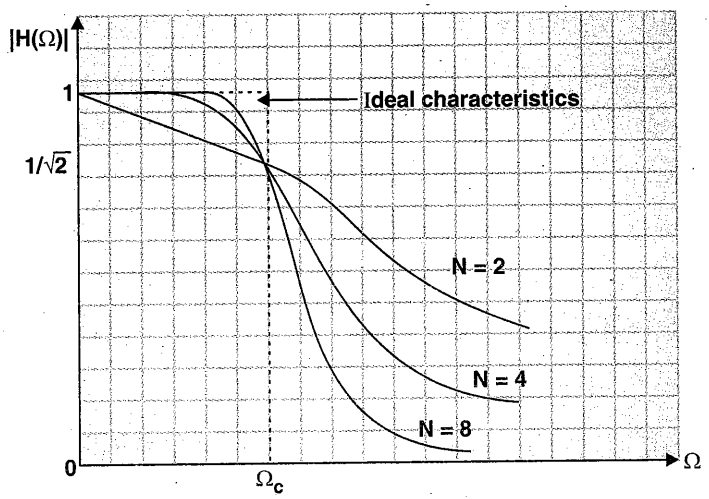


Fig. K-6 : Effect of N on frequency response characteristics

Salient features of low pass butterworth filter :

1. The magnitude response is nearly constant (equal to 1) at lower frequencies. That means pass band is maximally flat.
2. There are no ripples in the pass band and stop band.
3. The maximum gain occurs at $\Omega = 0$ and it is $|H(0)| = 1$.
4. The magnitude response is monotonically decreasing.

Design equations and design steps :

Let A_p = Attenuation in pass band.

A_s = Attenuation in stop band.

Ω_p = Pass band edge frequency.

Ω_c = Cut-off frequency.

Ω_s = Stop band edge frequen

In the problem the specifications of required digital filter will be given and it will be asked to design a particular discrete time butterworth filter. Then the following steps should be used :

Step I : From the given specifications of digital filter ; obtain equivalent analog filter as follows :

(a) For impulse invariance method :

$$\Omega = \frac{\omega}{T_s}$$

(b) For bilinear transformation method :

$$\Omega = \frac{2}{T_s} \tan \frac{\omega}{2}$$

Here Ω = Frequency of analog filter ω = Frequency of digital filter

T_s = Sampling time

Step II : Calculate the order 'N' of filter using the equation,

$$N = \frac{1}{2} \times \frac{\log \left[\frac{\left(\frac{1}{A_s^2} - 1 \right)}{\left(\frac{1}{A_p^2} - 1 \right)} \right]}{\log \left(\frac{\Omega_s}{\Omega_p} \right)}$$

or

$$N = \frac{\log \left[\left(1/\delta_s^2 \right) - 1 \right]}{2 \log \left(\Omega_s / \Omega_c \right)}$$

Here δ_s = Attenuation in stop band.

If the specifications are given in decibels (dB) then use the equation,

$$N = \frac{1}{2} \frac{\log \left[\frac{10^{0.1 A_s (dB)} - 1}{10^{0.1 A_p (dB)} - 1} \right]}{\log \left(\frac{\Omega_s}{\Omega_p} \right)}$$

Step III : Calculation of cut-off frequency (Ω_c) :

The cut-off frequency (Ω_c) of analog filter is calculated as :

(a) For impulse invariance method :

$$\Omega_c = \frac{\omega_c}{T_s}$$

(b) For bilinear transformation method :

$$\Omega_c = \frac{2}{T_s} \tan \frac{\omega_c}{2}$$

When ω_c is not given then use the equation,

(i) $$\Omega_c = \frac{\Omega_p}{\left(\frac{1}{A_p^2} - 1\right)^{1/2N}}$$ and if specifications are in dB then.

(ii) $$\Omega_c = \frac{\Omega_p}{10^{0.1 A_p} - 1}$$

Step IV : Calculate the poles using,

$$P_k = \pm \Omega_c e^{j(N+2k+1)\pi/2N}, \quad k = 0, 1, 2, \ldots N-1.$$

If the poles are complex conjugate then organize the poles (P_k) as complex conjugate pairs that means,

s_1 and s_1^*, s_2 and s_2^* etc.

Step V : Calculate the system transfer function of analog filter using,

$$H(s) = \frac{\Omega_c^N}{(s - P_1)(s - P_2)\ldots}$$

and if poles are complex conjugate then,

$$H(s) = \frac{\Omega_c^N}{(s - s_1)(s - s_1^*)(s - s_2)(s - s_2^*)}$$

Step VI : Design the digital filter using impulse invariance method or bilinear transformation method.

Solved Problems :

Prob. 1 : A digital filter has frequency specification as :

Passband frequency = $\omega_p = 0.2\pi$

Stopband frequency = $\omega_s = 0.3\pi$

What are the corresponding specifications for pass band and stop band frequencies in analog domain if,

(I) Impulse invariance technique is used for designing.

(II) Bilinear transformation is used for designing.

Soln. : Assume sampling time $T_s = 1$.

(I) For impulse invariance method we have,

$$\Omega_p = \frac{\omega_p}{T_s}$$

$$\therefore \quad \Omega_p = 0.2\,\pi = 0.63 \text{ rad / sec.}$$

and $\Omega_s = \dfrac{\omega_s}{T_s}$ $\qquad \therefore \quad \Omega_s = \dfrac{0.3\,\pi}{T_s} = 0.3\,\pi$

$$\therefore \quad \Omega_s = 0.94 \text{ rad/sec.}$$

(II) For bilinear transformation we have,

$$\Omega_p = \frac{2}{T_s} \tan\left(\frac{\omega_p}{2}\right)$$

$$\therefore \quad \Omega_p = 2 \tan\left(\frac{0.2\,\pi}{2}\right) = 2 \tan\left(\frac{0.2 \times 180}{2}\right)$$

$$\therefore \quad \Omega_p = 0.65 \text{ rad / sec.}$$

and $\Omega_s = \dfrac{2}{T_s} \tan\left(\dfrac{\omega_s}{2}\right)$

$$\therefore \quad \omega_s = 2 \tan\left(\frac{0.3\,\pi}{2}\right) = 2 \tan\left(\frac{0.3 \times 180}{2}\right)$$

$$\therefore \quad \Omega_s = 1.019 \text{ rad/sec.}$$

Prob. 2 : Design a second order DT butterworth filter with cut-off frequency of 1 KHz and sampling frequency of 10^4 samples/sec. by bilinear transformation.

Soln. :

In this problem, the specifications of digital filter are not given directly. So first we have to obtain required design specifications for digital filter. Then for butterworth approximation we have to convert these specifications into specifications of equivalent analog filter. Finally using bilinear transformation we have to obtain H (Z).

Given specifications of analog filter :

Order of filter, N = 2

Cut-off frequency of analog filter, F_c = 1 KHz = 1000 Hz.

Sampling frequency, $F_s = 10^4$ samples/sec. = 10,000 Hz.

Part A : Calculation of the required design specification of digital filter :

We have the equation to convert continuous frequency (F) into discrete frequency (f). It is,

$$f = \frac{F}{F_s} \qquad \therefore \quad f_c = \frac{F_c}{F_s}$$

$$\therefore \quad f_c = \frac{1000}{10{,}000} = 0.1 \text{ cycles/sample.}$$

Now the angular frequency (frequency of digital filter) is,

$$\omega_c = 2\pi f_c = 2\pi \times 0.1 \qquad \therefore \quad \omega_c = 0.2\pi \text{ radians/sample.}$$

Part B : Calculation of specifications of analog filter for butterworth approximation :

Step I : For bilinear transformation we have,

$$\Omega = \frac{2}{T_s} \tan \frac{\omega}{2} \qquad \therefore \quad \Omega_c = \frac{2}{T_s} \tan \frac{\omega_c}{2}$$

Here T_s = sampling time $= \frac{1}{F_s} = \frac{1}{10{,}000}$

$$\therefore \quad \Omega_c = (2 \times 10{,}000) \tan\left(\frac{0.2\pi}{2}\right)$$

$$\therefore \quad \Omega_c = 6498.39 \text{ radians/sec.}$$

The value of 'N' is given so it is not necessary to calculate it. Thus the specifications of analog filter are,

(i) Cut-off frequency = Ω_c = 6498.39 radians/sec.

(ii) Order of filter = N = 2.

Step II : Calculate the poles using,

$$P_k = \pm \Omega_c e^{j(N+2k+1)\pi/2N} \quad \ldots k = 0, 1, \ldots N-1$$

Here N = 2. Thus k = 0 to N – 1 means k = 0 and k = 1.

(i) For k = 0 :

$$P_0 = \pm \Omega_c e^{j(N+1)\pi/2N} = \pm 6498.39\, e^{j3\pi/4}$$

$$= \pm 6498.39\left[\cos\left(\frac{3\pi}{4}\right) + j \sin\left(\frac{3\pi}{4}\right)\right]$$

$$= \pm 6498.39\,[-0.707 + j\,0.707]$$

$$\therefore \quad P_0 = 6498.39\,[-0.707 + j\,0.707] \text{ and } -6498.39\,[-0.707 + j\,0.707]$$

$$\therefore \quad P_0 = -4595.05 + j\,4595.05 \text{ and } 4595.05 - j\,4595.05$$

(ii) For k = 1 :

$$P_1 = \pm \Omega_c e^{j(N+2+1)\pi/2N}$$

$$\therefore \quad P_1 = \pm 6498.39\, e^{j(5\pi/4)}$$

$$\therefore \quad P_1 = \pm 6498.39\left[\cos\left(\frac{5\pi}{4}\right) + j\sin\left(\frac{5\pi}{4}\right)\right]$$

$$\therefore \quad P_1 = \pm 6498.39\,[-0.707 - j\,0.707]$$

$$\therefore \quad P_1 = 6498.39\,[-0.707 - j\,0.707] \text{ and } -6498.39\,[-0.707 - j\,0.707]$$

$$\therefore \quad P_1 = -4595.05 - j\,4595.05 \text{ and } 4595.05 + j\,4595.05$$

Now we will plot these poles as shown in Fig. K-7.

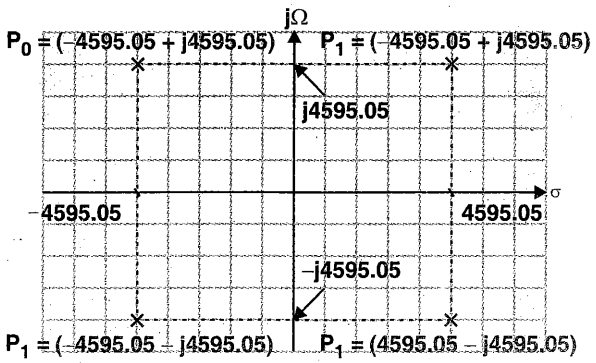


Fig. K-7 : Plot of poles, P_k

Important Note : We know that analog filter is stable if the poles lie on the L.H.S. of s-plane. So for the filter to be stable; we have to select the poles which are on the L.H.S. of s-plane. Observe Fig. k-7. We will denote these poles by s_1 and s_1^*, because these poles are complex conjugate of each other.

$$\therefore \quad s_1 = -4595.05 + j\,4595.05$$

$$\text{and} \quad s_1^* = -4595.05 - j\,4595.05$$

Step III : The transfer function of analog filter is obtained by using the equation.

$$H(s) = \frac{\Omega_c^N}{(s - s_1)(s - s_1^*)}$$

$$\therefore \quad H(s) = \frac{(6498.39)^2}{(s + 4595.05 - j\,4595.05)(s + 4595.05 + j\,4595.05)}$$

$$\therefore H(s) = \frac{(6498.39)^2}{s^2 + 4595.05s + js\,4595.05 + 4595.05s + (21.11 \times 10^6) + j\,(21.11 \times 10^6) - js\,4505.05 - j\,(21.11 \times 10^6) + (21.11 \times 10^6)}$$

$$\therefore \quad H(s) = \frac{(6498.39)^2}{s^2 + 9190.1\,s + 42.22 \times 10^6}$$

Part C : Designing digital filter using bilinear transformation :

In case of bilinear transformation. $H(Z)$ is obtained by putting $s = \frac{2}{T_s}\left[\frac{Z-1}{Z+1}\right]$ in the equation of $H(s)$

Here $T_s = \frac{1}{10,000}$. Thus we have to put,

$$s = 2\times 10^{+4}\left[\frac{Z-1}{Z+1}\right] \text{ in the equation of } H(S)$$

$$\therefore \quad H(Z) = \frac{(6498.39)^2}{\left[2\times 10^4\left(\frac{Z-1}{Z+1}\right)\right]^2 + 183.802\times 10^6\left(\frac{Z-1}{Z+1}\right) + 42.22\times 10^6}$$

This is the required transfer function for digital filter.

Prob. 3 : Using bilinear transformation, design a butterworth filter which satisfies the following conditions :

$0.8 \le |H(e^{j\omega})| \le 1 \qquad 0 \le \omega \le 0.2\pi$

$|H(e^{j\omega})| \le 0.2 \qquad 0.6\pi \le \omega \le \pi$

Soln. :

$|H(e^{j\omega})|$ is same as $|H(\omega)|$; so the specifications are given for the required digital filter. These specifications are shown in Fig. K-8.

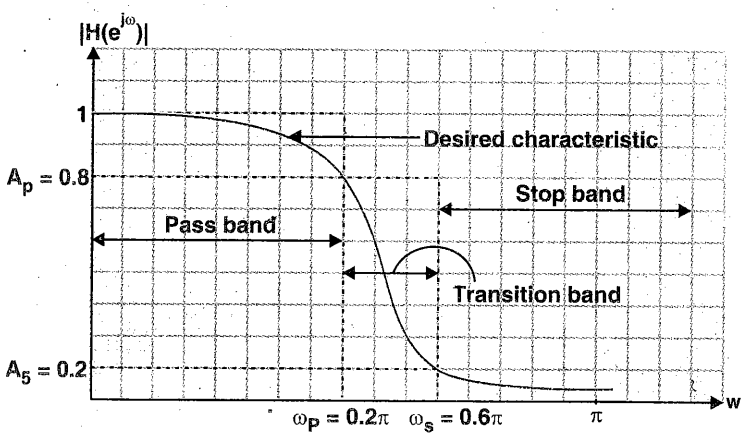


Fig. K-8 : Characteristic plot for digital filter

Given specifications of digital filter are :

$A_p = 0.8, \quad \omega_p = 0.2\pi$

$A_s = 0.2, \quad \omega_s = 0.6\pi$

Part A : Calculating specifications of analog filter :

Step I : We will obtain the values of Ω_p and Ω_s. We have to use bilinear transformation method. Thus for bilinear transformation we have,

$$\Omega = \frac{2}{T_s} \tan \frac{\omega}{2}$$

$$\therefore \quad \Omega_p = \frac{2}{T_s} \tan \frac{\omega_p}{2}$$

The value of sampling time T_s is not given. So assume $T_s = 1$.

$$\therefore \quad \Omega_p = 2 \tan \frac{0.2\,\pi}{2}$$

$$\therefore \quad \Omega_p = 0.65$$

$$\text{Similarly,} \quad \Omega_s = \frac{2}{T_s} \tan \frac{\omega_s}{2} = 2 \tan \frac{0.6\,\pi}{2}$$

$$\therefore \quad \Omega_s = 2.75$$

Thus specifications of analog filter are :

$A_p = 0.8, \qquad \Omega_p = 0.65$

$A_s = 0.2, \qquad \Omega_s = 2.75$

Step II : Calculation of order of filter, N

It is given by,

$$N = \frac{1}{2} \times \frac{\log \left[\left(\frac{1}{A_s^2} - 1 \right) \Big/ \left(\frac{1}{A_p^2} - 1 \right) \right]}{\log \left(\frac{\Omega_s}{\Omega_p} \right)}$$

$$\therefore \quad N = \frac{1}{2} \times \frac{\log \left[\left(\frac{1}{(0.2)^2} - 1 \right) \Big/ \left(\frac{1}{(0.8)^2} - 1 \right) \right]}{\log \left(\frac{2.75}{0.65} \right)}$$

$$\therefore \quad N = \frac{1}{2} \times \frac{\log \left[\frac{24}{0.5625} \right]}{\log \left(\frac{2.75}{0.65} \right)}$$

$$\therefore \quad N = 1.3$$

Thus approximately we can take N = 2.

Step III : Calculation of cut-off frequency, Ω_c.

ω_c is not given so use the equation,

$$\Omega_c = \frac{\Omega_p}{\left(\frac{1}{A_p^2} - 1\right)^{1/2N}}$$

$$\therefore \quad \Omega_c = \frac{0.65}{\left[\frac{1}{(0.8)^2} - 1\right]^{\frac{1}{4}}}$$

$$\therefore \quad \Omega_c = 0.75$$

Step IV : Calculations of the poles of **H (s)** :

The poles of H (s) are given by,

$$P_k = \pm \Omega_c e^{j(N+2k+1)\pi/2N} \quad , k = 0, 1, 2, ... N-1$$

$$\therefore \quad P_k = \pm 0.75 \; e^{j(2+2k+1)\pi/4}$$

$$\therefore \quad P_k = \pm 0.75 \, e^{j(3+2k)\pi/4}$$

Here range of k is from 0 to N – 1 and N = 2. Thus the value of k is, k = 0 and k = 1.

For k = 0 :

$$P_0 = \pm 0.75 \, e^{j(3)\pi/4} = \pm 0.75 \left[\cos\left(\frac{3\pi}{4}\right) + j \sin\left(\frac{3\pi}{4}\right)\right]$$

$$\therefore \quad P_0 = \pm 0.75 \, [-0.707 + j\, 0.707 \,]$$

$$\therefore \quad P_0 = \pm [-0.53 + j\, 0.53 \,]$$

$$\therefore \quad P_0 = -0.53 + j\, 0.53 \quad \text{and} \quad 0.53 - j\, 0.53$$

For k = 1 :

$$P_1 = \pm 0.75 \, e^{j(3+2)\pi/4} = \pm 0.75 \, e^{j\,5\pi/4}$$

$$\therefore \quad P_1 = \pm 0.75 \left[\cos\frac{5\pi}{4} + j \sin\frac{5\pi}{4} \right]$$

$$\therefore \quad P_1 = \pm 0.75 \, [-0.707 - j\, 0.707 \,]$$

$$\therefore \quad P_1 = -0.53 - j\, 0.53 \text{ and } 0.53 + j\, 0.53$$

The location of poles are as shown in Fig. K-9.

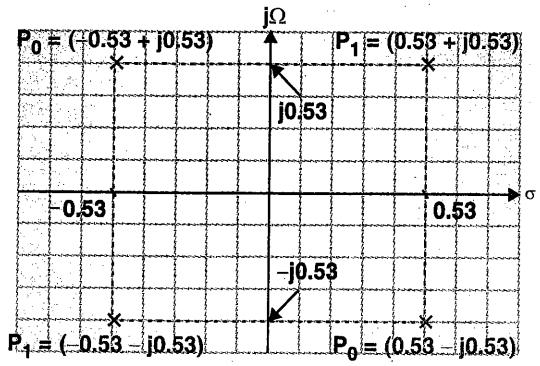


Fig. K-9 : Location of poles

We know that, for the stability; the poles which are at the L.H.S. of s-plane should be selected. Thus we will select the poles $-0.53 + j\,0.53$ and $-0.53 - j\,0.53$. Note that these poles are complex conjugate of each other. We will denote the poles by,

$$s_1 = -0.53 + j\,0.53 \quad \text{and} \quad s_1^* = -0.53 - j\,0.53$$

Step V : Calculation of **H (s)** :

The transfer function of analog filter is given by,

$$H(s) = \frac{\Omega_c^N}{(s - s_1)(s - s_1^*)}$$

$$\therefore \quad H(s) = \frac{(0.75)^2}{(s + 0.53 - j\,0.53)(s + 0.53 + j\,0.53)}$$

$$\therefore \quad H(s) = \frac{(0.75)^2}{s^2 + 0.53\,s + j\,0.53\,s + 0.53\,s + 0.28 + j\,0.28 - j\,0.53\,s - j\,0.28 + 0.28}$$

$$\therefore \quad H(s) = \frac{(0.75)^2}{s^2 + 1.06\,s + 0.56}$$

$$\therefore \quad H(s) = \frac{0.56}{s^2 + 1.06\,s + 0.56}$$

Part B : Obtaining transfer function for digital filter :

We have to use bilinear transformation method. Then H (Z) is obtained by putting $s = \frac{2}{T_s}\left(\frac{Z-1}{Z+1}\right)$ in the equation of H (s).

We have assumed sampling time $T_s = 1$.

$$\therefore \quad H(Z) = \frac{0.56}{4\left(\frac{Z-1}{Z+1}\right)^2 + (2\times 1.06)\left(\frac{Z-1}{Z+1}\right) + 0.56}$$

$$\therefore \quad H(Z) = \frac{0.56}{\frac{4(Z-1)^2}{(Z+1)^2} + \frac{2.12Z-2.12}{Z+1} + 0.56}$$

$$\therefore \quad H(Z) = \frac{0.56(Z+1)^2}{4(Z^2-2Z+1)+(2.12Z-2.12)(Z+1)+0.56(Z+1)^2}$$

$$\therefore \quad H(Z) = \frac{0.56(Z+1)^2}{4Z^2-8Z+4+2.12Z^2+2.12Z-2.12Z-2.12+0.56Z^2+1.12Z+0.56}$$

$$\therefore \quad H(Z) = \frac{0.56(Z+1)^2}{6.68Z^2-6.88Z+2.44}$$

This is the required transfer function to design the digital filter.

Prob. 4 : (a) Find the order and cut-off frequency of a digital filter with following specifications :

$0.89 \le |H(e^{j\omega})| \le 1, \qquad 0 \le \omega \le 0.4\pi$

$|H(e^{j\omega})| \le 0.18, \qquad 0.6\pi \le \omega \le \pi$

Use impulse invariance method.

(b) Draw its poles.

Soln. :

(a) The given specifications are :

$$0.89 \le |H(e^{j\omega})| \le 1, \qquad 0 \le \omega \le 0.4\pi$$

$$|H(e^{j\omega})| \le 0.18, \qquad 0.6\pi \le \omega \le \pi$$

Thus,

$A_p = 0.89, \qquad \omega_p = 0.4\pi$

$A_s = 0.18, \qquad \omega_s = 0.6\pi$

Calculation of order of filter :

Order of filter is given by,

$$N = \frac{1}{2}\frac{\log\left[\frac{\left(\frac{1}{(A_s)^2}-1\right)}{\left(\frac{1}{(Ap^2)}-1\right)}\right]}{\log\left(\frac{\Omega_s}{\Omega_p}\right)} \qquad ...(1)$$

First we will obtain values of Ω_s and Ω_p.

For impulse invariance method,

$$\Omega_p = \frac{\omega_p}{T_s} \quad \text{and} \quad \Omega_s = \frac{\omega_s}{T_s}.$$

Here sampling time T_s is not given. So assume $T_s = 1$ sec.

$\therefore \Omega_p = \omega_p = 0.4\,\pi$ and $\Omega_s = \omega_s = 0.6\,\pi$.

Putting these values in Equation (1) we get,

$$N = \frac{1}{2} \cdot \frac{\log\left[\dfrac{\dfrac{1}{(0.18)^2} - 1}{\dfrac{1}{(0.89)^2} - 1}\right]}{\log\left(\dfrac{0.6\,\pi}{0.4\,\pi}\right)}$$

$$\therefore \quad N = \frac{1}{2} \cdot \frac{2.056}{0.176} = 5.83$$

Thus order of filter = N = 6

Calculation of cut-off frequency :

The cut-off frequency is given by,

$$\Omega_c = \frac{\Omega_p}{\left(\dfrac{1}{A_p^2} - 1\right)^{\frac{1}{2N}}}$$

$$\therefore \quad \Omega_c = \frac{0.4\,\pi}{\left[\dfrac{1}{(0.89)^2} - 1\right]^{\frac{1}{2\times 6}}}$$

$$\therefore \ \Omega_c = 1.405$$

(b) The poles are calculated using,

$$P_k = \pm \Omega_c\, e^{j(N+2k+1)\pi/2N}, \quad k = 0, 1 \ldots N-1.$$

We have N = 6. Thus k varies from 0 to 5.

For k = 0 :

$$P_0 = \pm 1.405\, e^{j(6+0+1)\frac{\pi}{12}} = \pm 1.405\, e^{j\frac{7\pi}{12}}$$

$$\therefore \quad P_0 = \pm 1.405\left[\cos\frac{7\pi}{12} + i \sin\frac{7\pi}{12}\right] = \pm 1.405\,[-0.26 + j0.97]$$

$$\therefore \quad P_0 = -0.37 + j\,1.37,\ \ 0.37 - j\,1.37$$

For k = 1 :

$$P_1 = \pm 1.405\, e^{j(6+2+1)\frac{\pi}{12}} = \pm 1.405\, e^{\frac{j9\pi}{12}}$$

$$= \pm 1.405\left[\cos\frac{9\pi}{12} + j\sin\frac{9\pi}{12}\right] = \pm 1.405\ [\ -0.707 + j\sin 707\ ]$$

$$\therefore \quad P_1 = -1 + j1,\ 1 - j1$$

For k = 2 :

$$P_2 = \pm 1.405\, e^{j(6+4+1)\frac{\pi}{12}} = \pm 1.405\, e^{j\frac{11\pi}{12}}$$

$$= \pm 1.405\left[\cos\frac{11\pi}{12} + j\sin\frac{11\pi}{12}\right] = \pm 1.405\ [\ -0.97 + 0.26\ ]$$

$$\therefore \quad P_2 = -1.36 + j\,0.36,\ \ 1.36 - j\,0.36$$

For k = 3 :

$$P_3 = \pm 1.405\, e^{j(6+6+1)\frac{\pi}{12}} = \pm 1.405\, e^{j\frac{13\pi}{12}}$$

$$= \pm 1.405\left[\cos\frac{13\pi}{12} + j\sin\frac{13\pi}{12}\right] = \pm 1.405\ [\ -0.97 - j\,0.26\ ]$$

$$\therefore \quad P_3 = -1.36 - j\,0.3.6,\ 1.36 + j\,0.36$$

For k = 4 :

$$P_4 = \pm 1.405\, e^{j(6+8+1)\frac{\pi}{12}} = \pm 1.405\, e^{\frac{15\pi}{12}}$$

$$= \pm 1.405\left[\cos\frac{15\pi}{12} + j\sin\frac{15\pi}{12}\right] = \pm 1.405\ [\ -707 - j\,0.707\ ]$$

$$\therefore \quad P_4 = -1 - j\,1,\ 1 + j1$$

For k = 5 :

$$P_5 = \pm 1.405\, e^{j(6+10+1)\frac{\pi}{12}} = \pm 1.405\, e^{j\frac{17\pi}{12}}$$

$$= \pm 1.405\left[\cos\frac{17\pi}{12} + j\sin\frac{17\pi}{12}\right] = \pm 1.405\ [\ -0.26 - 0.97\ ]$$

$$\therefore \quad P_s = -0.37 - j\,1.37,\ 0.37 + j\,1.37$$

The plot of poles is shown in Fig. K-10.

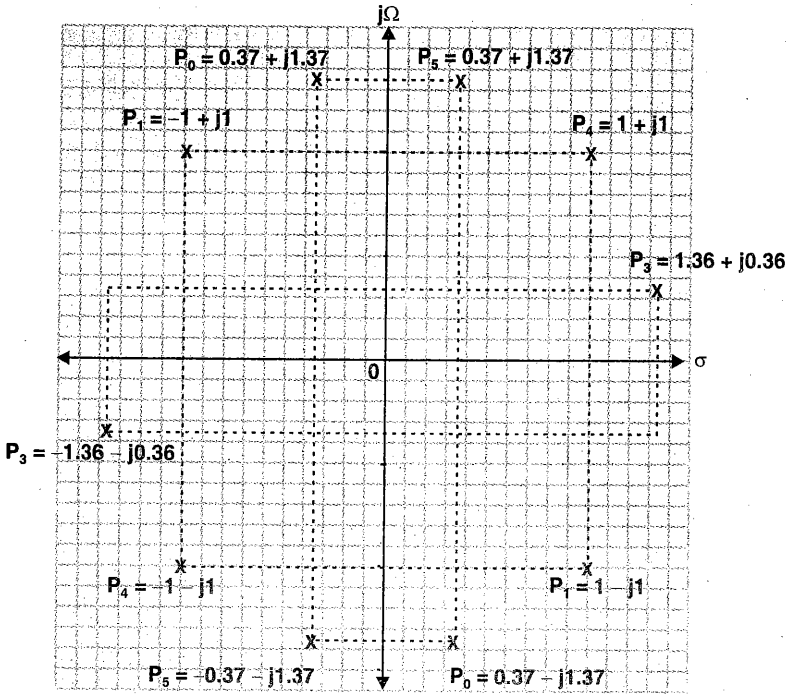


Fig. K-10

Prob. 5 : A digital low pass IIR filter is to be designed with butter- worth approximation using bilinear transformation technique. Find the order of filter to meet the following specifications :

(i) Passband magnitude is constant within 1 dB for frequencies below 0.2 π.

(ii) Stopband attenuation is greater than 15 dB for frequencies between 0.3 π to π.

Soln. : The given specifications for the desired digital IIR filter are :

$$A_p = 1 \text{ dB}, \qquad \omega_p = 0.2\,\pi$$

and $$A_s = 15 \text{ dB}, \qquad \omega_s = 0.3\,\pi$$

Part A : Calculation of specifications for analog filter :

For the bilinear transformation we have,

$$\Omega = \frac{2}{T_s} \tan \frac{\omega}{2}$$

The values of T_s is not given, so assume $T_s = 1$.

$$\therefore \quad \Omega = 2 \tan \frac{\omega}{2}$$

Thus we can write,

$$\Omega_p = 2 \tan \frac{\omega_p}{2} = 2 \tan \frac{0.2\pi}{2}$$

$$\therefore \quad \Omega_p = 0.65$$

$$\text{Similarly, } \Omega_s = 2 \tan \frac{\omega_s}{2} = 2 \tan \frac{0.3\pi}{2}$$

$$\therefore \quad \Omega_s = 1.02$$

Thus specifications of analog filter are :

$$A_p = 1 \text{ dB}, \qquad \Omega_p = 0.65$$

$$\text{and } A_s = 15 \text{ dB}, \qquad \Omega_s = 1.02$$

Part B : Calculation of the order of filter (N) :

The specifications are given in dB; so we have to use the formula,

$$N = \frac{1}{2} \frac{\log \left[\frac{10^{0.1 A_s \text{ dB}} - 1}{10^{0.1 A_p \text{ dB}} - 1} \right]}{\log \left(\frac{\Omega_s}{\Omega_p} \right)}$$

$$\therefore \quad N = \frac{1}{2} \frac{\log \left[\frac{10^{(0.1 \times 15)} - 1}{10^{(0.1 \times 1)} - 1} \right]}{\log \left(\frac{1.02}{0.65} \right)}$$

$$\therefore \quad N = \frac{1}{2} \times \frac{2.073}{0.1957}$$

$$\therefore \quad N = 5.296$$

Thus the order of filter is N = 6.

Prob. 6 : An IIR low pass filter is required to meet the following specifications :

Passband peak to peak ripple	:	≤ 1 dB
Stopband edge	:	1.2 kHz
Stopband attenuation	:	≥ 40 dB
Stopband edge	:	2.5 kHz
Sample rate	:	8 kHz

The filter is to be designed by performing BLT on an analog system function of required order butterworth filter so as to meet the specifications in the implementation.

Soln. : Given specifications are,

Attenuation in passband (peak to peak ripple) = $A_p = 1$ dB

Attenuation in stopband = $A_s = 40$ dB

Passband edge frequency = F_{PB} = 1.2 kHz

Stopband edge frequency = F_{SB} = 2.5 kHz

Sampling frequency = F_s = 8 kHz

Part A : Calculation of the required design specifications of digital filter :

We have, $f_{PB} = \frac{F_{PB}}{F_s} = \frac{1.2\text{ kHz}}{8\text{ kHz}} = 0.15$

$\therefore \omega_p = 2\pi f_{PB} = 2\pi \times 0.15 = 0.94$

Similarly $f_{SB} = \frac{F_{SB}}{F_s} = \frac{2.5\text{ kHz}}{8\text{ kHz}} = 0.31$

$\therefore \omega_s = 2\pi f_{SB} = 2\pi \times 0.31 = 1.96$

Thus specifications of digital filter are,

A_p = 1 dB, $\omega_p = 0.94$

A_s = 40 dB, $\omega_s = 1.96$

Part B : Calculation of specifications of analog filter for butterworth approximation :

Step I : For BLT we have,

$$\Omega = \frac{2}{T_s}\tan\frac{\omega}{2} \qquad \therefore \Omega_p = \frac{2}{T_s}\tan\frac{\omega_p}{2}$$

Here $T_s = \frac{1}{F_s} = \frac{1}{8}$

$$\therefore \Omega_p = 16\tan\frac{0.94}{2} \qquad \therefore \Omega_p = 8.13$$

and $\Omega_s = \frac{2}{T_s}\tan\frac{\omega_s}{2}$

$$\therefore \Omega_s = (2\times 8)\tan\frac{1.96}{2} \qquad \therefore \Omega_s = 23.85$$

Thus specification of analog filter are,

A_p = 1 dB, Ω_p = 8.13

A_s = 40 dB, Ω_s = 23.85

Step II : Calculation of order of filter :

The order of filter is given by,

$$N = \frac{1}{2}\frac{\log\left[\frac{10^{0.1A_s}-1}{10^{0.1A_p}-1}\right]}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$\therefore\ N = \frac{1}{2}\frac{\log\left[\dfrac{10^{0.1(40)}-1}{10^{0.1(1)}-1}\right]}{\log\left(\dfrac{23.85}{8.13}\right)} \qquad \therefore\ N = \frac{1}{2}\times\frac{4.586}{0.467}$$

$$\therefore\ N = 4.9 \approx 5$$

Step III : Calculation of cut-off frequency :

The cut-off frequency (Ω_c) is,

$$\Omega_c = \frac{\Omega_p}{10^{0.1\,A_p}-1} \qquad \Omega_c = \frac{8.13}{10^{0.1(1)}-1}$$

$$\therefore\ \Omega_c = 31.4$$

Step IV : Calculation of poles of H(s) :

The poles of H (s) are given by,

$$P_k = \pm\Omega_c\, e^{j(N+2k+1)\pi/2N} \qquad ...k = 0, 1, 2, ... N-1$$

Here $N = 5$

$$\therefore\ P_k = \pm 31.4\, e^{j(6+2k)\pi/10} \qquad ...k = 0, 1, 2, 3, 4$$

For k = 0 :

$$P_0 = \pm 31.4\, e^{j\frac{6\pi}{10}} = \pm 31.4\, e^{j\,1.88}$$

$$\therefore\ P_0 = \pm 31.4\,[\cos 1.88 + j \sin 1.88]$$

$$\therefore\ P_0 = \pm 31.4\,[-0.3 + j\,0.95]$$

$$\therefore\ P_0 = -9.42 + j\,29.83 \text{ and } 9.42 - j\,29.83$$

For k = 1 :

$$P_1 = \pm 31.4\, e^{j(5+2+1)\pi/10} = \pm 31.4\, e^{j\frac{8\pi}{10}}$$

$$\therefore\ P_1 = \pm 31.4\left[\cos\frac{8\pi}{10} + j \sin\frac{8\pi}{10}\right]$$

$$\therefore\ P_1 = \pm 31.4\,[-0.8 + j\,0.59]$$

$$\therefore\ P_1 = -25.12 + j\,18.5 \text{ and } 25.12 - j\,18.5$$

For k = 2 :

$$P_2 = \pm 31.4\, e^{j(5+4+1)\pi/10} = \pm 31.4\, e^{j\pi}$$

$$\therefore \quad P_2 = \pm 31.4\,[\cos\pi + j\sin\pi]$$

$$\therefore \quad P_2 = \pm 31.4\,[-1 + j\,0] = \pm 31.4\,[-1]$$

$$\therefore \quad P_2 = 31.4 + j\,0 \text{ and } -31.4 + j\,0$$

For k = 3 :

$$P_3 = \pm 31.4\, e^{j(5+6+1)\pi/10} = \pm 31.4\, e^{12\pi/10}$$

$$\therefore \quad P_3 = \pm 31.4\left[\cos\left(\frac{12\pi}{10}\right) + j\sin\left(\frac{12\pi}{10}\right)\right]$$

$$\therefore \quad P_3 = \pm 31.4\,[-0.8 - j\,0.59]$$

$$\therefore \quad P_3 = -25.12 - j\,18.52 \text{ and } 25.12 + j\,18.52$$

For k = 4 :

$$P_4 = \pm 31.4\, e^{j(5+8+1)\pi/10} = \pm 31.4\, e^{\frac{14\pi}{10}}$$

$$\therefore \quad P_4 = \pm 31.4\left[\cos\frac{14\pi}{10} + j\sin\frac{14\pi}{10}\right]$$

$$\therefore \quad P_4 = \pm 31.4\,[-0.3 - j\,0.95]$$

$$\therefore \quad P_4 = -9.42 - j\,29.83 \text{ and } 9.42 + j\,29.83$$

These poles are plotted as shown in Fig. K-11.

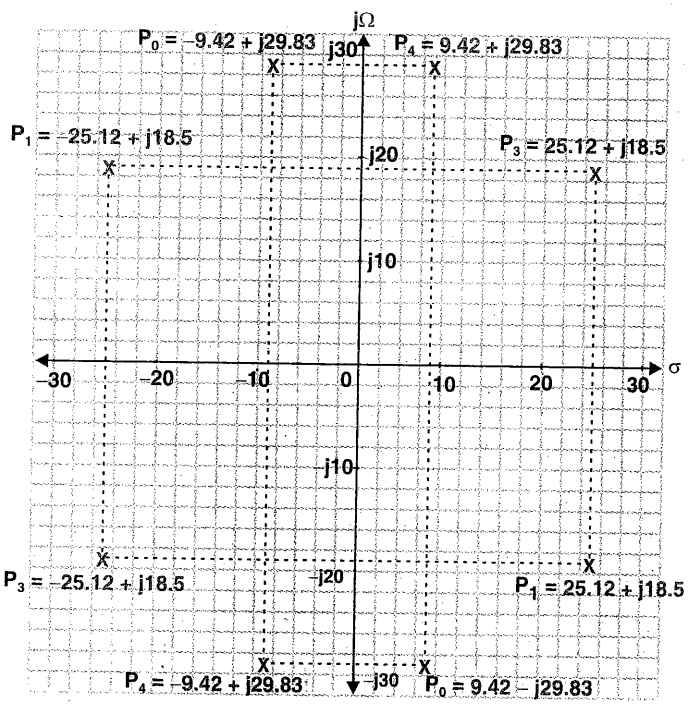


Fig. K-11 : Poles of H (s)

We know that for the stability of filter; we have to select the poles which are on the L.H.S. of s-plane. So we will select the following poles :

$$s_1 = -9.42 + j\,29.83 \qquad s_1^* = -9.42 - j\,29.83$$

$$s_2 = -25.12 + j\,18.5 \qquad s_2^* = -25.12 - j\,18.5$$

$$\text{and } s_3 = -31.4 + j\,0$$

Step V : Calculation of H(s) :

The transfer function of analog filter is given by,

$$H(s) = \frac{\Omega_c^N}{(s-s_1)(s-s_1^*)(s-s_2)(s-s_2^*)(s-s_3)}$$

$$\therefore H(s) = \frac{(31.4)^5}{(s+9.42-j\,29.83)(s+9.42+j\,29.83)(s+25.12-j\,18.5)(s+25.12+j\,18.5)(s+31.4)}$$

$$\therefore H(s) = \frac{(31.4)^5}{(s^2+18.84\,s+978.5)(s^2+50.24\,s+973.26)(s+31.4)}$$

Part C : Obtaining transfer function for digital filter :

The transfer function of digital filter is obtained by putting,

$$s = \frac{2}{T_s}\left[\frac{Z-1}{Z+1}\right] \qquad \text{Here } T_s = \frac{1}{F_s} = \frac{1}{8}$$

$$\therefore s = 16\left[\frac{Z-1}{Z+1}\right]$$

Thus we get,

$$H(Z) = \frac{(31.4)^5}{\left[256\left(\frac{Z-1}{Z+1}\right)^2 + 301.5\left(\frac{Z-1}{Z+1}\right) + 978.5\right]\left[256\left(\frac{Z-1}{Z+1}\right)^2 + 803.84\left(\frac{Z-1}{Z+1}\right) + 973.26\right]\left[16\left(\frac{Z-1}{Z+1}\right) + 31.4\right]}$$

1.4 Chebyshev Filter Approximation :

Using chebyshev filter design, there are two subgroups,

1. Type-1 chebyshev filter
2. Type-2 chebyshev filter

1. Type-1 Chebyshev filter :

These filters are all pole filters. In the passband, these filters show equiripple behaviour and they have monotonic characteristics in the stopband.

The filter characteristics for odd and even values of N are shown in Fig. K-12.

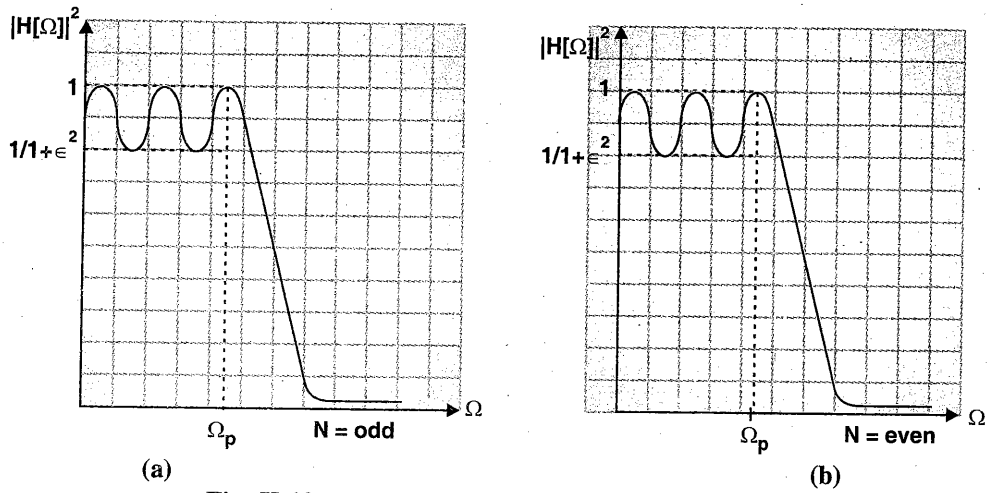


Fig. K-12 : Type-1 chebyshev filter characteristics

The magnitude squared frequency response is given by,

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2 C_N^2(\Omega/\Omega_p)} \quad ...(1)$$

Here ϵ = Ripple parameter in the passband

Ω_p = Passband frequency

$C_N(x)$ = Chebyshev polynomial of order N.

The chebyshev polynomials are determined by using the equation,

$$C_{N+1}(x) = 2x\,C_N(x) - C_{N-1}(x) \quad ...(2)$$

with $C_0(x) = 1$ and $C_1(x) = x$

The different polynomials are given in Table K-1.

Table K-1

N	$C_N(x)$
0	1
1	x
2	$2x^2 - 1$
3	$4x^3 - 3x$
4	$8x^4 - 8x^2 + 1$
5	$16x^5 - 20x^3 + 5x$

The major difference between butterworth and chebyshev filter is that the poles of butterworth filter lie on the circle, while the poles of chebyshev filter lie on ellipse.

2. Type-2 Chebyshev filter :

This filter contains zeros as well as poles. The characteristic of such filters for even and odd values of N is shown in Fig. K-13.

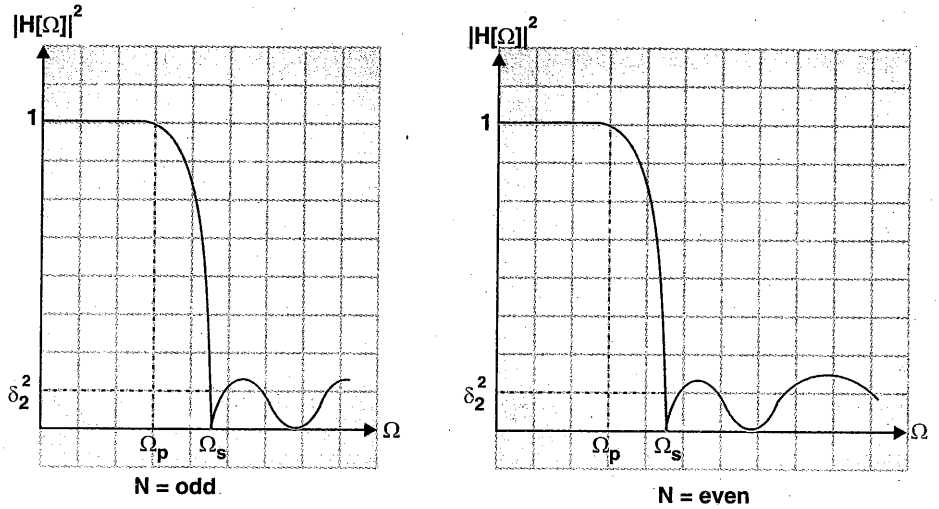


Fig. K-13 : Type-2 chebyshev filter characteristic

The magnitude squared response is given by,

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2\left[C_N^2\left(\frac{\Omega_s}{\Omega_p}\right)\Big/C_N^2\left(\frac{\Omega_s}{\Omega}\right)\right]}$$

Here $C_N(x)$ = N^{th} order polynomial

Ω_s = Stopband frequency

Ω_p = Passband frequency

Chebyshev low pass filter design :

(1) Frequency response :

The magnitude squared frequency response of chebyshev filter is given by,

$$|H(j\Omega)|^2 = \frac{1}{1+\epsilon^2 C_N^2\left(\frac{\Omega}{\Omega_p}\right)}$$

Here $C_N\left(\frac{\Omega}{\Omega_p}\right)$ = Chebyshev polynomial of order M.

(2) Parameter ϵ :

It represents ripple parameter in the passband. It is given by,

$$\epsilon = \left[10^{0.1 A_p \text{ dB}} - 1 \right]^{1/2}$$

If A_p is not in dB then ϵ is calculated using the equation,

$$\epsilon = \left[\frac{1}{A_p^2} - 1 \right]^{1/2}$$

(3) At cut-off frequency $\Omega_c = 1$, the magnitude is given by,

$$|H(j\Omega)| = \frac{1}{\sqrt{1+\epsilon^2}}$$

(4) Order of a filter :

When magnitude is expressed in dB then order (N) of a filter is obtained by using the equation.

$$|H(j\Omega)| \text{ in dB} = -20 \log_{10} \epsilon \; -6(N-1) \Rightarrow -20 N \log \Omega_s$$

(5) Position of poles of chebyshev filter :

First calculate regular position of poles of butterworth filter using

$$\theta_k = \frac{\pi}{2} + \frac{(2k+1)\pi}{2N}, \quad k = 0, 1, 2, \ldots N-1$$

The position of poles of chebyshev filter lie on ellipse at co-ordinates x_k and y_k given by,

$$x_k = r \cos \theta_k, \quad k = 0, 1, 2, \ldots N-1$$

$$y_k = R \sin \theta_k, \quad k = 0, 1, 2, \ldots N-1$$

Here r represents minor axis of ellipse and is given by,

$$r = \Omega_p \frac{\beta^2 - 1}{2\beta}$$

and R represents major axis of ellipse and is given by,

$$R = \Omega_p \frac{\beta^2 + 1}{2\beta}$$

Here the parameter β is given by,

$$\beta = \left[\frac{\sqrt{1+\epsilon^2} + 1}{\epsilon} \right]^{\frac{1}{N}}$$

Then the pole positions are denoted by s_p and,

$$s_p = r \cos \theta_k + jR \sin \theta_k$$

(6) System transfer function :

The system transfer function of analog filter is given by,

$$H(s) = \frac{k}{(s-s_0)(s-s_1)(s-s_2)\ldots}$$

After simplification this equation can be written as,

$$H(s) = \frac{k}{s^N + b_{N-1}s^{N-1} + \ldots + b_0}$$

Here b_0 = Constant term in the denominator

Now the value of k is calculated as follows

$$k = \begin{cases} b_0 & \text{for 'N' odd} \\ \dfrac{b_0}{\sqrt{1+\epsilon^2}} & \text{for 'N' even} \end{cases}$$

Prob. 1 : Design a low-pass 1 rad/sec bandwidth chebyshev filter with following characteristics.

(i) Acceptable pass band ripple of 2 dB.

(ii) Cut-off radian frequency of 1 rad/sec.

(iii) Stop band attenuation of 20 dB or greater beyond 1.3 rad/sec.

Soln. :

Given data :

Given cut-off frequency is 1 rad/sec. That means it is normalized low pass chebyshev filter.

Pass band ripple $= A_p = 2$ dB

Stop band attenuation $= A_s = 20$ dB

Stop band frequency $= \Omega_s = 1.3$ rad/sec.

Step I : Calculation of parameter ϵ :

We have,

$$\epsilon = \left[10^{0.1 A_{p\,dB}} - 1\right]^{\frac{1}{2}}$$

$$\therefore \epsilon = \left[10^{0.1(2)} - 1\right]^{\frac{1}{2}}$$

$$\therefore \epsilon = 0.765$$

Step II : Calculation of order N of filter :

Given stopband attenuation = 20 dB for $\Omega_s = 1.3$ rad/sec. Since it is attenuation we can write,

$$|H(j\Omega)| = -20 \text{ dB for } \Omega = \Omega_s = 1.3 \text{ rad/sec.}$$

The order N of filter is calculated using the equation,

$$|H(j\Omega)| \text{ in dB} = -20\log_{10}\in -6(N-1) - 20N\log_{10}\Omega_s$$

$$\therefore \quad -20 = -20\log_{10}(0.765) - 6(N-1) - 20N\log_{10}(1.3)$$

$$\therefore \quad 6(N-1) + 20N\log_{10}1.3 = 20 - 20\log_{10}(0.765)$$

$$6N - 6 + 2.28N = 22.327$$

$$\therefore \quad N = 3.421$$

Since N is order of filter we can write,

$$N \approx 4$$

Step III : Calculation of poles :

The pole positions are given by,

$$S_P = r\cos\theta_k + jR\sin\theta_k \qquad ...(1)$$

First we will calculate parameter β

$$\beta = \left[\frac{\sqrt{1+\in^2}+1}{\in}\right]^{\frac{1}{N}}$$

$$\therefore \quad \beta = \left[\frac{\sqrt{1+(0.765)^2}+1}{0.765}\right]^{\frac{1}{4}}$$

$$\therefore \quad \beta = 1.31 \qquad ...(2)$$

Now we will calculate values of r and R.

This is normalized filter. So $\Omega_P = 1$ rad/sec.

We have, minor axis of ellipse $= r = \Omega_P \cdot \frac{\beta^2 - 1}{2\beta}$

$$\therefore \quad r = 1 \cdot \frac{(1.31)^2 - 1}{2(1.31)}$$

$$\therefore \quad r = 0.273 \qquad ...(3)$$

The major axis of an ellipse is given by,

$$R = \Omega_P \frac{\beta^2 + 1}{2\beta}$$

$$\therefore \quad R = 1 \cdot \frac{(1.31)^2 + 1}{2(1.31)}$$

$$\therefore \quad R = 1.04 \qquad ...(4)$$

Now we will calculate values of θ_k as follows :

$$\theta_k = \frac{\pi}{2} + \frac{(2k+1)\pi}{2N}, \quad k = 0, 1, 2, 3.$$

For k = 0 : $\theta_0 = \frac{\pi}{2} + \frac{\pi}{8} = \frac{5\pi}{8}$

For k = 1 : $\theta_1 = \frac{\pi}{2} + \frac{3\pi}{8} = \frac{7\pi}{8}$

For k = 2 : $\theta_2 = \frac{\pi}{2} + \frac{5\pi}{8} = \frac{9\pi}{8}$

For k = 3 : $\theta_3 = \frac{\pi}{2} + \frac{7\pi}{8} = \frac{11\pi}{8}$

Using Equation (1), we can calculate pole positions as follows :

For k = 0 : $s_0 = r\cos\theta_0 + jR\sin\theta_0$

$$\therefore s_0 = 0.273\cos\left(\frac{5\pi}{8}\right) + j\,(1.04)\sin\left(\frac{5\pi}{8}\right) \quad ...(1)$$

$$\therefore s_0 = -0.1 + j\,0.96$$

For k = 1 : $s_1 = r\cos\theta_1 + jR\sin\theta_1$

$$\therefore s_1 = 0.273\cos\left(\frac{7\pi}{8}\right) + j\,(1.04)\sin\left(\frac{7\pi}{8}\right)$$

$$\therefore s_1 = -0.25 + j\,0.4$$

For k = 2 : $s_2 = r\cos\theta_2 + jR\sin\theta_2$

$$\therefore s_2 = 0.273\cos\left(\frac{9\pi}{8}\right) + j\,(1.04)\sin\left(\frac{9\pi}{8}\right) \quad ...(2)$$

$$\therefore s_2 = -0.25 - j\,0.4$$

For k = 3 : $s_3 = r\cos\theta_3 + jR\sin\theta_3$

$$\therefore s_3 = 0.273\cos\left(\frac{11\pi}{8}\right) + j\,(1.04)\sin\left(\frac{11\pi}{8}\right)$$

$$\therefore s_3 = -0.1 - j\,0.96$$

Note : Observe that the poles are complex conjugate of each other.

Step IV : Calculation of system function H(s) :

The system function H (s) is given by,

$$H(s) = \frac{k}{(s - s_0)(s - s_1)(s - s_2)(s - s_3)} \quad ...(3)$$

$$\therefore H(s) = \frac{k}{(s + 0.1 - j\,0.96)(s + 0.25 - j\,0.4)(s + 0.25 + j\,0.4)(s + 0.1 + j\,0.96)} \quad ...(4)$$

$$\therefore H(s) = \frac{k}{(s^2 + 0.2s + 0.9316)(s^2 + 0.5s + 0.2225)}$$

$$\therefore \quad H(S) = \frac{k}{s^4 + 0.7\,s^3 + 1.25\,s^2 + 0.51s + 0.207}$$

Here N = 4 which is even. Thus

$$k = \frac{b_0}{\sqrt{1+\epsilon^2}}$$

b_0 = Constant term in the denominator = 0.207.

$$\therefore \quad k = \frac{0.207}{\sqrt{1+(0.765)^2}} = 0.16$$

$$\therefore \quad H(s) = \frac{0.16}{s^4 + 0.7\,s^3 + 1.25\,s^2 + 0.51s + 0.207}$$

This is the required transfer function.

Prob. 2 : Design a Chebyshev analog filter with maximum pass band attenuation of 2.5 dB at Ω_P = 20 rad/sec and stop band attenuation of 30 dB at Ω_S = 50 rad/sec.

Soln. : Given data :

Pass band attenuation = A_P = 2.5 dB

Pass band frequency = Ω_P = 20 rad/sec

Stop band attenuation = A_s = 30 dB

Stop band frequency = Ω_s = 50 rad/sec

Step I : Calculation of parameter ϵ :

We have,

$$\epsilon = \left[10^{0.1\,A_P\,dB} - 1\right]^{\frac{1}{2}}$$

$$\therefore \quad \epsilon = \left[10^{0.1\,(2.5)} - 1\right]^{\frac{1}{2}}$$

$$\therefore \quad \epsilon = 0.882.$$

Step II : Calculation of order N of the filter :

Given stop band attenuation = 30 dB for Ω_s = 50 rad/sec

$$\therefore \quad |H(j\Omega)| = -30 \text{ dB for } \Omega = \Omega_s = 50 \text{ rad/sec}$$

The order 'N' of filter is calculated using the equation,

$$|H(j\Omega)| \text{ in dB} = -20\log_{10}\epsilon - 6(N-1) - 20N\log_{10}\Omega_s$$

$$\therefore \quad -30 = -20\log_{10}(0.882) - 6(N-1) - 20\,N\log_{10}(50)$$

$$\therefore \quad -30 = 1.09 - 6(N-1) - 33.98\,N$$

$$\therefore \quad -30 = 1.09 - 6N + 6 - 33.98\,N$$

$$\therefore \quad 39.98N = 1.09 + 30 + 6$$

$$\therefore \quad N = 0.95$$

Thus order of filter $= N \approx 1$

Step III : Calculation of poles :

The pole positions are given by,

$$S_P = r\cos\theta_k + jR\sin\theta_k \qquad ...(1)$$

First we will calculate parameter β

$$\beta = \left[\frac{\sqrt{1+\epsilon^2}+1}{\epsilon}\right]^{\frac{1}{N}}$$

$$\therefore \quad \beta = \left[\frac{\sqrt{1+(0.882)^2}+1}{0.882}\right]^{1}$$

$$\therefore \quad \beta = 2.64 \qquad ...(2)$$

We will calculate values of r and R.

We have,

$$\text{Minor axis of ellipse} = r = \Omega_P \cdot \frac{\beta^2 - 1}{2\beta}$$

$$\therefore \quad r = 20 \cdot \frac{(2.64)^2 - 1}{2(2.64)}$$

$$\therefore \quad r = 22.6 \qquad ...(3)$$

The major axis of an ellipse is given by,

$$R = \Omega_P \frac{\beta^2 + 1}{2\beta}$$

$$\therefore \quad R = 20\,\frac{(2.64)^2 + 1}{2(2.64)}$$

$$\therefore \quad R = 30.19 \qquad ...(4)$$

Now we will calculate values of θ_k as follows.

$$\theta_k = \frac{\pi}{2} + \frac{(2k+1)\pi}{2N}, \qquad k = 0 \text{ to } N-1 \text{ that means for } k = 0.$$

$$\therefore \quad \theta_0 = \frac{\pi}{2} + \frac{(0+1)\pi}{2\times 1}$$

$$\therefore \quad \theta_0 = \pi$$

Using Equation (1) we can write pole positions as,

$$s_0 = r\cos\theta_0 + jR\sin\theta_0$$

$$\therefore \quad s_0 = 22.6 \cos \pi + j\,(30.19) \sin \pi$$

$$\therefore \quad s_0 = -22.6$$

Step IV : Calculation of system function H(s) :

The system function is given by,

$$H(s) = \frac{k}{(s - s_0)}$$

$$\therefore \quad H(s) = \frac{k}{s + 22.6}$$

Here for N = 1 we have,

$$k = b_0 = 22.6$$

$$\therefore \quad H(s) = \frac{22.6}{s + 22.6}$$

This is the required transfer function.

Prob. 3 : Design a digital low pass filter using chebyshev filter design procedure that meets the following specifications : Passband magnitude characteristics that is constant to within 1 dB for frequencies below $\omega = 0.2\,\pi$ and stop band attenuation of atleast 15 dB for frequencies between $\omega = 0.3\,\pi$ and π. Use bilinear transformation.

Soln. : In this problem, the specifications of digital filter are given. The given specifications are

Passband ripple = $A_p = 1$ dB

Passband edge frequency = $\omega_p = 0.2\,\pi$

Stop band attenuation = $A_s = 15$ dB

Stop band edge frequency = $\omega_s = 0.3\,\pi$

First we will calculate the values of edge frequencies for the analog filter.

Step I : Calculation of analog filter's edge frequencies :

For the bilinear transformation we have,

$$\Omega_p = \frac{2}{T_s} \tan \frac{\omega_p}{2}$$

Note : Sampling time is not given. So we will assume sampling time $T_s = 1$ sec.

$$\therefore \quad \Omega_p = 2 \tan \frac{0.2\,\pi}{2}$$

$$\therefore \quad \Omega_p = 0.6498$$

$$\text{and} \quad \Omega_s = \frac{2}{T_s} \tan \frac{\omega_s}{2}$$

Again $T_s = 1$ sec.

$$\therefore \quad \Omega_s = 2 \tan \frac{0.3\,\pi}{2}$$

$$\therefore \quad \Omega_s = 1.019$$

Step II : Calculation of parameter ϵ :

It is given by,

$$\epsilon = [\,10^{0.1\,A_p\,dB} - 1\,]^{\frac{1}{2}}$$

$$\therefore \quad \epsilon = [\,10^{0.1\,(1)} - 1\,]^{\frac{1}{2}}$$

$$\therefore \quad \epsilon = 0.5088$$

Step III : Calculation of order (N) of a filter :

It is calculated using the equation,

$$|\,H(j\,\Omega)\,| \text{ in dB} = -20 \log_{10} \epsilon - 6\,(N-1) - 20\,N \log_{10}(\Omega_s)$$

Given attenuation is 15 dB.

$$\therefore \quad -15 \text{ dB} = -20 \log_{10} 0.5088 - 6\,(N-1) - 20\,N \log_{10}(1.019)$$

$$\therefore \quad -15 = 5.87 - 6N + 6 - 0.1634$$

$$\therefore \quad 6.1634\,N = 26.87$$

$$\therefore \quad N = 4.35$$

Thus order of filter $= N \approx 5$

Step IV : Calculation of pole positions :

The pole positions are given by,

$$s_p = r \cos \theta_k + jR \sin \theta_k \qquad \ldots(1)$$

Now we will calculate values of r, R and θ_k.

$$\text{Minor axis of ellipse} = r = \Omega_p \frac{\beta^2 - 1}{2\,\beta}$$

$$\text{and} \quad \beta = \left[\frac{\sqrt{1+\epsilon^2}+1}{\epsilon}\right]^{\frac{1}{N}}$$

$$\therefore \quad \beta = \left[\frac{\sqrt{1+(0.5088)^2}+1}{0.5088}\right]^{\frac{1}{5}}$$

$$\therefore \quad \beta = 1.33$$

$$\therefore \quad r = 0.6498 \cdot \frac{(1.33)^2 - 1}{2 \times 1.33} = 0.186 \qquad \ldots(2)$$

And Major axis of ellipse $= R = \Omega_p \cdot \dfrac{\beta^2 + 1}{2\beta}$

$$\therefore \quad R = 0.6498 \frac{(1.33)^2 + 1}{2 \times 1.33}$$

$$\therefore \quad R = 0.67 \qquad \ldots(3)$$

Now we will calculate values of θ_k.

We have,

$$\theta_k = \frac{\pi}{2} + \frac{(2k+1)\pi}{2N}, \qquad k = 0, 1, 2, \ldots N-1$$

that means $k = 0, 1, 2, 3, 4$

For k = 0 : $\theta_0 = \dfrac{\pi}{2} + \dfrac{\pi}{10} = \dfrac{6\pi}{10}$

For k = 1 : $\theta_1 = \dfrac{\pi}{2} + \dfrac{(2+1)\pi}{10} = \dfrac{\pi}{2} + \dfrac{3\pi}{10} = \dfrac{8\pi}{10}$

For k = 2 : $\theta_2 = \dfrac{\pi}{2} + \dfrac{5\pi}{10} = \pi$

For k = 3 : $\theta_3 = \dfrac{\pi}{2} + \dfrac{7\pi}{10} = \dfrac{12\pi}{10}$

For k = 4 : $\theta_4 = \dfrac{\pi}{2} + \dfrac{9\pi}{10} = \dfrac{14\pi}{10}$

Now putting these values in Equation (1) we get,

For k = 0 : $s_0 = 0.186 \cos\theta_0 + j\,0.67 \sin\theta_0$

$$\therefore \quad s_0 = 0.186 \cos\left(\frac{6\pi}{10}\right) + j\,0.67 \sin\left(\frac{6\pi}{10}\right)$$

$$\therefore \quad s_0 = -0.057 + j\,0.637$$

For k = 1 : $s_1 = 0.186 \cos\theta_1 + j\,0.67 \sin\theta_1$

$$\therefore \quad s_1 = 0.186 \cos\left(\frac{8\pi}{10}\right) + j\,0.67 \sin\left(\frac{8\pi}{10}\right)$$

$$\therefore \quad s_1 = -0.15 + j\,0.39$$

For k = 2 : $s_2 = 0.186 \cos\theta_2 + j\,0.67 \sin\theta_2$

$$\therefore \quad s_2 = 0.186 \cos\pi + j\,0.67 \sin\pi$$

$$\therefore \quad s_2 = -0.186$$

For k = 3 : $s_3 = 0.186 \cos\theta_3 + j\,0.67 \sin\theta_3$

$$\therefore \quad s_3 = 0.186 \cos\left(\frac{12\pi}{10}\right) + j\,0.67 \sin\left(\frac{12\pi}{10}\right)$$

$$\therefore \quad s_3 = -0.15 - j\,0.39$$

For k = 4 : $s_4 = 0.186 \cos\theta_4 + j\,0.67 \sin\theta_4$

$$\therefore \quad s_4 = 0.186 \cos\left(\frac{14\pi}{10}\right) + j\,0.67 \sin\left(\frac{14\pi}{10}\right)$$

$$\therefore \quad s_4 = -0.057 - j\,0.637$$

Step V : Calculation of system transfer function of analog filter :

It is given by,

$$H(s) = \frac{k}{(s-s_0)(s-s_1)(s-s_2)(s-s_3)(s-s_4)}$$

$$\therefore \quad H(s) = \frac{k}{(s+0.057-j\,0.637)(s+0.15-j\,0.39)(s+0.186)(s+0.15+j\,0.39)(s+0.057+j\,0.637)}$$

$$\therefore \quad H(s) = \frac{k}{(s+0.186)(s^2+0.3s+0.1746)(s^2+0.114s+0.409)}$$

Now N = 5 thus,

$k = b_0$ = Constant term in the denominator

$$\therefore \quad k = b_0 = (0.186)(0.1746)(0.409) = 0.0135$$

$$\therefore \quad H(s) = \frac{0.0132}{(s+0.186)(s^2+0.3s+0.1746)(s^2+0.114s+0.409)}$$

Step VI : Calculation of transfer function for digital filter :

By using bilinear transformation, the transfer function for digital filter H (Z) is obtained from H (s) by putting,

$$s = \frac{2}{T_s}\left[\frac{Z-1}{Z+1}\right]$$

Here T_s = Sampling time = 1 sec. (assumed)

$$\therefore \quad H(Z) = \frac{0.0132}{\left[2\left(\frac{Z-1}{Z+1}\right)+0.186\right]\left\{\left[2\left(\frac{Z-1}{Z+1}\right)\right]^2+0.3\left[2\left(\frac{Z-1}{Z+1}\right)\right]+0.01746\right\}\left\{\left[2\left(\frac{Z-1}{Z+1}\right)\right]^2+0.114\left[2\left(\frac{Z-1}{Z+1}\right)\right]+0.409\right\}}$$

1.5 Frequency Transformations :

Uptil now we have studied the design of low pass filter only. Now if it is asked to design other filter like high pass, band pass or bandreject filter then we have to use the frequency transformation.

If the cut-off frequency of LPF is equal to 1 that means if $\Omega_c = 1$ then, it is called as normalized filter. To design the other types of filters; first the system function of normalized LPF is obtained. Then using frequency transformation we can get the system function of the required filter. The following formulae are used for the frequency transformation.

Let, we have a normalized L.P.F. having cut-off frequency Ω_c.

1. Low pass to low pass :

Suppose it is asked to design another LPF with new passband edge frequency Ω_{LP}. Then use the transformation,

$$s \rightarrow \frac{\Omega_c}{\Omega_{LP}} s$$

That means replace 's' by $\frac{\Omega_c}{\Omega_{LP}} s$ in the given equation of H (s).

2. Low pass to high pass :

Suppose we have to design HPF with cutoff frequency Ω_{HP} then use,

$$s \rightarrow \frac{\Omega_c \, \Omega_{HP}}{s}$$

3. Low pass to bandpass :

Suppose we have to design bandpass filter with higher cutoff frequency Ω_u and lower cut-off frequency Ω_l then use the transformation.

$$s \rightarrow \Omega_c \frac{s^2 + \Omega_l \, \Omega_u}{s (\Omega_u - \Omega_l)}$$

4. Lowpass to bandstop or notch or band reject :

Suppose we have to design notch filter with higher cut-off frequency Ω_u and lower cutoff frequency Ω_l then use the transformation,

$$s \rightarrow \Omega_c \frac{s (\Omega_u - \Omega_l)}{s^2 + \Omega_u \, \Omega_l}$$

The following Table K-2 gives the summary of frequency transformation :

Table K-2

Sr. No.	Conversion type	Transformation
1.	Low pass	$s \rightarrow \frac{\Omega_c}{\Omega_{LP}} s$
2.	High pass	$s \rightarrow \frac{\Omega_c \, \Omega_{HP}}{s}$
3.	Band pass	$s \rightarrow \Omega_c \times \frac{s^2 + \Omega_l \, \Omega_u}{s (\Omega_u - \Omega_l)}$
4.	Band stop	$s \rightarrow \Omega_c \times \frac{s (\Omega_u \cdot \Omega_l)}{s^2 + \Omega_u \cdot \Omega_l}$

Note : These formulae are applicable for analog frequency transformation. Similarly, we can transform the LPF in digital domain by using the formulae for digital frequency transformation.

Frequency transformation in digital domain :

Similar to the frequency transformation in analog domain, following formulae are used for the frequency transformation in digital domain.

(1) Low pass to low pass :

$$Z^{-1} \longrightarrow \frac{Z^{-1}-\alpha}{1-\alpha Z^{-1}}; \qquad \alpha = \frac{\sin\left[(w_p - w_{Lp})/2\right]}{\sin\left[(w_{Hp} + w_p)/2\right]}$$

Here w_p = Passband edge frequency of low pass filter.

w_{Lp} = Passband edge frequency of new low pass filter.

(2) Low pass to high pass :

$$Z^{-1} \longrightarrow -\left[\frac{Z^{-1}+\alpha}{1+\alpha Z^{-1}}\right]; \qquad \alpha = -\frac{\cos\left[(w_{Hp} + w_p)/2\right]}{\cos\left[(w_{Hp} - w_p)/2\right]}$$

Here w_p = Passband edge frequency of low pass filter.

w_{Hp} = Passband edge frequency of high pass filter.

(3) Low pass to Band pass :

$$Z^{-1} \longrightarrow \frac{-\left[Z^{-2} - \frac{2\alpha k}{1+k}Z^{-1} + \frac{k-1}{k+1}\right]}{\left[\frac{k-1}{k-1}Z^{-2} - \frac{2\alpha k}{k+1}Z^{-1} + 1\right]}$$

Here $\alpha = \dfrac{\cos\left[(w_u + w_l)/2\right]}{\cos\left[(w_u - w_l)/2\right]}$

$$\text{and } k = \cot\left[\frac{w_u - w_l}{2}\right]\tan\frac{w_p}{2}$$

w_p = Passband edge frequency of low pass filter.

w_u = Upper cut-off frequency of bandpass filter.

w_l = Lower cut-off frequency of bandpass filter.

(4) Low pass to bandstop or Bandreject or Notch :

$$Z^{-1} \longrightarrow \frac{Z^{-2} - \frac{2\alpha}{1+k}Z^{-1} + \frac{1-k}{1+k}}{\frac{1-k}{1+k}Z^{-1} - \frac{2\alpha}{1+k}Z^{-1} + 1}$$

$$\alpha = \frac{\cos\left[(w_u + w_l)/2\right]}{\cos\left[(w_u - w_l)/2\right]}, \quad k = \tan\left[(w_u - w_l)/2\right]\tan\frac{w_p}{2}$$

w_p = Passband edge frequency of low pass filter.

w_u = Upper cut-off frequency of bandstop filter.

w_l = Lower cut-off frequency of bandstop filter.

Solved Problems :

Prob. 1 : Design second order bandpass digital butterworth filter with passband of 300 Hz to 500 Hz and sampling frequency of 1500 Hz using bilinear transformation method; given prototype LPF with transfer function $H(s) = \frac{\Omega_c}{s + \Omega_c}$.

Soln. :

Part A : Calculation of specifications for digital filter :

First we will calculate the required specification of digital filter. That means specifications in terms of ω.

The given specification are

Higher cut-off frequency = F_u = 500 Hz

Lower cut-off frequency = F_l = 300 Hz.

We want the specifications in terms of ω_u and ω_l. For that, first we will calculate values of f_u and f_l by using sampling frequency F_s = 1500 Hz. We have,

$$f_u = \frac{F_u}{F_s} = \frac{500}{1500} = 0.33$$

Now $\omega_u = 2\pi f_u$ $\therefore \omega_u = 2\pi \times 0.33 = 2.073$ radians/sample

$$\text{Similarly} \quad f_l = \frac{F_l}{F_s} = \frac{300}{1500} = 0.2$$

$$\therefore \quad \omega_l = 2\pi f_l$$

$$\therefore \quad \omega_l = 2\pi \times 0.2$$

$$\therefore \quad \omega_l = 1.257 \text{ radians/sample}$$

Thus specifications of digital filter are :

(i) Higher cut-off frequency = ω_u = 2.073 radians/sample

(ii) Lower cut-off frequency = ω_l = 1.257 radians/sample

(iii) Order of filter = N = 2.

Part B : Calculations of specifications for analog filter :

To use the butterworth approximation; we will calculate the specifications of equivalent analog filter using BLT method. The frequency relationship for impulse invariance method is given by,

$$\Omega = \frac{2}{T_s} \tan \frac{\omega}{2}$$

Here T_s is the sampling time.

$$\therefore \quad \Omega_u = \frac{2}{T_s} \tan \frac{\omega_u}{2}$$

$$\therefore \quad \Omega_u = (2 \times 1500) \tan \frac{2.073}{2} = 5070.11$$

$$\text{and} \quad \Omega_l = \frac{2}{T_s} \tan \frac{\omega_l}{2}$$

$$\therefore \quad \Omega_l = (2 \times 1500) \tan \frac{1.257}{2} = 2180.46$$

Part C : Calculation of system transfer function for analog BPF :

The given transfer function of LPF is,

$$H(s) = \frac{\Omega_c}{s + \Omega_c} \qquad \ldots(1)$$

for the normalized filter $\Omega_c = 1$.

$$\therefore \quad H(s) = \frac{1}{s+1} \qquad \ldots(2)$$

Now H (s) BPF is obtained by using the transformation equation,

$$s \rightarrow \Omega_c \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_c - \Omega_l)}$$

Putting $\Omega_c = 1$ and putting the values of Ω_u and Ω_l we get,

$$s \rightarrow 1 \times \frac{s^2 + (2180.46 \times 5070.11)}{s(5070.11 - 2180.46)}$$

$$\therefore \quad s \rightarrow \frac{s^2 + 11.055 \times 10^6}{2889.65\, s}$$

Putting this value of 's' in Equation (2) we get,

$$H(s) = \frac{1}{\frac{s^2 + 11.055 \times 10^6}{2889.65\, s} + 1}$$

$$\therefore \quad H(s) = \frac{2889.65\, s}{s^2 + 2889.65\, s + 11.055 \times 10^6}$$

This is the transfer function for analog BPF.

Part D : Calculation H (Z) for digital filter :

H (Z) can be obtained by putting $s = \frac{2}{T_s}\left[\frac{Z-1}{Z+1}\right]$ in the equation of H (s).

$$\therefore \quad H(Z) = \frac{2889.65 \times 2 \times 1500 \left(\frac{Z-1}{Z+1}\right)}{(2\times 1500)^2\left(\frac{Z-1}{Z+1}\right)^2 + 2889.65 \times 2 \times 1500\left(\frac{Z-1}{Z+1}\right) + 11.055 \times 10^6}$$

$$\therefore \quad H(Z) = \frac{8.67 \times 10^6 \left(\frac{Z-1}{Z+1}\right)}{9 \times 10^6\left(\frac{Z-1}{Z+1}\right)^2 + 8.67 \times 10^6\left(\frac{Z-1}{Z+1}\right) + 11.055 \times 10^6}$$

$$\therefore \quad H(Z) = \frac{8.67\left(\frac{Z-1}{Z+1}\right)}{9\left(\frac{Z-1}{Z+1}\right)^2 + 8.67\left(\frac{Z-1}{Z+1}\right) + 11.055}$$

$$\therefore \quad H(Z) = \frac{8.67\left(\frac{Z-1}{Z+1}\right)\times (Z+1)^2}{9(Z-1)^2 + 8.67(Z-1)(Z+1) + 11.055(Z+1)^2}$$

$$\therefore \quad H(Z) = \frac{8.67(Z-1)(Z+1)}{9(Z-1)^2 + 8.67(Z^2-1) + 11.055(Z+1)^2}$$

Prob. 2 : Design the digital high pass filter for cut-off frequency of 30 Hz and sampling frequency of 150 Hz using BLT.

Soln. :

Part A : Calculation of specification for digital filter :

We have $f_{HP} = \frac{F_{HP}}{F_s}$

Here F_{HP} = Given cut-off frequency = 30 Hz

F_s = Sampling frequency = 150 Hz

$\therefore \quad f_{HP} = \frac{30}{150} = 0.2$ cycles/samples.

Now $\omega_{HP} = 2\pi f_{HP} = 2\pi \times 0.2 = 0.4\pi$

The order of filter is not given; so we can assume order of N = 1.

Part B : Calculation of specifications for analog filter :

Step I : Calculation of Ω_{HP}

For the bilinear transformation we have,

$$\Omega = \frac{2}{T_s} \tan \frac{\omega}{2}$$

$$\therefore \quad \Omega_{HP} = \frac{2}{\frac{1}{150}} \tan \frac{\omega_{HP}}{2} = 300 \tan \frac{0.4\pi}{2}$$

$$\therefore \quad \Omega_{HP} = 217.96 \text{ radians/sec.}$$

and we have assumed N = 1.

Step II : Calculation of pole positions :

We have,

$$P_k = \pm \Omega_c e^{j(N+2k+1)\pi/2N}, \quad k = 0, 1 \ldots\ldots N-1$$

We have to use normalized low pass filter to design the required high pass filter. For normalized low pass filter.

$$\Omega_c = 1$$

$$\therefore \quad P_k = \pm e^{j(1+2k+1)\pi/2N}, \quad k = 0 \text{ to } N-1$$

The range of k is 0 to N – 1 and N = 1; thus we get k = 0.

$$\therefore \quad P_0 = \pm e^{j\frac{(2)\pi}{2N}} = \pm e^{\frac{j2\pi}{2}} = \pm e^{j\pi}$$

$$\therefore \quad P_0 = \cos\pi + j\sin\pi$$

$$\therefore \quad P_0 = -1$$

$$\therefore \quad s_1 = -1$$

Thus pole is at -1. So we can write the system transfer function for analog filter.

$$H(s) = \frac{1}{s - s_1} = \frac{1}{s+1} \qquad \ldots(1)$$

Part C : Use frequency transformation :

Now using frequency transformation to obtain transfer function of analog high pass filter. We have,

$$s \rightarrow \frac{\Omega_c \Omega_{HP}}{s}$$

But $\Omega_c = 1$

$$\therefore \quad s \rightarrow \frac{\Omega_{HP}}{s}$$

$$\therefore \quad s \rightarrow \frac{217.96}{s}$$

Putting this value in Equation (1) we get,

$$H(s) = \frac{1}{\frac{217.96}{s} + 1} = \frac{s}{s + 217.96} \qquad ...(2)$$

Part D : Calculation of H (Z) for digital filter :

Using bilinear transformation, H (Z) is obtained by putting,

$$s = \frac{2}{T_s}\left[\frac{Z-1}{Z+1}\right] \quad \text{in Equation (2).}$$

$$\therefore \quad H(Z) = \frac{2 \times 150\left(\frac{Z-1}{Z+1}\right)}{2 \times 150\left(\frac{Z-1}{Z+1}\right) + 217.96}$$

$$\therefore \quad H(Z) = \frac{300\left(\frac{Z-1}{Z+1}\right)}{300\left(\frac{Z-1}{Z+1}\right) + 217.96}$$

$$\therefore \quad H(Z) = \frac{\left(\frac{Z-1}{Z+1}\right)}{\left(\frac{Z-1}{Z+1}\right) + 0.73}$$

$$\therefore \quad H(Z) = \frac{Z-1}{Z-1+0.73\,Z+0.73}$$

$$\therefore \quad H(Z) = \frac{Z-1}{1.73\,Z-0.27}$$

Prob. 3 : Design the second order low pass digital filter of Butterworth type using BLT for the specifications given below :

(i) Analog transfer function of the filter

$$H(s) = 1/(s^2 + \sqrt{2}\,s + 1)$$

(ii) Cutoff frequency = 1 kHz

(iii) Sampling frequency = 10 kHz.

Soln. : Given,

Cut-off frequency $= F_c = 1$ kHz

and sampling frequency $F_s = 10$ kHz

$\therefore$ Sampling time $T_s = \dfrac{1}{10\,\text{kHz}}$

Given transfer function is,

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1} \qquad ...(1)$$

We know that prewarping frequency is,

$$\Omega_p^* = \frac{2}{T_s}\tan\left(\frac{\omega_c T_s}{2}\right) \qquad ...(2)$$

Here we will assume $\dfrac{2}{T_s} = 1$ because in BLT we have to put $s = \dfrac{2}{T_s}\left(\dfrac{Z-1}{Z+1}\right)$; so in the final stage $\dfrac{2}{T_s}$ will cancel.

$$\therefore \quad \Omega_p^* = \tan\left(\frac{\omega_c T_s}{2}\right)$$

$$= \tan\left(\frac{2\pi \times 1 \times 10^3}{2 \times 10 \times 10^3}\right)$$

$$\therefore \quad \Omega_p^* = 0.324 \qquad ...(3)$$

Now we will use frequency transformation as follows :

$$H^*(s) = H(s)\Big|_{s=\frac{s}{\Omega_p^*}}$$

$$\therefore \quad H^*(s) = \frac{1}{\left(\dfrac{s}{0.324}\right)^2 + \sqrt{2}\left(\dfrac{s}{0.324}\right) + 1}$$

$$\therefore \quad H^*(s) = \frac{(0.324)^2}{s^2 + \sqrt{2}\, s \times 0.324 + (0.324)^2}$$

$$\therefore \quad H^*(s) = \frac{(0.324)^2}{s^2 + 0.458 + (0.324)^2} \qquad ...(4)$$

Using BLT we can obtain $H(Z)$ as follows :

$$H(Z) = H^*(s)\Big|_{s=\frac{2}{T_s}\left(\frac{Z-1}{Z+1}\right)}$$

But we have assumed $\dfrac{2}{T_s} = 1$

$$\therefore \quad H(Z) = H^*(s)\Big|_{s=\left(\frac{Z-1}{Z+1}\right)}$$

Thus Equation (4) becomes,

$$H(Z) = \frac{(0.324)^2}{\left(\frac{Z-1}{Z+1}\right)^2 + 0.458\left(\frac{Z-1}{Z+1}\right) + (0.324)^2}$$

$$\therefore \quad H(Z) = \frac{(0.324)^2 (Z+1)^2}{(Z-1)^2 + 0.458 (Z-1)(Z+1) + (0.324)^2 (Z+1)^2}$$

$$\therefore \quad H(Z) = \frac{0.1049 Z^2 + 0.2099 Z + 0.1049}{Z^2 - 2Z + 1 + 0.458 Z^2 - 0.458 + 0.1049 Z^2 + 0.2099 Z + 0.1049}$$

$$\therefore \quad H(Z) = \frac{0.1049 Z^2 + 0.2099 Z + 0.1049}{1.563 Z^2 - 1.79 Z + 0.647}$$

To convert it into negative powers of Z, multiply numerator and denominator by Z^{-2},

$$\therefore \quad H(Z) = \frac{0.1049 Z^{-2} + 0.2099 Z^{-1} + 0.1049}{0.647 Z^{-2} - 1.79 Z^{-1} + 1.563}$$

1.6 Elliptic Filters :

Elliptic filters are also called as cauer filters. Such filters have equiripple passband and stopband. The magnitude squared frequency response of elliptic filter is shown in Fig. K-14.

The magnitude squared response of elliptic filter is obtained by using,

$$|H(\Omega)|^2 = \frac{1}{1 + \epsilon^2 U_N\left(\frac{\Omega}{\Omega_p}\right)} \qquad ..(1)$$

Here ϵ = Ripple parameter in passband

Ω_p = Passband edge frequency

$U_N(x)$ = Jacobian elliptic function of order N.

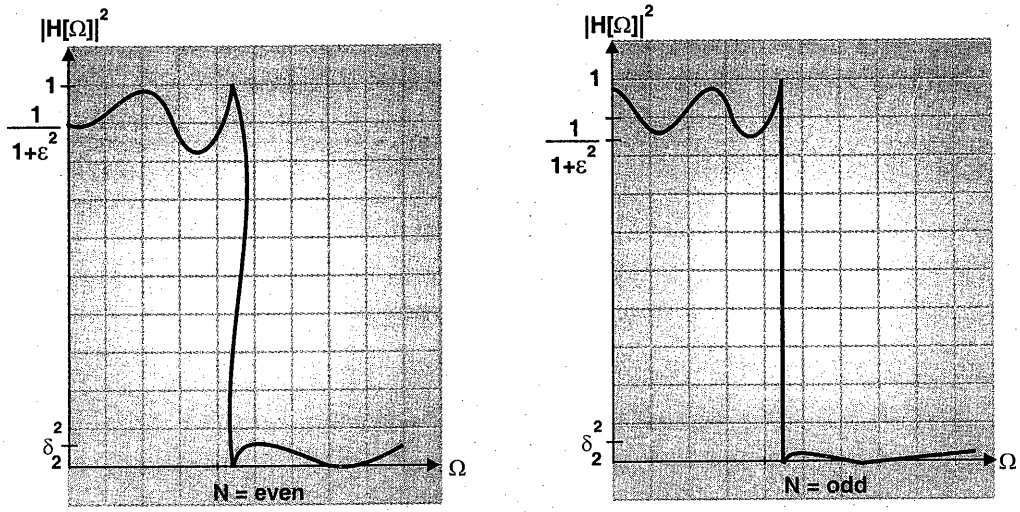


Fig. K-14 : Elliptic filter characteristics

The order 'N' of elliptic filter is given by,

$$N = \frac{K(\Omega_p/\Omega_s)\,K\left(\sqrt{1-(\epsilon^2/\delta^2)}\right)}{K(\epsilon/\delta)\,K\left(\sqrt{1-(\Omega_p/\Omega_s)^2}\right)}$$

Here δ is related to δ_1 and δ_2 as follows :

$$\delta_2 = \sqrt{1+\delta^2} \qquad ..(2)$$

where δ_2 = stopband ripple

and $\delta_1 = 10\log_{10}(1+\epsilon^2)$

where δ_1 = stopband ripple

The function K (x) is called as elliptic integral of first kind.

Advantages of elliptic filters :

1. Elliptic filter is more efficient because it provides the smallest order filter for given set of specifications.
2. It has smallest transition bandwidth.

Disadvantage of elliptic filters :

The phase response is more non-linear in passband.

Theory Questions

Q. 1 For a bilinear transformation method, discuss the warping effect. Explain the design steps of IIR filter by the bilinear transformation method.

Q. 2 Write short note on : Bilinear transformation.

Q. 3 Explain BLT method for designing IIR filter? What is warping effect?

Q. 4 What do you mean by impulse-invariance? Using this method can you design high pass and band reject filters? Justify.

Q. 5 Explain the design steps of IIR filter by the impulse invariance method?

Q. 6 How the problem of frequency warping occurs in BLT method of IIR filter design? How it is compensated? Explain the design steps of BLT method.

Q. 7 What are the advantages and disadvantages of digital filter ?

Q. 8 Explain design of IIR filters using approximation of derivatives.

Q. 9 Write short note on :

(a) Butterworth filter approximation.
(b) Chebyshev filter approximation
(c) Elliptic filter approximation

Unsolved Numericals

Q. 1 Using impulse invariance method find $H(Z)$ at $T_s = 1$ sec.

Given $H(s) = \dfrac{2}{s^2 + 8s + 15}$

Ans. :

$$H(Z) = \frac{1}{1 - 0.49 Z^{-1}} - \frac{1}{1 - (6.74 \times 10^{-3}) Z^{-1}}$$

Q. 2 Obtain equation of $H(Z)$ using impulse invariance method, given,

$$H(s) = \frac{s+2}{(s+1)(s+3)}$$

Ans. :

$$H(Z) = \frac{\frac{1}{2}\left[2 - Z^{-1}\left(e^{-Ts} + e^{-3Ts}\right)\right]}{\left(1 - e^{-Ts} \cdot Z^{-1}\right)\left(1 - e^{-3Ts} \cdot Z^{-1}\right)}$$

Q. 3 Convert analog filter with system function $H(s) = \dfrac{s + 0.1}{(s + 0.1)^2 + 25}$ into digital IIR filter by means of impulse invariance method.

Ans. :

$$H(Z) = \frac{1 - \left(e^{-0.1\,Ts} \cos 5\,Ts\right) Z^{-1}}{1 - \left(2\,e^{-0.1\,Ts} \cos 5\,Ts\right) Z^{-1} + e^{-0.27} Z^{-1}}$$

Q. 4 Convert analog filter with transfer function $H(s) = \frac{2}{(s+2)(s+3)}$ into digital filter.

Ans. :

$$H(Z) = 2\left[\frac{1}{1-e^{-2Ts}\cdot Z^{-1}} - \frac{1}{1-e^{-3Ts}\cdot Z^{-1}}\right]$$

Q. 5 Using bilinear transformation convert to Z domain at T_s = 1 sec

$$H(s) = \frac{1}{(s+1)(s+2)}$$

Ans. :

$$H(Z) = \frac{(Z+1)^2}{(3Z-1)(4Z)}$$

Q. 6 Use bilinear transformation to design first order butterworth LPF with 3 dB cutoff frequency of $0.2\,\pi$.

Ans. :

$$H(Z) = \frac{0.6498}{2\left[\frac{1-Z^{-1}}{1+Z^{-1}}\right] + 0.6498}$$

Q. 7 Given that $|H_a(j\Omega)|^2 = \frac{1}{1+64\,\Omega^6}$; determine analog filter system function $H_a(s)$.

Hint : $\left|H_a(j\Omega)\right|^2 = \frac{1}{1+64\,\Omega^6} = \frac{1}{1+\left(\frac{\Omega}{112}\right)^6} = \frac{1}{1+\left(\frac{\Omega}{112}\right)^{2\times 3}}$

Compare it with standard equation $|H_a(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$

$\therefore \quad \Omega_c = \frac{1}{2}$ and N = 2

Ans. :

$$H(s) = \frac{0.125}{(s+0.5)(s^2+0.5\,s+0.25)}$$

Q. 8 Design digital filter using BLT for following analog transfer function $H(s) = \frac{1}{s^2+\sqrt{2}\,s+1}$ using prewarping procedure

Given : ω_p = 150 Hz and Fs = 1.28 kHz

$$H(Z) = \frac{0.0878 + 0.1756\,Z^{-1} + 0.0878\,Z^{-2}}{1 - 1.0048\,Z^{-1} + 0.3561\,Z^{-2}}$$

Q. 9 A transfer function of notch filter with 1 rad/sec frequency is $H(s) = \frac{s^2+1}{s^2+s+1}$. Design digital notch filter with following specifications.

(i) Notch frequency = 60 Hz

(ii) Sampling frequency = 960 sps use prewaring concept.

Hint: Given notch frequency = 60 Hz $\therefore \omega_p = 2\pi \times 60$ rad / sec

Ans. :

$$H(Z) = \frac{9.6603 + 2Z^{-1} - 7.66Z^{-2}}{4.6923 - 4Z^{-1} + 3.3Z^{-2}}$$

Q. 10 If $H(s) = \frac{s+2}{(s+1)(s+3)}$, convert it to Z transform using matched Z transform technique.

Ans. :

$$H(Z) = \frac{1 - e^{-2Ts} \cdot Z^{-1}}{\left(1 - e^{-Ts} \cdot Z^{-1}\right)\left(1 - e^{-3Ts} \cdot Z^{-1}\right)}$$

Q. 11 Given for low pass butterworth filter, $A_p = -1$ dB at 0.2π and As = – 15 dB at 0.3π

Design digital filter using BLT and using prewarping procedure.

Ans. :

$$H(Z) = \frac{0.0007378\,(1+Z^{-1})^6}{(1 - 1.26Z^{-1} + 0.7Z^{-2})(1 - 1.01Z^{-1} + 0.35Z^{-2})(1 - 0.9Z^{-1} + 0.21Z^{-2})}$$

Q. 12 Specification of chebyshev low pass analog filter are,

passband : $-1 < |H(\Omega)|$ dB ≤ 0 for $0 \leq \Omega \leq 1404\,\pi$ rad/sec.

Stopband : $|H(\Omega)|_{dB} < -60\ \Omega s \geq 8268\,\pi$ rad/sec

Design digital filter using impulse invariance method

Ans. :

$$H(Z) = \frac{7.23 \times 10^{-3}}{(1 - 1.687Z^{-1} + 0.734Z^{-2})(1 - 1.707Z^{-1} + 0.88Z^{-2})}$$

Q. 13 Design second order butterworth bandpass filter, using prewarping procedure to meet the following specifications :

Lower cutoff frequency = 210 Hz
Upper cutoff frequency = 330 Hz
Sampling frequency = 960 Hz

Ans. :

$$H(Z) = \frac{1.0502\,(Z^2 - 1)}{3.586Z^2 + 1.0708Z + 1.48542}$$

❑❑❑

DSP Processors Architecture

Section No.	Name of the Topic
1.1	Introduction
1.2	Computer Architecture
1.3	Internal Architecture of ADSP 21xx Family
1.4	Features of ADSP 21xx Family of Processors
1.5	Instruction Set of ADSP 21xx Series Processor
1.6	Texas Instruments TMS 320 C5X
1.7	Comparison of General Purpose Microprocessor with DSP Processors

• Code : DSP13

1.1 Introduction :

Digital signal processing is a method of processing real world signals (represented by a sequence of numbers) using mathematical techniques to perform transformations or extract information.

You don't speak in a digital signal. A digital signal is a language of 1s and 0s that can be processed by mathematics. We speak in real-world, analog signals. Analog signals are real world signals that you and I experience everyday - sound, light, temperature and pressure. A digital signal is a numerical representation of the analog signal. It may be easier and more cost effective to process these signals in the digital world. In the real world, we can convert these signals into digital signals through our analog-to-digital conversion process, process the signals, and if needed, bring the signals back out to the analog world through the digital-to-analog converter.

A digital signal processor (DSP) is a type of microprocessor - one that is incredibly fast and powerful. A DSP is unique because it processes data in real time. This real-time capability makes a DSP perfect for applications that cannot tolerate any delays. For example, did you ever talk on a cell phone where two people couldn't talk at once? You had to wait until the other person finished talking. If you both spoke simultaneously, the signal was cut—you didn't hear the other person. With today's digital cell phones, which use DSP, you can talk normally. The DSP processors inside cell phones process sounds so rapidly you hear them as quickly as you can speak - in real time. Here are just some of the advantages of designing with DSPs over other microprocessors:

- Single-cycle multiply-accumulate operations
- Real-time performance, simulation and emulation
- Flexibility
- Reliability
- Increased system performance
- Reduced system cost

Fig. M-1 shows a graphical representation of digital signal processing carried out with a digital signal.



Fig. M-1 : Block diagram of digital processing

The result is crystal clear sound, with no annoying echoes. That's a basic explanation of what a DSP does. It takes a digital signal and processes it to improve the signal. The improvement may be clearer sound, sharper images, or faster data. And that ability to improve signals is making new breakthroughs such as Internet music and broadband to the home possible.

1.2 Computer Architecture :

Computer architecture is the theory behind the design of a computer. In the same way as a building architect sets the principles and goals of a building project as the basis for the draftsman's plans, so too, a computer architect sets out the computer architecture as a basis for the actual design specifications.

There are three types of architectures :

1. Von Neumann Architecture :

It refers to computer architectures that use the same data storage for their instructions and data (in contrast to the *Harvard architecture*).

A Von Neumann Architecture computer has five parts: an arithmetic-logic unit, a control unit, a memory, some form of input/output and a bus that provides a data path between these parts.

A Von Neumann Architecture computer performs or emulates the following sequence of steps :

1. Fetch the next instruction from memory at the address in the program counter.
2. Add 1 to the program counter.
3. Decode the instruction using the control unit. The control unit commands the rest of the computer to perform some operation. The instruction may change the address in the program counter, permitting repetitive operations. The instruction may also change the program counter only if some arithmetic condition is true, giving the effect of a decision, which can be calculated to any degree of complexity by the preceding arithmetic and logic.
4. Go back to step 1.

Von Neumann computers spend a lot of time moving data to and from the memory, and this slows the computer (this problem is called Von Neumann bottleneck) So, engineers often separate the bus into two or more buses, usually one for instructions, and the other for data.

This type of architecture is cheap, requiring less pins that the Harvard architecture, and simple to use because the programmer can place instructions or data anywhere throughout the available memory. But it does not permit multiple memory accesses.

2. Harvard architecture :

The term **Harvardrchitecture** originally referred to computer architectures that used separate data storage for their instructions and data (in contrast to the Von Neumann architecture). The term originated from the Harvard Mark I relay based computer, which stored instructions on punched tape and data in relay latches.

The term Harvard architecture is usually used now to refer to a particular computer architecture design philosophy where separate data paths exist for the transfer of instructions and data. Refer Fig. M-2.

All computers consist primary of two parts, the CPU which processes data, and the memory which holds the data. The memory in turn has two aspects to it, the data itself, and the location where it is found - known as the address. Both are important to the CPU, as many common instructions boil down to something like "take the data in this address and add it to the data in that address", without actually knowing what the data itself is.

In recent years the speed of the CPU has grown many times in comparison to the memory it talks to, so care needs to be taken to reduce the numbers of times you access it in order to keep performance. If, for instance, every instruction run in the CPU requires an access to memory, the computer gains nothing for increased CPU speed - a problem referred to as being memory bound.

Memory can be made much faster, but only at high cost. The solution then is to provide a small amount of very fast memory known as a cache. As long as the memory the CPU needs is in the cache, the performance hit is very much less than it is if the cache then has to turn around and get the data from the main memory. Tuning the cache is an important aspect of computer design.

The Harvard architecture refers to one particular solution to this problem.

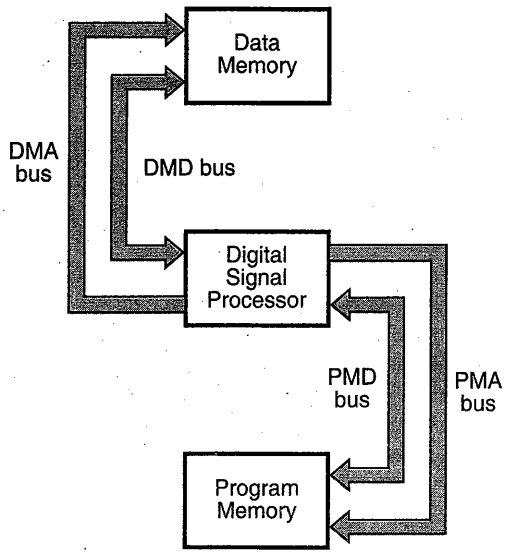


Fig. M-2 : Block diagram of Harvard Architecutre

The principal feature of the Harvard Architecture is that the program and data memories lie in two separate spaces, permitting a full overlap of instruction fetch and execution. Standard microprocessors, are characterized by a single bus structure for both data and instructions.

Suppose that, we wish to read a value op1 at address ADR1 in memory into the accumulator and then store it at two other addresses, ADR2 and ADR3. The instructions could be

LDA	ADR1	load the operand op1 into the accumulator from ADR1.
STA	ADR2	store op1 in address ADR2.
STA	ADR3	store op1 in address ADR3.

In a standard processor, without Harvard architecture, the program instructions and the data are held in one memory space. Thus the fetching of the next instruction while the current one is executing is not allowed, because the fetch and execution phases each require memory access.

In a Harvard architecture, since the program instructions and data lie in separate memory spaces the fetching of the next instruction can overlap the execution of the current instruction; Refer Fig. M-3

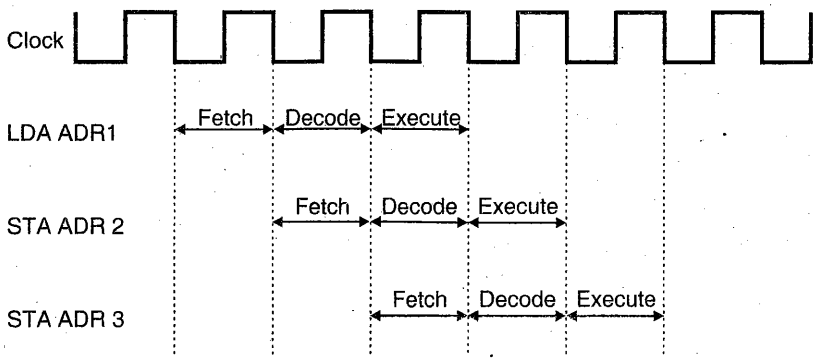


Fig. M-3 : An illustration of instruction overlap in Harvard architecture

Thus, the instruction fetch involves fetching the next instruction from memory, and instruction execute involves either reading or writing data into memory.

Instructions and data are stored in separate caches to improve performance. However this has the disadvantage of halving the amount of cache available to either one, so it works best only if the CPU reads instructions and data at about the same frequency.

The Harvard architecture requires two memory buses. This makes it expensive to bring off the chip - for example a DSP using 32 bit words and with a 32 bit address space requires at least 64 pins for each memory bus - a total of 128 pins if the Harvard architecture is brought off the chip. This results in very large chips, which are difficult to design into a circuit.

3. Modified Harvard architecture : The true Harvard architecture dedicates one bus for fetching instructions, with the other available to fetch operands. This is inadequate for DSP operations, which usually involve at least two operands. So DSP Harvard architectures usually permit the 'program' bus to be used also for access of operands. Note that it is often necessary to fetch three things - the instruction plus two operands - and the Harvard architecture is inadequate to support this: so DSP Harvard architectures often also include a cache memory which can be used to store instructions which will be reused, leaving both Harvard buses free for fetching operands. This extension - Harvard architecture plus cache - is sometimes called an extended Harvard architecture or Super Harvard Architecture (SHARC). Refer Fig. M-4.

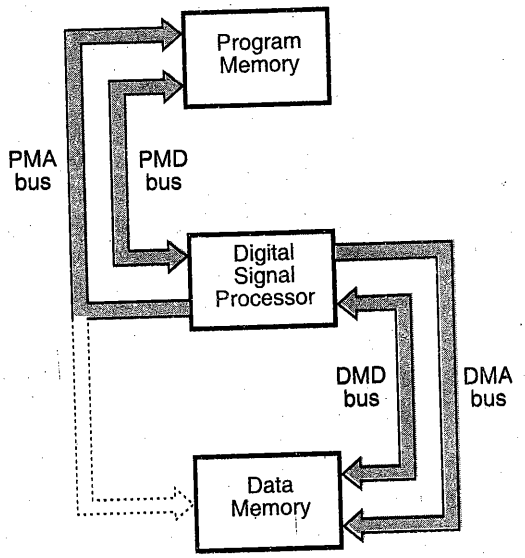


Fig. M-4 : Block diagram of modified Harvard Architecutre

1.3 Internal Architecture of ADSP 21xx Family :

The architecture contains the following main units :

- Arithmetic logic unit (ALU)
- Multiplier accumulator (MAC)
- Barrel shifter
- Register file

- Data address generators
- Program sequencer
- Cache memory and
- Bus exchange

Along with the main units there are 5 internal buses :

(i) Program memory address (PMA) bus
(ii) Program Memory Data (PMD) bus
(iii) Data memory Address (DMA) bus
(iv) Data Memory Data (DMD) bus
(v) Result bus (R)

Brief description of the architecture :

ANALOG DEVICES ADSP-2100 :

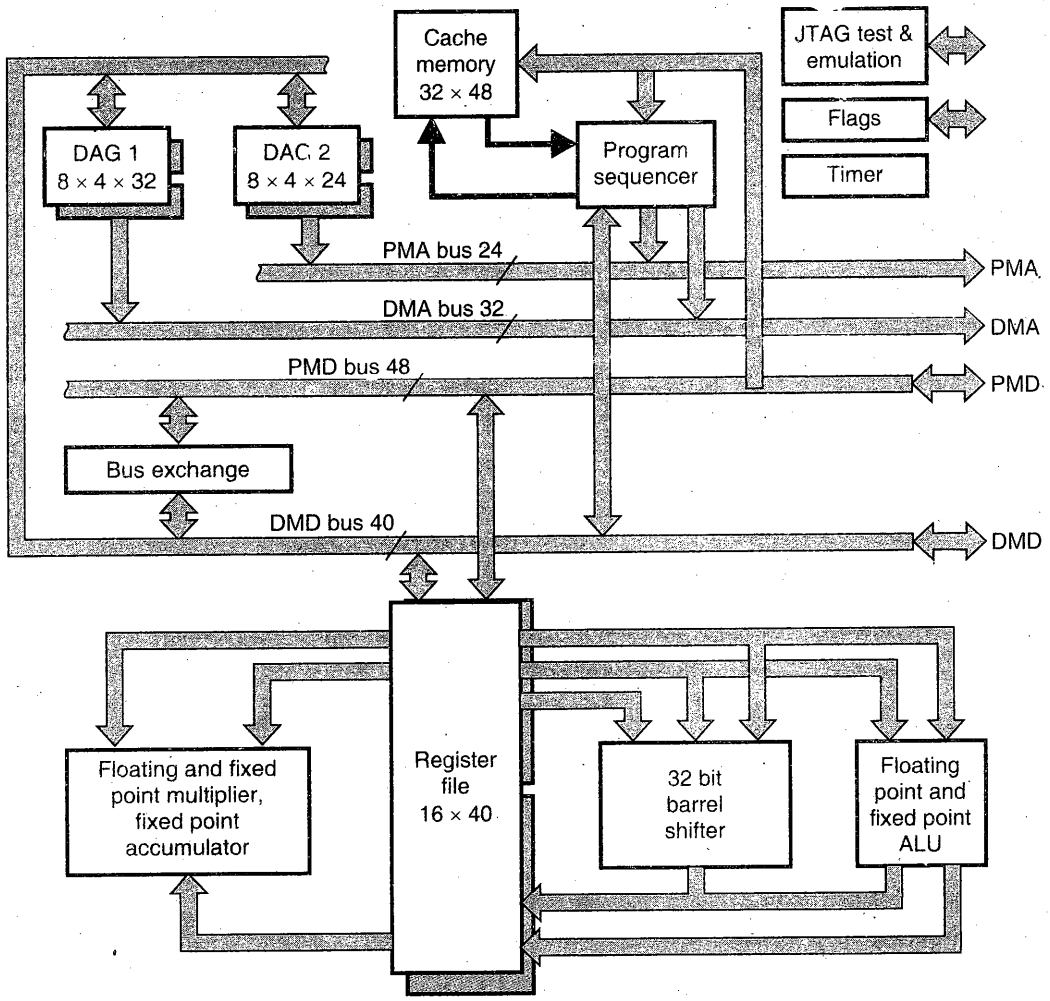


Fig. M-5 : Internal Architecture of ADSP 2100

The ADSP-21xx is the first DSP processor family from Analog Devices. The family consists of a large number of processors based on a common 16-bit fixed-point architecture with a 24-bit instruction word. The ADSP-21xx is also available as a licensable core for use in ASICs. As of October 1998, the fastest members of the family operate at 75 MIPS at 2.5 volts, 52 MIPS at 3.3 volts, and 40 MIPS at 5.0 volts. Analog Devices has recently announced the ADSP-219x series, which offers projected speeds of up to 300 MIPS, as well as architectural enhancements. ADSP-21xx processors are targeted at modem, audio, PC multimedia, and digital cellular applications.

ADSP-21xx processors use a modified Harvard architecture with separate memory spaces and on-chip bus sets for program and data. All processors in the ADSP-21xx family, except the ADSP-2100A, include on-chip program RAM or ROM and on-chip data RAM. The ADSP-2100A has no on-chip memory; instead it has a 16-word instruction cache and two external bus sets.

On-chip program memory can be used for both instructions and data, and it can be accessed via a 14-bit address bus and a 24-bit data bus. On-chip program memory is dual-ported to allow the processor to fetch both a data operand and the next instruction in a single instruction cycle. The on-chip data memory can be accessed via a 14-bit address bus and a 16-bit data bus. One access to the on-chip data memory can be performed in a single instruction cycle. Three memory accesses (one instruction and two data operands) can be performed in one instruction cycle (except on the ADSP-2100A which has no on-chip memory). On a 75 MIPS ADSP-2189M this results in a maximum sustainable on-chip data memory bandwidth of 150 16-bit M words /second.

Both of the on-chip memory spaces can be extended off-chip. All ADSP-21xx processors except the ADSP-2100A have one external memory interface, providing a 14-bit address bus and a 24-bit data bus. This external interface is multiplexed between program and data memory accesses.

The ADSP-21xx data path consists of three separate arithmetic execution units: an arithmetic/logic unit (ALU), a multiplier/accumulator (MAC), and a barrel shifter. Each unit is capable of single-cycle execution, but only one of these units can be active during a single instruction cycle. The multiply-accumulate (MAC) unit, ALU, and barrel shifter are separate but cannot execute in parallel. The ALU operates on 16-bit data. In addition to the usual ALU operations, the ALU provides increment/decrement, absolute value, and add-with-carry functions. ALU results are saturated upon overflow if the programmer sets the appropriate configuration bit. The MAC unit includes a 16x16->32-bit multiplier, four input registers (two for each memory space, PM and DM), a feedback register, a 40-bit adder, and a single 40-bit result register/accumulator providing eight guard bits. Besides signed operands, the multiplier can operate on unsigned/unsigned or on signed/unsigned operands, thus supporting multi-precision arithmetic. The barrel shifter shifts 16-bit inputs from an input register or from the ALU/MAC/barrel shifter result registers into a 32-bit result register. Logical and arithmetic shifts are supported left or right up to 32 bits. The barrel shifter also supports block floating-point arithmetic with block exponent detect (which determines a maximum exponent of a block of data), single-word exponent detect, normalize, and exponent adjust instructions.

The ADSP-2100 family's CPU handles general processing needs and executes all instructions in a single cycle. All of Analog Devices 16-bit DSPs are code-compatible and feature algebraic programming syntax. The processor can execute multiple operations per cycle. Secondary registers shadow each execution unit's registers, allowing fast context switching for interrupt processing. If you need extended precision, you can address the MAC unit's 40-bit accumulator (includes 8 guard bits) as two16-bit and one 8-bit register and individually copy the contents to another register. The barrel shifter moves 16-bitinputs left or right into a 32-bit register .The shifter also include hardware support to perform logical and arithmetic shifts in addition to exponent detection and normalization for block floating point and increasing the precision of a16-bit DSP. Algorithms such as FFTs in

which bits grow from stage to stage use block-floating point. A programmer may use the shifter to convert between fixed-and floating-point numbers. ADSP-2100 family members have X and Y data-address generators (DAGs) and separate program and data buses. Two DAGs provide addresses for simultaneous dual-operand fetches (from data and program memory). Each DAG maintains and updates four address Pointers. You may associate a length value with each pointer to implement automatic modulo addressing for circular buffers. While executing from the on-chip memory, the buses feed the X- and Y-data values for each MAC cycle. Thus, you can use program memory as data memory to hold constants for single-cycle MAC processing. The program bus is free for MAC use when the CPU executes from on-chip program memory. The dual-ported program memory allows two memory accesses in one cycle. For access to external memory, the ADSP-2100 has a programmable wait-state Generator for zero to seven waits states. Analog Devices designers opted for a 16-bit-wide data word and a 24-bit-wide instruction word. The wider instruction word lets the device use more complex instructions and offers more flexibility than does a 16-bit operation code. For external-memory design, the different memory widths mean that if you let three 8-bit-wide memory chips share program and data, you sacrifice every third byte of the data-mem ry area. Analog Devices integrates as much as 64 k bytes of SRAM around its DSP core to help increase data-transfer efficiency. Many ADSP-2100s also integrate DMA ports that connect to external hosts or external memory.

These bi-directional, byte-wide ports can directly access as much as 4 Mbytes of external memory for off-chip storage of program overlays or data tables. The bus connect block is connected between the PMD and DMD bus. This block is used to exchange data between PMD and DMD bus. The PMD bus is 48bits and the DMD bus 40 bits wide. While exchanging the 8-bit discrepancy between two buses is overcome by the Px register in bus connect unit.

Addressing modes :

The ADSP-2100 includes immediate, register-direct, memory-direct and register-indirect addressing modes. The program sequencer features internal loop counters and loop stacks, enabling looped code to execute with zero overhead. Each address generator supports as many as four circular buffers, each with three registers. The registers define the end, lengt and access address. One address generator provides bit-reversed addressing for data only.

Special instructions :

The ADSP-2100 can conditionally execute most instructions .A do-until command establishes a sequence of instructions that can be arbitrary in length and nested four deep for repeat operations. In addition to the standard arithmetic and logic instructions, the ALU supports division primitives. Because the ADSP-2100 is a non-pipelined machine, it incurs no penalties for jumps and calls.

The ADSP 21XX Register groups are :

1. **Universal registers :**

   Register File (d reg)
   Data Address Generator
   Bus Exchange (Px)
   Timer
   System Registers (s reg)
   Program Sequencer

2. **Computation unit registers :**

   Multiplier Results (MR)

3. I/O processor register :

Memory mapped

Universal registers :

Universal registers are accessible to other universal Registers as well as to data memory.

Nearly every register in the core processor of the ADSP-21k Family is classified as a universal register

Data register file features :

- For transferring data between Computation Units and Memory, And for storing intermediate results
- Unconstrained data flow between Computation Units and Memory
- Transfers 9 words per cycle:
  - 2 memory transfers
  - 4 registers reads, 3 register writes with Computation Units
- 16 registers by 40 bits wide for Extended Floating-Point, or by 32 For Single-Precision Floating-Point and Fixed Point storage
- 16 additional Secondary Registers for Fast Context Switch
- Instruction set allows maximum data flow flexibility, i.e.. Radix-4 FFT butterfly.

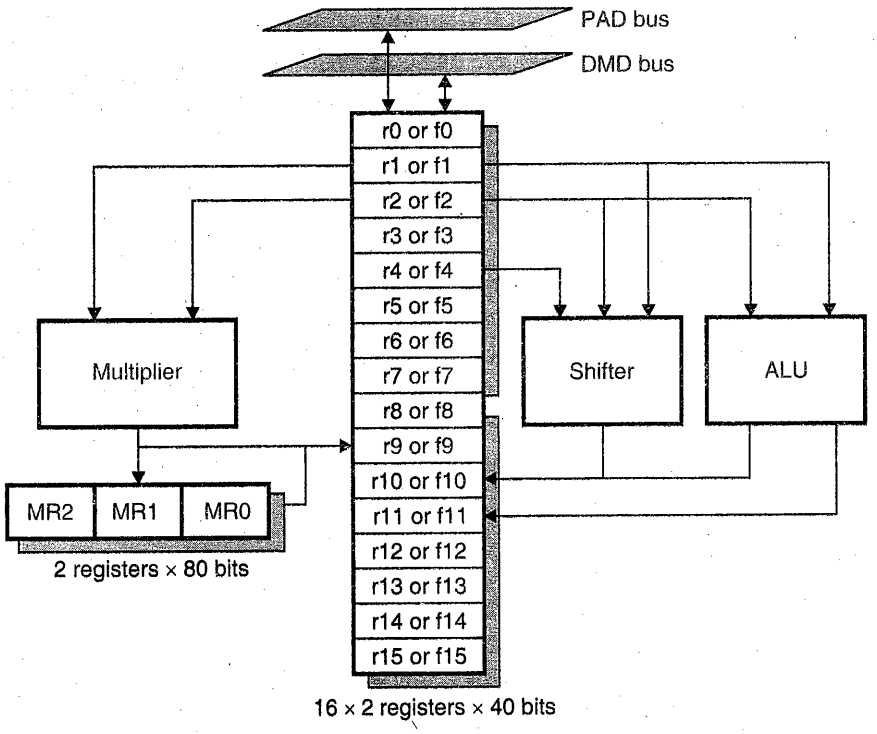


Fig. M-6 : ADSP-21000 register file

Data types :

- Floating Point : "f" register number prefix (f0 - f15)
- Fixed-Point : "r" register number prefix (r0 - r15)

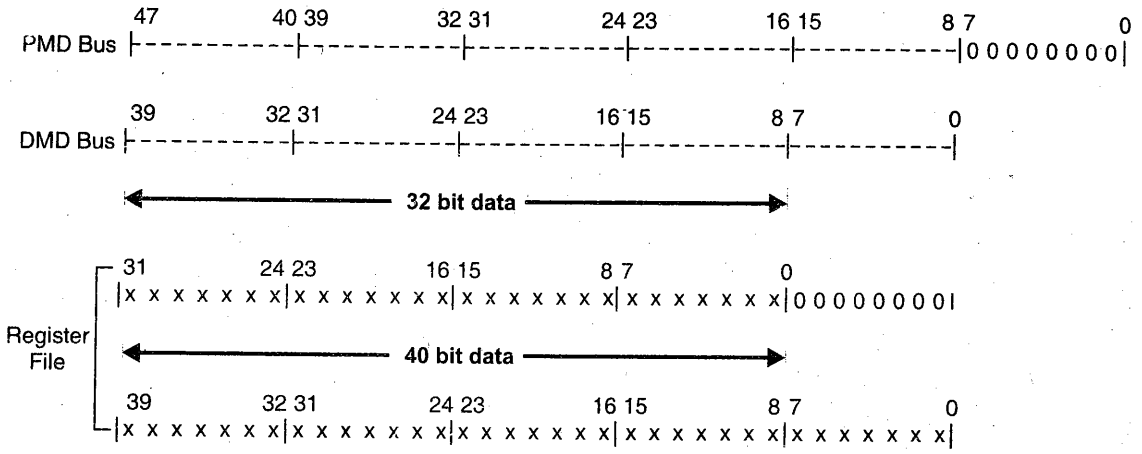


Fig. M-7 : Data alignment for data registers

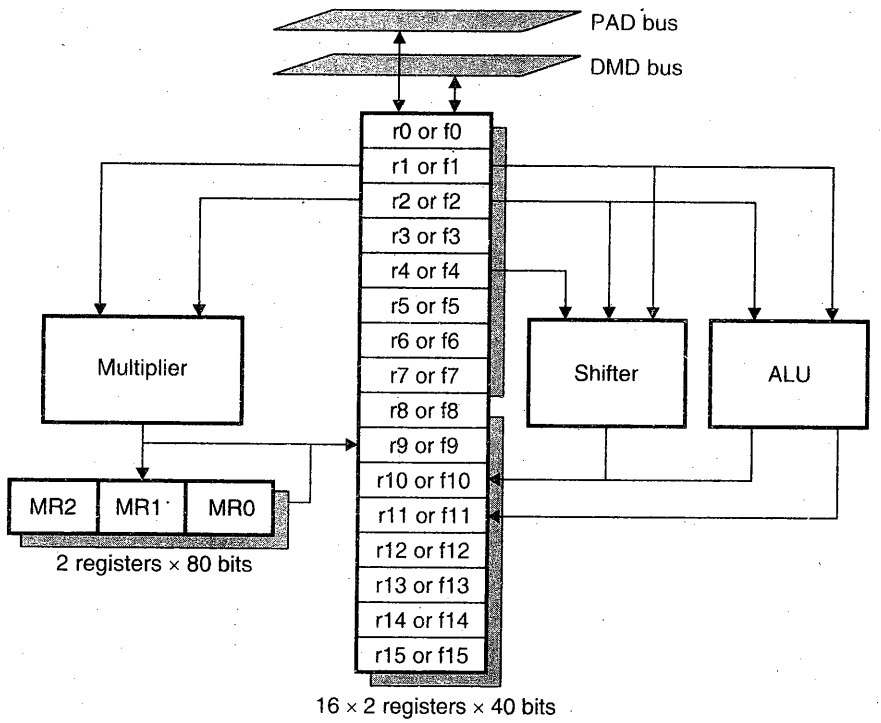


Fig. M-8 : ADSP - 21000 ALU

1.3.1 Arithmetic Logic Unit (ALU) :

ALU Features/ Summary :

- Sample ALU instructions
  (32/40-bit Floating/Fixed-Point operations):
  - A+B, A – B
  - Simultaneous ADD/SUB: A ± B
  - Absolute value |A|
  - Average: [A+B]/2
  - Logical: AND, OR, XOR, NOT
  - MIN / MAX of any two register values
  - Saturate A by |B|
  - Extract exponent or mantissa; modify exponent
  - Scaled conversions between all formats
- 8-bit accumulated status for sequential compares

  The ALU generates 6 status Flags :

  AZ - ALU result zero or floating-point underflow

  AV- ALU overflow

  AN- ALU result negative

  AC -ALU fixed-point carry

  AS -ALU X input sign (ABS and MANT operations)

  AI -ALU floating-point invalid operation

  AF -Last ALU operation was a floating-point operation

  CACC -Compare Accumulation register (results of last eight compare operations)

Theory of ALU :

Performance of ALU is some as any processors performing logical as well as arithmetic functions.

- It is 16 bit wide.
- It has two input ports X and Y
- It has one output port R
- ALU generates six status signals and one CI (carry In) bit which are stored in ASTAT (arithmetic status) register.

Data from PMD or DMD Bus are taken by AY through MUX, this AY register file consist of two 16 bit register AY0 and AY1 and the output of this AY register file in dual ported one port to give data to MUX and other port to send data to DMD Bus data from this AY register file goes to Y input port of ALU through MUX. Y input port can also accept data from AF register.

The AF register gets data from the output port 'R' of ALU and can be called also as feedback register.

Data from DMD Bus goes to AX registers, unlike AY registers which also takes data from PMD Bus. AX registers also has two 16 bit registers, AX registers are also dual ported, it gives data to X input port of ALU as well as can drive DMD Bus simultaneously X input port gets data from MUX and the data to MUX either by from AX registers or from R Bus.

The output port of ALU 'R' port gives data to MUX, which is at output port and also to AF register (which in turn goes to MUX at Y input port).

MUX at 'R' port gets data either from 'R' port or from DMD Bus and gives to AR register AR register output connected to DMD Bus and R-Bus.

The extra registers present in AX, AY and AF registers are used in case of interrupt service routines otherwise only one set of register are normally used. This increases the speed of operation.

ALU Conditions are as follows :

- EQ-ALU equal zero
- LT-ALU less than zero
- LE-ALU less then or equal zero
- AC-ALU carries
- AV-ALU overflow
- NE-ALU not equal to zero

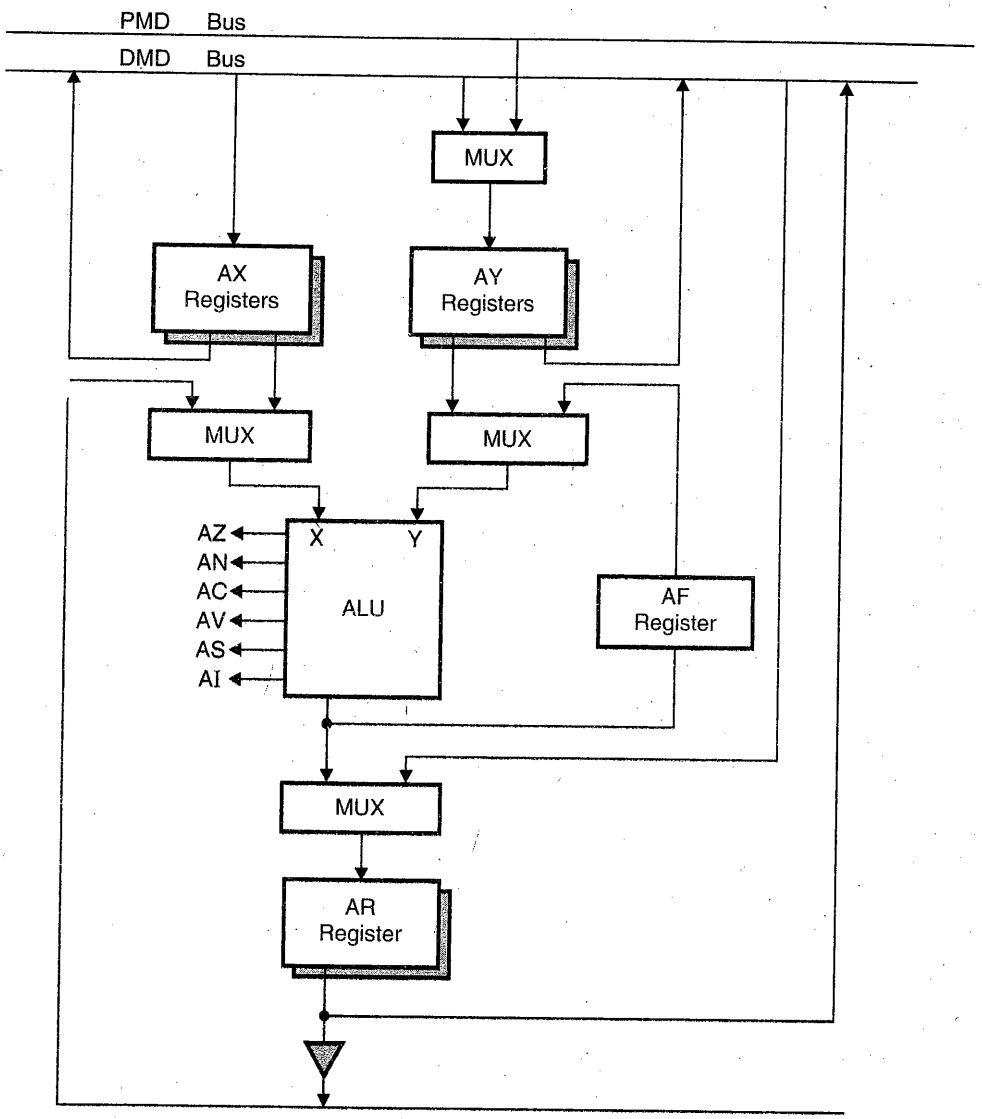


Fig. M-9 : Block diagram of ALU

- GE-ALU greater than or equal to zero
- GT-ALU greater than zero
- NOT AC-Not ALU carry
- NOT AV-Not ALU overflow

1.3.2 Multiplier - Accumulator :

Multiplier/MAC features :

- 32/40-bit Floating-Point Multiplier
  - Multiplier used with ALU for 1-cycle pipelined MAC
  - 32/40-bit floating-point multiplication
    40-bit results, optional round to 32
- 32-bit Fixed-Point MAC
  - 64-bit results, integer and fractional
  - 80-bit dual accumulators
  - 32-bit results (stored to register file)

Theory of MAC :

Multiplier Accumulator MAC : MAC is used for high speed multiplication.

- Has two 16 bit input ports 'X' and 'Y'.
- Has one 32 bit output port 'P'.

Data from PMD Bus or DMD bus first come to MY register through MUX, MY registers has two 16 bit registers and are dual ported, one goes to Y input of MAC through MUX and other drives DMD bus simultaneously.

MUX at the 'Y' input port takes data either from MY registers or from MF register. MF register is 16 bit register which gets data from 'R1' output port of ADDER-SUB.

DAta from DMD Bus goes to X port of multiplier through MX register MX registers have two 16 bit register and is again dual ported one goes to 'X' port through MUX while second drives DMD Bus.

MUX at the 'X' input port gets data from MX registers or from R- Bus.

Data from output port 'P', which is 32 bit goes to Adder Subtractor. Which in turn has three output R_0, R_1 and R_2 and also generates status bit MV (multiply overflow) which is set when accumulator overflows beyond 32 bit and also takes output of MR register.

R_0, R_1 and R_2 output ports of Adder-subtractor gives data to MR0, MR1 and MR2 registers respectively MR0, MR1 are 16 bit while MR2 in 8 bit wide. There register can also be loaded through MUX from DMD bus and can also give data to DMD bus again through MUX.

Output of MR registers goes to ADDER-SUB, DMD Bus and R-Bus.

Multiplier status flags are as follows :

ASTAT

MN - Multiplier result negative
MV - Multiplier overflow
MU - Multiplier underflow
M I - Multiplier floating-point invalid operation

STKY

MOS - Multiplier fixed-point overflow

MVS - Multiplier floating-point overflow

MUS - Multiplier underflow

MIS - Multiplier floating-point invalid operation

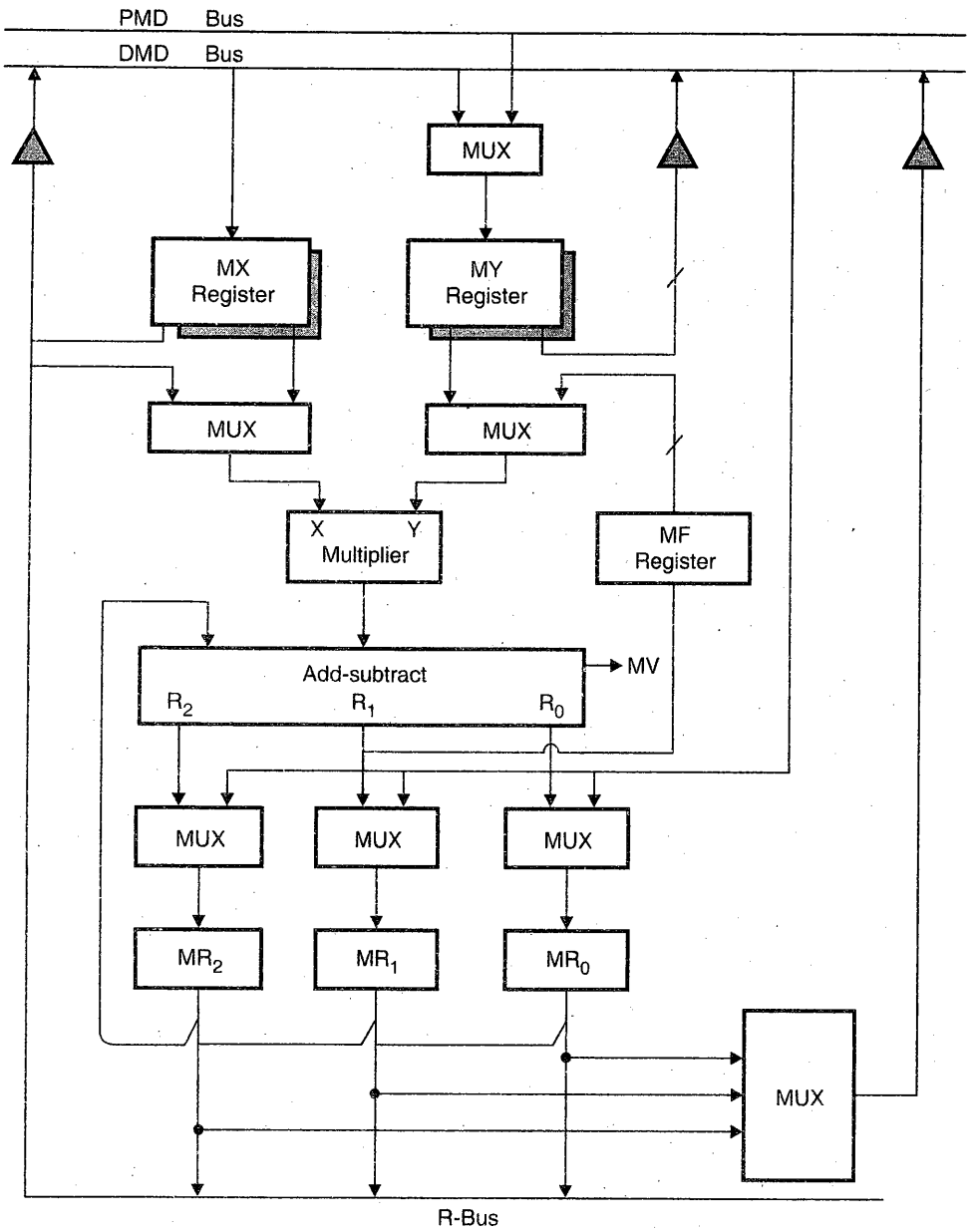


Fig. M-10 : Block diagram of MAC

1.3.3 Barrel Shifter :

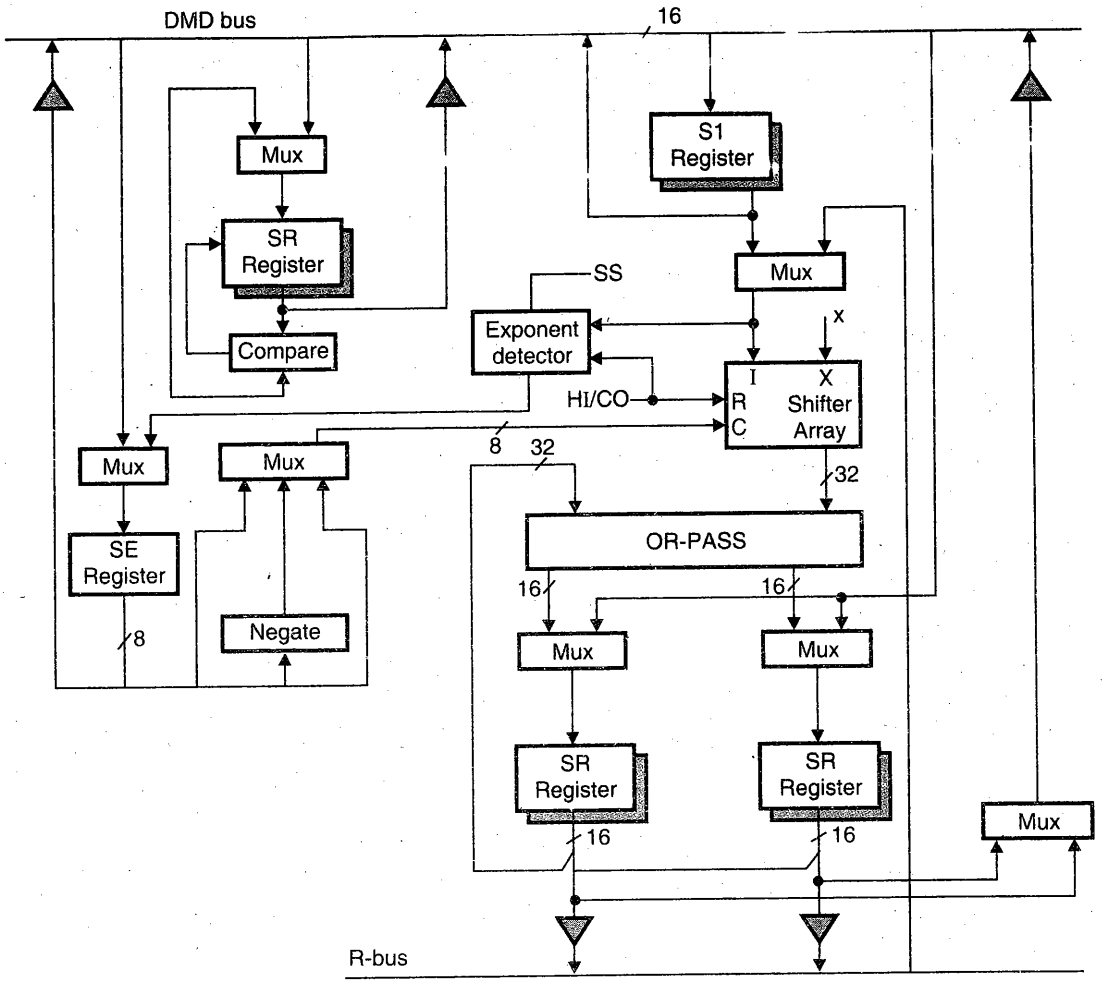


Fig. M-11 : Block diagram of barrel shifter

ADSP-21k 32-bit barrel shifter features :

- Arithmetic and logical shifts
- Word rotates
- Field manipulations (extract, deposit)
- Bit set / clear / toggle / test
- Exponent extract for 2100 family fixed-point compatibility

Theory of barrel shifter :

As shown in the Fig. M-11 shifter has four main components. They are : Shifter array, OR/PASS logic, exponent detector and exponent compare logic.

The shifter array is 16 by 32 bit barrel shifter. This barrel shifter accepts 16 bit input and places anywhere in the 32 bit output field in a single cycle. the control code (C) and H1/L0 are reference signals given to the barrel shifter, thus controlling the shifting operations.

The data to be shifted is given to the shifter on it's I (input) port. This I input port can take data from R-bus or shifter input (SI) register through MUX. The SI register can read or write data to DMD bus. The R bus contains information from other computational units. This data can be given to the shifter array immediately at the beginning of next cycle.

The 'X' input to the shifter is called the extension bit. It is used to pad 0's and 1's in the shifted data.

The shifted sections of multiprecision numbers are combined into single quantity by OR/PASS logic.

The 32 bit output of the OR/PASS logic is written into the SR register file through MUX.

The SR register file consists of two 16 bit registers SR0 and SR1. These registers can also be loaded directly from DMD bus through MUX. The contents of SR register can also be given as input to OR/PASS logic.

When OR is selected then contents of SR registers are ORed with output of shifter array in the OR/PASS logic. When PASS is selected the contents of the shifter array are not modified by OR/PASS logic and is written directly into the SR registers. The contents of the SR registers can be given to OR-PASS logic, written to DMD and R bus.

The shifter exponent (SE) has 8 bits. It holds the exponent during normalize and denormalize operations. The SE register can read or write to lower 8 bits of DMD bus.

The shifter block (SB) register has 5 bits. It holds the block exponent value in floating point operations. This register can also read or write to lower 5 bits of DMD bus.

The exponent compare logic is used to determine the largest exponent value in the array of shifter input values.

The exponent detector derives an exponent for the shifter input value.

1.3.4 Data Address Generators :

There are two DAG's in the architecture of ADSP 21XX. Because of these two DAG's the program memory and data memory can be accessed simultaneously thus increasing the speed of operation.

The DAG's perform following functions :

- DAG1 provides optional bit reversal capability
- Automatic address modification
- Provide indirect addressing capability
- Modulo address modification for addressing circular buffers
- Independent and simultaneous address for both program and data

The DAG1 only generates address for the data memory. It provides optional bit reversal capability. In FFT algorithms this bit reversal is required.

The DAG's have three registers files :

1) Modify (M) register file
2) Index (I) register file
3) Length (L) register file

Each of the above register files contains four 14 bit registers. These registers can read and write on to the DMD bus.

1.3.5 Program Sequencer :

The program sequencer performs the following functions :

1) It generates addresses for the instructions
2) It provides flexible control of the program flow
3) It gives sophisticated interrupt processing
4) Conditional as well as unconditional zero overhead looping is provided
5) Similarly conditional as well as unconditional single cycle branching is also provided

The program sequencer has six status registers as follows :

a. Arithmetic status register (ASTAT)
b. Stack status register (SSTAT)
c. Mode status register (MSTAT)
d. Interrupt control register (ICNTL)
e. Interrupt mask register (IMASK)
f. Interrupt force and clear register (IFC)

1.3.6 Cache Memory :

In computer science it is a short-term memory in a computer with quick access. A cache is intended to speed up access to a set of data. The cache will be a piece of memory that is faster (hence more expensive, hence smaller) than the principal data storage area for the data in question. The cache operates by storing a part of the data, allowing that part to be accessed more quickly. A speed-up is achieved if many accesses to the data can access the data in the cache. The reason caches work at all is that many access patterns in typical computer applications have locality of reference. There are several sorts of locality, but we mainly mean that often the same data is accessed frequently or with accesses that are close together in time, or that data near to each other are accessed close together in time. Caches are also used between the CPU and the (slower) main RAM in modern PC. There are often several levels of cache, with different sizes and access speed. Generally the caches get smaller, faster and more expensive, the closer to the CPU they are.

Instruction cache :

It is used to store previous 16 executed instructions. The instruction fetched for the instruction is also stored in the instruction cache.

Cache memory monitor :

The cache memory monitor keeps track of program memory addresses of instruction stored in the cache memory.

1.4 Features of ADSP 21xx Family of Processors :

1. 16 bit fixed point DSP microprocessors
2. Enhanced Harvard architecture for three bus performance
3. Integrated timer
4. Low profile IDLE instruction
5. Separate on chip bus for program and data memory
6. Three edge or level sensitive interrupts
7. Dual data address generator with modulo and bit reverse addressing
8. On chip program and data memories that can be extended off chip
9. Independent three computational units-ALU, MAC and shifter
10. Automatic booting of on chip memory
11. Single cycle instruction execution true instruction cycle
12. Efficient program sequencing with zero overhead looping, single cycle loop set up.
13. 25 MIPS, 40ns Maximum Instruction rate.
14. Double-Buffered Serial Ports with Companding Hardware, Automatic Data Buffering and Multichannel operation.

1.5 Instruction Set of ADSP 21xx Series Processor :

The ADSP-21XX assembly language uses an algebric syntax for ease of coding and readability. The sources and destinations of computations and data movements are written explicitly in each assembly statement, eliminating cryptic assembler mnemonics. Every instruction is a single 24-bit word and executes in a single cycle.

There are five basic categories of instructions :

1. Data move instructions
2. Computational instructions
3. Multifunction instructions
4. Program flow control instructions
5. Miscellaneous instructions

The instruction set is summarized below,

Notation conventions :

Ix	Index registers for indirect addressing
My	Modify registers for indirect addressing
< data >	Immediate data value.
< addr >	Immediate address value.
< exp >	Exponent (shift value) in shift immediate instructions (8-bit signed number)
< ALU >	Any ALU instruction (except divide)
< MAC >	Any multiply-accumulate instruction
< SHIFT >	Any shift instruction (except shift immediate)
cond	Condition code for conditional instruction.
term	Termination code for DO UNTIL loop.
dreg	Data register (of ALU, MAC or shifter)

reg Any register (including dregs)
; A semicolon terminates the instruction.
, Commas separate multiple operations of a single instruction.
[ ] Optional part of instruction.
[, ...] Optional, multiple operations of an instruction.
option1|option2 List of options; choose one.

ALU instructions :

[IF cond] AR|AF	= xop + yop [+ C];	Add/Add with carry
	= xop – yop [+ C – 1];	Subtract X – Y/Subtract X – Y with Borrow.
	= yop – xop [+ C – 1];	Subtract Y – X/Subtract Y – X with Borrow.
	= xop AND yop;	AND
	= xop OR yop;	OR
	= xop XOR yop;	XOR
	= PASS xop;	Pass, Clear
	= – xop;	Negate
	= NOT xop;	NOT
	= ABS xop;	Absolute value
	= yop + 1;	Increment
	= yop – 1;	Decrement
	= DIVS yop, xop;	Divide
	= DIVQ xop;	

MAC instructions :

[IF cond] MR|MF=	xop * yop;	Multiply
	= MR + xop * yop;	Multiply/Accumulate
	= MR – xop * yop;	Multiply/Subtract
	= MR;	Transfer MR
	= 0;	Clear
IF MV SAT MR;		Conditional MR saturation

Shifter instructions :

[IF cond] SR	= [SR OR] ASHIFT xop;	Arithmetic Shift
[IF cond] SR	= [SR OR] LSHIFT xop;	Logical Shift
SR	= [SR OR] ASHIFT xop BY < exp >;	Arithmetic Shift Immediate
SR	= [SR OR] LSHIFT xop BY < exp >;	Logical Shift Immediate
[IF cond] SE	= EXP xop;	Derive Exponent
[IF cond] SB	= EXPADJ xop;	Block Exponent Adjust
[IF cond] SR	= [SR OR] NORM xop;	Normalize

Data move instructions :

reg = reg;	Register-to-Register Move.
reg = < data >;	Load Register Immediate.
reg = DM (< addr >);	Data Memory Read (Direct Address)
dreg = DM (Ix, My);	Data Memory Read (Indirect Address)
dreg = PM (Ix, My);	Program Memory Read (Indirect Address)
DM (< addr >) = reg;	Data Memory Write (Direct Address)
DM (Ix, My) = dreg;	Data Memory Write (Indirect Address)
PM (Ix, My) =dreg;	Program Memory Write (Indirect Address)

Multifunction instructions :

< ALU > | < MAC > | < SHIFT >, dreg =dreg;	Computation with Register-to-Register Move.
< ALU > | < MAC > | < SHIFT >, dreg = DM (Ix, My);	Computation with Data Memory Read
< ALU > | < MAC > | < SHIFT >, dreg = PM (Ix, My);	Computation with Program Memory Read
DM (Ix, My) = dreg, < ALU > | < MAC > | < SHIFT >;	Computation with Data Memory Write
PM (Ix, My) = dreg, < ALU > | < MAC > | < SHIFT >;	Computation with Program Memory Write
dreg = DM (Ix, My), dreg = PM (Ix, My);	Data and Program Memory Read
< ALU > | < MAC >, dreg = DM (Ix, My), dreg = PM (Ix, My);	ALU/MAC with Data and Program Memory Read.

Program flow instructions :

DO < addr > [UNTIL term];	Do Until Loop
[IF cond] JUMP (Ix);	Jump
[IF cond] JUMP < addr >;	
[IF cond] CALL (Ix);	Call Subroutine
[IF cond] CALL < addr >;	
IF [NOT] FLAG_IN JUMP < addr >;	Jump/Call on Flag In Pin
IF [NOT] FLAG_IN CALL < addr >;	
[IF cond] SET|RESET|TOGGLE FLAG_OUT [, ...];	Modify Flag Out Pin
[IF cond] RTS;	Return from Subroutine
[IF cond] RTI;	Return from Interrupt Service Routine
IDLE [(n)];	Idle

Miscellaneous instructions :

NOP;		No Operation
MODIFY (Ix, My);		Modify Address Register
[PUSH STS] [, POP CNIR] [, POP PC] [, POP LOOP];		Stack Control
ENA|DIS	SEC_REG [, ...];	Mode Control
	BIT_REV	
	AV_LATCH	
	AR_SAT	
	M_MODE	
	TIMER	
	G_MODE	

Assembly code example :

The following example is a code fragment that performs the filter tap update for an adaptive filter based on a least-mean-squared algorithm.

```
        MF = MX0 * MY1( RND ), MX0 = DM (I2, M1);        { MF = error * beta }
        MR = MX0 * MF ( RND ),  AY0 = PM (I6, M5);
        DO adapt UNTIL CE;
            AR = MR1+AY0, MX0 = DM (I2, M1), AY0 = PM (I6, M7);
adapt :     PM (I6, M6); = AR, MR = MX0 * MF (RND);

        MODIFY (I2, M3);                                 { Point to oldest data }
        MODIFY (I6, M7);                                 { Point to start of data }
```

1.6 Texas Instruments TMS 320 C5X :

The TMS320C5x, which is source-code-compatible with the C2x, operates as fast as 100 MHz with a 20-nsec instruction cycle. The static CMOS TMS320C5x is both an accumulator-and a register-based processor. It has a fixed-point multiply-accumulate (MAC) circuit with are registered 16×16-bit multiplier loading a 32-bit product register. The product register, in turn, feeds a 32-bit accumulator without guard bits. The C5x also has two parallel functional units feeding off the data bus: an independent ALU with are register file of eight auxiliary registers and a bit-manipulation, or parallel-logic, unit (PLU). Multiply and accumulate take one cycle each. The basic MAC cycle involves putting a value into a temporary register, fetching a second value, multiplying into a holder register and accumulating the result in the next cycle. However, if the TMS320C5x executes a MAC instruction within a hardware loop, the DSP can achieve single-cycle execution. For single-instruction cycle context switching, the C5x has a separate one-deep shadow-register stack for the major registers (accumulator, accumulator buffer, product and status registers, three temporary registers, index register and Auxiliary compare register). For control applications that need bit manipulation, the PLU runs in parallel with the MAC and ALU circuits. The PLU operations can set, clear, test, or toggle multiple bits in a control/status register or a data-memory location without altering the accumulator contents. The C5x also has 0-to 16-bit left- and right-data barrel shifters .A power-down mode minimizes power by shutting down the CPU or the CPU and the peripherals. Pulling down the Hold pin can also force the chip into power-down mode. An interrupt brings the chip up to normal run conditions .The C5x integrates a synchronous, double-buffered serial port that operates as fast as 12.5 Mbps with independent transmit and receive sections. The time-division multiplexed (TDM) serial port has all of the same features as the standard serial port, yet TDM features make the serial port better for inter processor communication. The buffered serial port operates as fast as 40 Mbps without CPU intervention . The 8-bit, parallel host-port interface interfaces a host processor to the C5x.

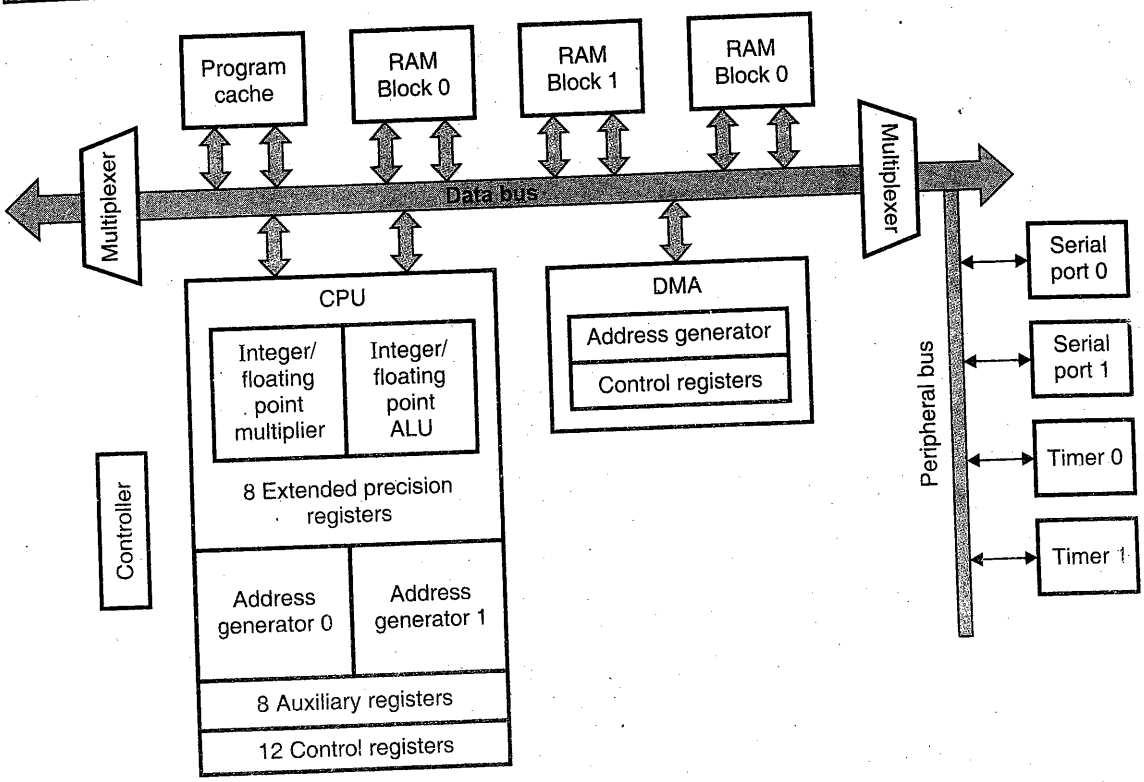


Fig. M-12 : Block diagram of TMS 320 C5X

Addressing modes :

The C5x sup-ports paged-memory direct addressing ,in which 7 bits in instruction concatenate with a 9-bit data-page pointer for accessing data RAM (128 words each page).It also supports indirect, immediate, dedicated-register and memory-mapped-register addressing. The processor supports automatic circular-buffer addressing for two buffers. The addressing mechanism supports buffer wraparound if the address-generation unit steps on the end of the buffer but not on over-shoot.

Special instructions :

The C5x sup-ports single and block repeat, load T(multiply) register, and accumulate previous product; load T register, accumulate previous product and move data; multiply and accumulate; multiply and accumulate previous product; square and accumulate; square and subtract previous product; call subroutine indirect; block move with repeat instruction and program to data, data-to-data memory; table read/write and test and manipulate bit in memory.

Support :

The C5x has a JTAG port for chip testing and in-circuit-emulator-like debugging control and monitoring .TI supplies a DSP starter kit, an evaluation module, and an emulator based on the C5x's built-in emulation logic. The EVM is a PC plug-in card that includes a TMS32C50 DSP with 10k words of on-chip RAM, 2k words of on-chip ROM, and 64 k bytes of external SRAM. The EVM uses embedded scan-based emulation via the onboard test-bus-controller IC. The debugger software can run and halt the C5x and scan in and out all of the bits of each register. TI supplies a

C compiler, a source-level C assembler/debugger ,an assembler/linker, a simulator, a profiler and an application library .Third-party hardware and software tools are also available.

Note : TMS320C5X is compatible with 'C1X', 'C2XX', 'C54XX'. So it has downward compatibility.

1.7 Comparison of General Purpose Microprocessor with DSP Processors :

Sr. No.	Parameter	DSP processor	Microprocessor
(1)	Instruction cycle	Instructions are executed in single cycle of the clock	Multiple clock cycles are required for execution of one instructions
(2)	Instruction execution	Parallel execution is possible	Execution of instructions is always sequential
(3)	Operand fetch and memory	Multiple operands are fetched simultaneously	Operands are fetched sequentially
(4)	Memories	Separate data and program memories	Normally no such separate memories are present
(5)	On chip/off chip memories	Program and data memories are present on chip and extendable off chip	Normally on chip cache memory present. Main memory is off chip
(6)	Program flow control	Program sequencer and instruction register takes care of program flow	Program counter takes care of flow of execution
(7)	Queuing/pipelining	Queuing is implicate through instruction register and instruction cache	Queuing is performed explicate by one queue registers for pipelining of instructions
(8)	Address generation	Address are generated combine by DAG's and program sequencer	Program counter is incremented sequentially to generate address
(9)	Address/data bus multiplexing	Address and data bus are not multiplexed. They are separate on chip as well as off chip	Address/data bus maybe separate on chip but are multiplexed off chip
(10)	Computational units	Three separate computational unit:ALU, MAC and Shifter	Only one main unit:ALU
(11)	On chip address and data buses	Separate address and data buses for program and data memory i.e. DMA, DMD, PMD, PMA and R bus	Address and data bus are the two bus on the chip
(12)	Addressing modes	Direct and indirect addressing modes	Direct, indirect, register,register indirect, immediate etc
(13)	Suitable for	Array processing operations	General purpose processing

Theory Questions

Q. 1 What are the differences between DSP and RISC processors ?

Q. 2 Discuss salient features of architecture for TMS 320XX/ADSP 21XX series DSP processor and functional blocks.

Q. 3 How fetch and execute instruction is carried out in the simplified harvard architecture ?

Q. 4 State the desired features of special purpose DSP processor.

Q. 5 Explain the following features of a typical DSP chip.

(i) ALU

(ii) MAC

(iii) Barrel shifter

(iv) Program sequencer.

Q. 6 Explain distinct features of DSP processor.

Q. 7 Describe architecture of the microprocessor used to carry out DSP applications.

Q. 8 Compare the features of DSP processor and a microprocessor with respect to architecture.

Q. 9 What are desirable features of DSP processors ? Compare architectural features of DSP and microprocessors.

Q. 10 Discuss features of DSP TMS 320C2X/C3X.

Q. 11 Explain the significance of the following :

(i) Barrel shifter

(ii) MAC

(iii) ALU

❑❑❑

Applications of DSP

Section No.	Name of the Topic
1.1	Introduction
1.2	Application of DSP in Image Processing
1.3	Speech and Voice Processing
1.4	Echo Cancellation
1.5	Reverberation
1.6	Oversampling A/D Converter
1.7	Oversampling D/A Converter
1.8	Applications of Musical Sound Processing
1.9	Applications of Multi Rate Signals
1.10	Applications of DSP in Vibration Analysis
1.11	Applications of DSP in Bio-medical Engineering

• Code : DSP14

1.1 Introduction :

In this chapter we will discuss the different application areas of digital signal processing. The major application areas of DSP are as follows :

(1) Image processing
(2) Instrumentation and control
(3) Speech and audio signals
(4) Biomedical
(5) Telecommunication

1.2 Application of DSP in Image Processing :

Image processing is the major application in DSP. Whenever an image is digitized, we get digital image. Digital images are in the form of two dimensional arrays say 128×128, 256×256 and so on. These 2-D arrays are further processed by DSP processor. Each array element is called as pixel.

Processing of image takes place in three steps. They are :

(A) Image enhancement
(B) Image restoration
(C) Image coding

Let us take a look on each one by one.

(A) Image enhancement :

In image enhancement, following features of image are focussed :

(1) Image sharpness
(2) Filtering
(3) Histogram
(4) Contrast enhancement
(5) Edge enhancement : Here, lower frequency components of image are attenuated.

(B) Image restoration :

A new reconstruction of image is done under Image restorations. This is mainly affected by Noise and image gets blurred by movement of camera.

(C) Image compression :

Redundancy in the image can be reduced by Image compression. Moreover, compressing an image requires less memory size.

The various techniques of compression are :

(1) Transform coding
(2) Predictive signal coding
(3) Sub-band coding

1.2.1 Feature Extraction :

Feature Extraction consists of detecting particular features like edges, lines, etc. associated with the different objects in the given image.

1. Spatial features :

These features are measured by its grey levels and combined probability distribution, spatial distribution and amplitudes of physical properties of the object like its transmittivity, reflectivity etc. Common histogram features, measured over a small moving window, are mean, median, mode, dispersion, variance, mean square value, skewness, etc.

2. Transform features :

These features are obtained by filtering the input data through a zonal filter, refer Fig. N-1. These features are useful for frequency-domain analysis.

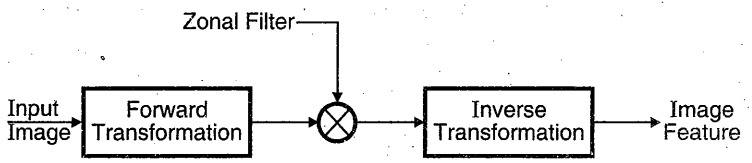


Fig. N-1 : Transform feature extraction

3. Edges and boundaries :

Edges are indicators of boundries. They are used for segmentation, registration and recognition of objects. Edge points are the pixels where the grey level changes suddenly. Edges in binary images are the black pixels having a minimum of one white nearest neighbour. Edge detection is done with edge detection operators such as,

- Gradient operators : which measure the gradient of image in two perpendicular direction.
- Compass operators : for gradient determination in a chosen number of directions.
- Laplacian operators : whose zero crossing enables edge detection.

Boundries are useful for calculating geometrical features like size and orientation. Boundries are detected using contour- following algorithms by ordering connective edge points.

4. Shape features :

- Shape Analysis-is employed for detecting and recognizing objects in an image.
- Shape Synthes-is applied to computer-aided design of parts of machines and for image simulation purpose in cartoon films, video games etc.

There are following methods of extracting shape features :

(i) Run-length codes :

The region occupied by an object is directly related to its shape. In run-length codes a sequence of alternating 1's and 0's is written which represents a binary image.

(ii) Quad-tree :

In this method, a region is confined in a rectangular area which is divided into four quadrants. These quadrants are then examined for black, white or grey levels.

(iii) Projections :

These represents two-dimensional shapes. A projection constitutes a histogram giving the number of pixels which project into a bin at a certain distance along an orientation direction.

5. Moments, structures and morphological processing :

Moment invariants are useful features for pattern recognition. It is an interesting field for representation of objects.

Objects like handwritten or printed text are often characterized by structures consisting of arcs or line patterns.

Morphological processing deals with operations in which an object is struck with a structuring elements to reveal its shape clearly. Two key operators viz. Erosion and Dilation operators are useful for studying the topology of objects from their images. The shape of the object is defined by Regenerative features like boundaries, moments, regions etc. and measurement features branching into Geometrical Features e.g. perimeter, area, etc. and moments such as centre of mass, orientation etc.

6. Texture :

Texture is the repetition of basic elements such as textels, each of which contains several pixels arranged periodically or in a random fashion. It is the local brightness variation from one pixel to another or within a small region. Texture is usually observed in surface patterns of wood, cloth, grain etc. It is divided into two classes,

(i) Statistical - random in nature,

(ii) Structural - comprising deterministic textels.

Textural features viz. coarseness are determined by linear filtering methods using image transforms. Two-dimensional histograms are very useful for texture studies.

1.2.2 Pattern Matching :

A pattern is essentially an arrangement. It is characterized by the order of the elements of which it is made, rather than by the instrinsic nature of these elements. A pattern class is a family of patterns that share some common properties. Recognition techniques based on matching represents each class by a prototype pattern vector. An unknown pattern is assigned to the class to which it is closest in terms of a predefined metric. There are following approaches of pattern matching,

1. Minimum distance classifier :

This approach computes the distance between the unknown and each of the prototype vectors. It chooses the smallest distance to give the result.

Suppose that we define the prototype of each pattern class to be the mean vector of the patterns of that class;

$$m_j = \frac{1}{N_j} \sum_{x \in \omega_j} X_j, \quad j = 1, 2, \ldots, W \qquad \ldots(1)$$

where,

N_j = Number of pattern vectors from class ω_j

W = Number of pattern classes.

The smallest distance implies the best match. Selecting the smallest distance is equivalent to evaluating the function,

$$d_j(X) = X^T m_j - \frac{1}{2} m_j^T m_j, \quad j = 1, 2, \ldots, W \qquad \ldots(2)$$

and assigning 'X' to class ω_i if $d_i(X)$ yields the largest numerical value.

Thus, the decision boundary between classes ω_i and ω_j for a minimum distance classifier is,

Form Equation (2),

$$\begin{aligned} d_{ij}(X) &= d_i(X) - d_j(X) \\ &= X^T (m_i - m_j) - \frac{1}{2} (m_i - m_j)^T (m_i - m_j) \\ &= 0 \end{aligned} \qquad \ldots(3)$$

The surface given by Equation (3) is the perpendicular bisector of the line segment joining m_i and m_j.

2. Pattern matching by correlation :

This approach can be formulated directly in terms of images. Here, we have to find matches of a subimage $\omega(x, y)$ of size $J \times K$ within an image $f(x, y)$ of size $M \times N$, here assumption is $J \leq M$ and $K \leq N$. Thus, correlation between $f(x, y)$ and $\omega(x, y)$ is,

$$c(x, y) = \sum_s \sum_t f(s, t)\, \omega(x + s, y + t) \qquad \ldots(4)$$

for $x = 0, 1, 2, \ldots, M - 1$

$y = 0, 1, 2, \ldots, N - 1$

and the summation is taken over the image region where ω and f overlap.

Fig. N-2 illustrates the procedure. Here assumption is that the origin of f is at its top left and the region of ω is at its centre. For one value of (x, y), let (x_o, y_o) inside f, using Equation (4), we have one value of c. As x and y are varied, ω moves around the image area, giving function $c(x, y)$. The maximum value of c indicates the position where ω best matches f.

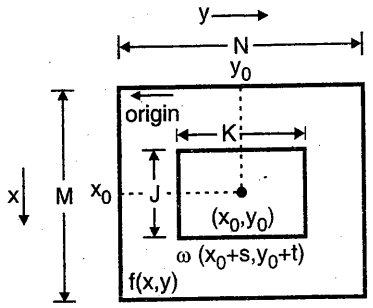


Fig. N-2 : Illustration of correlation of f and ω at point (x_0, y_0)

The correlation function given in Equation (4) has the disadvantages of being sensitive to changes in the amplitude of f and ω. To overcome this difficulty, matching via correlation :oeffient; given by Equation (5), is performed,

$$\gamma(x,y) = \frac{\sum_s \sum_t [f(s,t)-\bar{f}(s,t)][\omega(x+s,y+t)-\bar{\omega}]}{\left\{\sum_s \sum_t [f(s,t)-\bar{f}(s,t)]^2 \sum_s \sum_t [\omega(x+s,y+t)-\bar{\omega}]^2\right\}^{1/2}} \quad ...(5)$$

where, x = 0, 1, 2, ..., M − 1

y = 0, 1, 2, ..., y − 1

$\bar{\omega}$ = average value of the pixel in ω

$\bar{f}$ = average value of f in the region coincident with current location of ω.

1.3 Speech and Voice Processing :

Various applications of DSP in speech processing are as follows :

1.3.1 Acoustic Features of Speech :

- Human speech is generated through vocal cords and the vocal tract. The vocal cords, consisting of two bands of tough, elastic tissue, located at the opening of larynx, vibrate when the air from the lungs passes between them producing sound waves which are emitted from the lips and to some extent from the nose, and heard as speech.
- Speech sounds are of two types :

1. Voiced sounds :

There are produced by quasi-periodic pulses of air exciting the vocal tract. These pulses, in turn, are generated by forcing air through the epiglottis, with the tension of the vocal cords.

2. Unvoiced sound :

To produce these sounds, a constriction is formed at some point along the vocal tract, towards the mouth. Air is forced through through the constriction at a high speed resulting in turbulent flow. A broad-spectrum noise source is created which excites the vocal tract leading to the emission of a unvoiced sound.

Human sounds can be characterized in terms of three vital parameters as,

1. Pitch : related to the frequency of sound (in Hz),
2. Loudness : the physiological perception of the intensity of sound (in dB),
3. Quality : a property possessed by sound by virtue of its harmonic content.

Phonemes :

The smallest units of sound which can be recognized by contrast with their environment are called phonemes.

Dipones :

The sounds stretching from the middle of one phoneme to the centre of the next is termed as dipones.

Digitization of speech :

The acoustic energy of speech is converted into electrical energy by a electroacoustic transducer. First, the speech signal / waveform is passed through a low-pass filter restraining the components below a frequency W Hz. It is then sampled at a rate greater than 2W, quantised and converted into binary digits. Although the frequency W is determined by the intended application, it usually lies between 3 and 5 kHz. The sampling rate $\geq$ 8000 samples per second is adequate for digital representation of telephone bandwidth ~ 4 kHz speech signals. An encoding rate of 16 bits per second is used.

The digitzed signal is now ready for processing and various operations can be carried out on it, as demanded by the problem at hand.

1.3.2 Speech Recognition :

Speech recognition involves inputting of information into a computer using human voice. Computer then quantizes this information and then recognizes human speech.

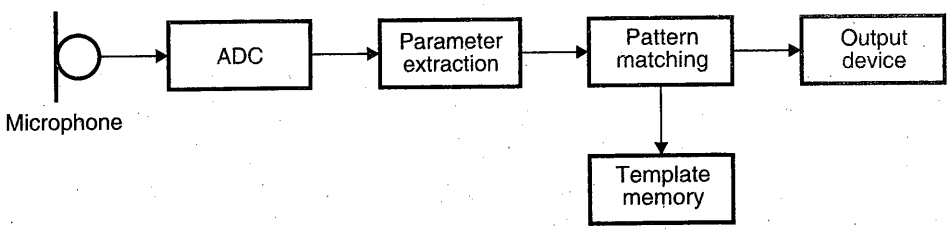


Fig. N-3 : Block diagram of speech recognition system

As shown in Fig. N-3, speech/voice is input through microphone. The input analog signal is digitized using A/D converter. This digitized output is stored in memory. Recognition process starts after storage. Then here again the spoken word is digitized and its template is compared with template of memory. As soon as the match occurs, this indicates that the word has been recognized and the same is informed to the user.

DSP processor plays an important role in extracting an important parameter called template from spoken word. The whole operation is done accurately by DSP processors.

Following performance parameters that greatly affect the above system too are overcome by DSP processor :

(1) Noise
(2) Microphone characteristics
(3) Pause taken between two words
(4) Correct pronunciation of word.

1.3.3 Speech Synthesis :

In this system, human voice is produced. To achieve this we use Linear Predictive Coding (LPC) technique.

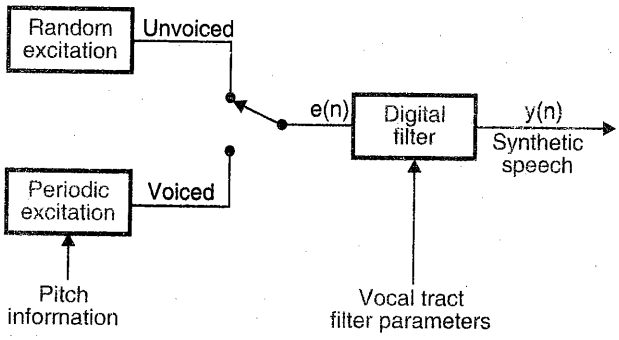


Fig. N-4 : Linear predictive coding of speech

In human speech, voiced sound and unvoiced sound are present. Voiced sound is nothing but the vowels. These represent the air flow through vocal cords. While unvoiced sound represents the noise created by forcing air in the vocal tract.

Unvoiced sound is generated by random excitation while voiced sound is generated by periodic excitation. Digital filters perfectly simulates the behaviour of the vocal tract. Thus, we get human speech.

1.3.4 Training Phase and Coding :

Coding algorithms seek to minimize the bit rate in the digital representation of a signal without an objctionable loss of signal quality in the process. Quality of speech refers to how natural it sounds as compared to the words uttered by a human being. High quality is attained at low bit rates by exploiting signal redundancy as well as the knowledge that certain types of coding

distortion are imperceptible because they are masked by the signal. Intelligibility and quality of speech are measured in terms of a performance index called Mean Opinion Score (MOS). To determine the MOS of a speech source, a large number of listeners are requested to rate the given sample of speech and allot marks as shown in Fig. N-5.

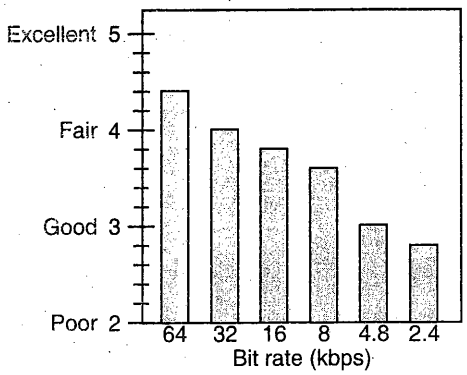


Fig. N-5 : The speech quality MOS for various bit rates

These numerical values, after averaging, yield the MOS score for the coder. For a high quality coder, the MOS values range between 4.0 and 4.5. Various Methods of speech coding are classified as follows :

(I) Waveform coding :

1. Pulse-Code Modulation (PCM),
2. Adaptive Pulse-Code Modulation (APCM),
3. Linear Predictive Coding :
   - (i) Differential Pulse-Code Modulation (DPCM),
   - (ii) Adaptive Differential Pulse-Code Modulation (ADPCM),
   - (iii) Delta Modulation (DM),
   - (iv) Adaptive Delta Modulation (ADM),
   - (v) Continuously Variable-Slope Delta Modulation (CVSDM).
4. Frequency-Domain Coding :
   - (i) Transform Coding (TC),
   - (ii) Adaptive Transform Coding (ATC),
   - (iii) Subband Coding (SBC).

(II) Base Band Coding (SBC)

(III) Narrowband Coding :

- (i) Pitch-Excited Coder,
- (ii) Optimal Scalar Quantisation (OSQ)
- (iii) Vector Quantisation (VQ)
- (iv) Segment Quantisation (SQ).

Let us take an overview of methods of speech coding.

(I) Waveform Coding :

Waveform coding aims at reproducing the speech waveform as faithfully as possible. The waveform coders are able to produce high-quality speech at high enough bit rates. Waveform coding includes :

1. Pulse-Code Modulation (PCM) :

PCM is the simplest coding system, a memoryless quantizer, and provides essentially transparent coding of telephone speech at 64 kbps. This waveform coding scheme employs uniform quantisation in which the range of signal value of x (n) is subdivided into small equal bins, each of width Δ, the step size, and all signal values falling within a bin are decoded to one value in that particular bin.

Dithering is used to weaken the signal dependent character of quantisation noise. In Dithering, a pseudorandom sequence is added to the signal prior to its quantisation, and subsequently at the receiver, the same sequence is subtracted from the decoder output yielding an almost white quantising error.

2. Adaptive Pulse-Code Modulation (APCM) :

Here the quantiser adjusts the step size according to the short- term amplitude of the signal. Fig. N-6 gives a schematic illustration of the Adaptive Pulse- Code Modulation system.

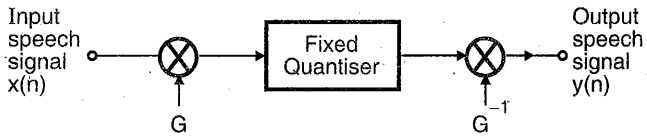


Fig. N-6 : APCM system

3. Linear Predictive Coding (LPC) :

In this scheme, the waveform coding is based on a linear prediction model of speech at any instant as a predicted value pulse an error term. For rates of 16 kbps and lower, high speech quality is achieved by using LPC.

The most familiar examples of LPC are as follows :

(i) Differential Pulse-Code Modulation (DPCM) :

In this system, predictor has the simple form as,

$$a(z) = 1 + a_1 z^{-1}$$

where a_1 is a fixed negative number calculated by determining the long-term average of the signal spectrum and computing the optimal a (z) for P = 1. Fixed predictors of higher order, P > 1, can also be used but beyond P = 4, the approach has not much utility.

(ii) Adaptive Differential Pulse-Code Modulation (ADPCM) :

In ADPCM, the predictor changes or modifies itself according to the need of signal. ADPCM provides high-quality speech at 32 kbps. The speech quality is slightly inferior to that of 64 kbps PCM. ADPCM at 32 kbps is widely used for expanding the number of speech channels by a factor of two, particularly in private networks and international circuits. It is also the basis of low-complexity speech coding in several proposals for personal communication networks.

(iii) Delta Modulation (DM) :

In DM, the waveform sampling rate is much larger than the Nyquist rate to increase the adjacent sample correlation, and the quantisation of the residual or difference signal into a one-bit or two-level strategy enabling the transmission of the difference only. Here, bit rate is equal to the sampling rate.

(iv) Adaptive Delta Modulation (ADM) :

In ADM, variable step size is used depending on the quantizer outputs. In a two-level quantizer observation of a single sample of quantiser output does not indicate slope overload distortion or granularity. So, a sequence of quantiser outputs is required for the desired step size adaptation.

(v) Continuously-Variable Slope-Delta (CVSD) Modulation :

CVSD modulation technique includes smooth adaptation of the step size with a time constant 5-10 ms. CVSD coders are more tolerant to channel errors and decrease the granular noise in output speech at the expense of a high slope overload distortion. CVSD coders give a clean-sounding speech at bit rates less than 24 kbps.

4. Frequency domain coding :

Frequency domain coders uses the non-flat short-term spectral characteristics of the speech signal, constituting the redundancy, and the signals at a level lower than the psychoacoustic threshold, which are irrelevant to the ear and are called the Irrelevancy.

Redundancy and irrelevancy removal in frequency domain coders is accomplished by decomposing the source spectrum into frequency bands containing uncorrelated spectral components of the signal. These components are quantized separately. At the receiver, the components are subjected to an inverse transformation to obtain the reconstructed signal.

These are two main forms of speech coding in the frequency domain as follows :

(i) Transform Coding (TC) :

TC is a frequency domain technique in which a block of input samples is taken and linearly transformed using DFT or DCT computation via a FFT algorithm. Then the transformation of the signal is efficiently coded by assigning bits to transform coefficients. At receiver, an inverse transformation is used to reconstruct the speech signal.

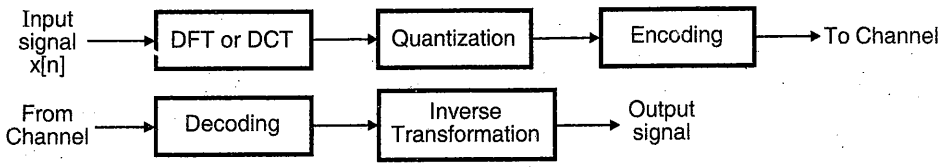


Fig. N-7 : Transform coding system

In TC, total number of bits available to quantize the transform coefficients remains constant whereas, in adaptive transform coding, the bit allocation to each coefficient changes from frame to frame. This dynamic bit allocation is controlled by the time varying characteristics of speech, which have to be transmitted as side information. The side information is also used to determine the step sizes of the various coefficient quantizer. The number of bits assigned to each transform coefficient is proportional to its corresponding spectral energy value.

(ii) Adaptive Transform Coding (ATC) :

In ATC, important thing is the determination of bit assignment or the number of bits used for quantising individual transform coefficient. In the presence of channel errors, the received side information may yield inaccurate bit assignment. This may lead to wrong decoding of the received bit stream into transform coefficients which spoils the quality of speech. Side information should therefore be safeguarded against channel noise.

(iii) Subband coding :

The speech signal is applied to an analysis filterbank consisting of a set of bandpass filters. This digital filtration divides the speech signal into a number of non-overlapping frequency band. By additive recombination of the set of subband signals, we can generate the original speech signal.

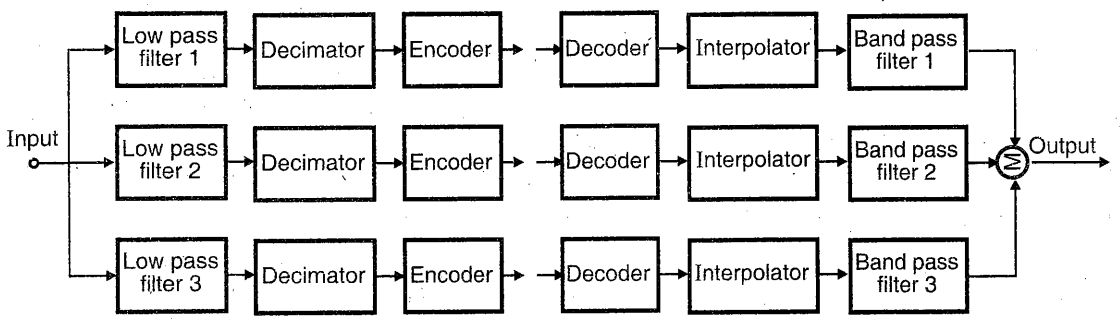


Fig. N-8 : Subband Coder

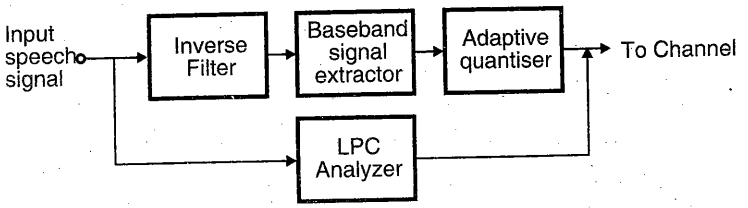
Each band of non-overlapping frequency is passed through a low- pass filter and decimated. Each band is separately quantized and coded using PCM, DPCM etc. and transmitted. In the receiver, the sampling rate of each band is raised to that of the source signal by inserting proper number of zero samples. Then, subband samples appear at the bandpass outputs of the synthesis

filter bank. The sum of bandpass outputs is equal to source signal without quantisation. Refer Fig. N-8.

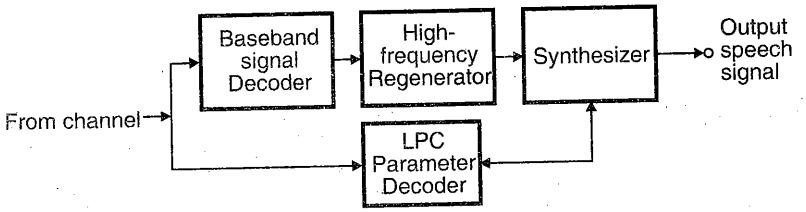
(ii) Baseband coding :

At bit rates < 12 kbps, the speech quality of waveform encoders falls down. If we wish to achieve higher speech quality at lower bit rates. We must consider input/output signal matching characteristic instead of quality resemblance of output speech with input speech.

In baseband coding, refer Fig. N-9, the input signal is inverse filtered to extract the baseband signal. The full signal is not transmitted, only a portion is sent for which a lower transmission rate suffices. In the receiver, the baseband signal is decoded.



(a) Transmitter



(b) Receiver

Fig. N-9 : BBC system

(iii) Narrowband coding :

Narrowband coding handles the speech signal with bit rate below 8 kbps. Main types of these coders include.

(i) Pitch-excited coder :

This method is based on a mixed source model of speech as shown in Fig. N-10. Here, a pulse source is used for exciting low frequencies. A white noise source, excites the high frequencies. The energy level of each frame is decided by a gain parameter. Generally, pitch is quantized to six bits and gain to five bits.

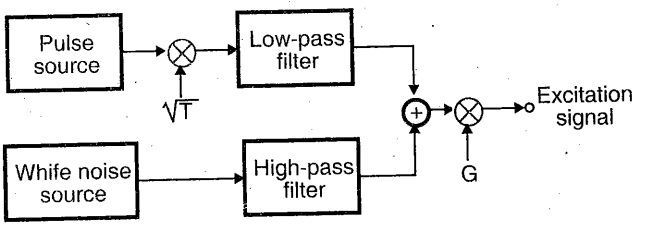


Fig. N-10 : Pitch-excited model

(ii) Optimal scalar quantisation :

This method exploits the correlation among a set of P parameters represented by a random vector $\vec{x}$ to be quantized to b bits, to decrease the bit rate. It is a three-step process,

- Parameter decorrelation : in which a new parameter vector $\vec{y}$ is formed using a matrix Q whose columns are the eigenvectors of the covariance matrix of random vector $\vec{x}$, as $\vec{y} = Q\vec{x}$.
- Bit allocation : in which the given b bits are assigned among the P components of $\vec{y}$.
- Scalar quantisation : in which each component y_i is quantized using b_i bits.

At the receiving station, the quantised vector $\vec{x'}$ is obtained from $\vec{x}$ by the inverse transformation

$$\vec{x} = Q\vec{y'}$$

(iii) Vector quantisation :

This method reduces the bit rate by statistical dependence among parameters beyond correlation. It is based on clustering procedure. Vector quantisers consume large amount of memory and computational space. They are used at low bit rates, one bit per parameter in the region where scalar quantisers are insufficient.

(iv) Segment quantisation :

Segment Quantisation utilizes interdependence between frames. Segments of a suitable size are obtained by dividing the speech into phoneme boundries. A diphone is the region separating the equilibrium state of one phoneme from that of the next. By warping, two segments can be made to posses an equivalent number of frames. While in Dynamic Time Warping, the distance between adjacent segments is minimized. By dynamic programming, the computational load is decreased.

1.4 Echo Cancellation :

Echo cancellation is achieved by using an hybrid circuit. This hybrid circuit provides impedance matching between two and four wire circuits.

Note : The subscribers are connected to telephone exchange by two wire circuit. The exchanges are connected by four wire circuits.

Fig. N-11 shows simplified long distance telephone circuit.

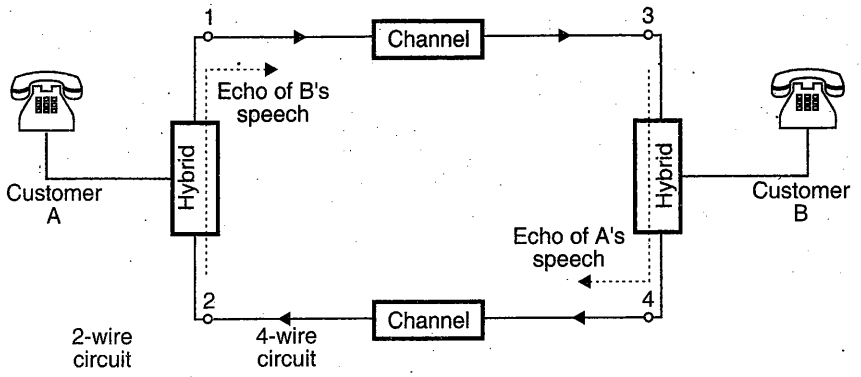


Fig. N-11 : Interface of two and four wire in telephone network

The hybrid circuit at exchange converts two wire circuit from subscriber to four wire and provides separate path for each direction of transmission.

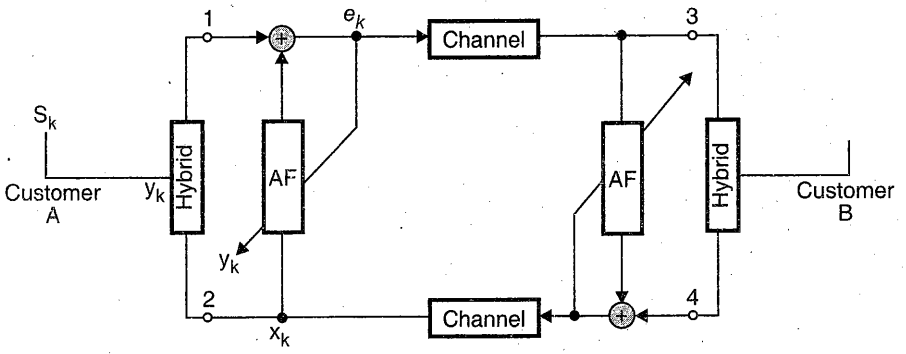


Fig. N-12 : Echo cancellation in long-distance voice telephony

Refer Fig. N-11. Speech signal from subscriber A travels along the upper transmission path to hybrid on right and from that to subscriber B. Same way speech signal from subscriber B travels along lower transmission path to subscriber A. Here one cannot exactly match impedances and therefore hybrid network allows some of incoming signal to leak into the output path and return to talker as echo. Problems arises when it is long distance call, in that one can listen echo after sometime is elapsed. To overcome this problem echo canceller should be present. The same is shown in Fig. N-12.

As shown in Fig. N-12, we have used adaptive filter (AF). Adaptive filter acts as an equaliser. It generates exact replica of the echo and subtracts the same from the signal in transmit path.

1.5 Reverberation :

Consider a music performed inside a room, such as concrete hall. In this case the sound waves propagate in all possible directions. These waves reach the listener from various directions and at various times. Some of these sound waves may fall on the walls of a room and get reflected.

Such reflected waves again reach the listener.

In such cases three types of sound waves reach the listener. These are as follows :

(1) Direct sounds
(2) Early reflections
(3) Reverberation

(1) Direct sounds :

As the name indicates; these are the sound waves coming directly to the listener. These sound waves are not reflected by the walls of a room. So these waves reach early compared to other sound waves and these waves reach at regular intervals of time.

(2) Early reflections :

Some of the sound waves get reflected from the walls of room. These waves are called as early reflections. Such sound waves are actually few number of closely spaced echoes. These waves reach the listener at irregular intervals of time.

(3) Reverberation :

Some of the sound waves, cause multiple reflections. So after the early reflection sound waves; more and more densely packed echoes reach the listener due to such multiple reflections. This group of echoes is called as reverberations.

The various types of echoes generated by a single sound source in a room are as shown in Fig. N-13.

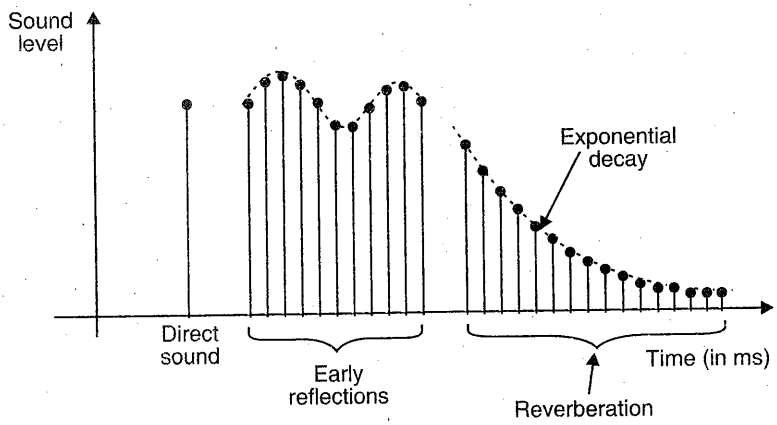


Fig. N-13

In case of reverberation attenuation in the sound level goes on increasing at each reflection. Thus amplitude of echoes decay exponentially with time as shown in Fig. N-13. The period of time in which the reverberation falls by 60 dB is called as reverberation time. Attenuation in an amplitude of sound waves takes place because of the absorption done by the material on which the waves are falling. Now absorption characteristic of different materials are not same at different frequencies thus the reverberation time varies from frequency to frequency.

Early reflections reach the listener, after certain delay compared to the direct sound waves. To make the recording in studio; such early reflection sound waves are artificially generated. Delay system with adjustable delay factors are used to generate early reflection waves. Such artificially generated early reflections are combined with electronically generated reverberation and are added to the recording made in studio. The block schematic of such a delay reverberation system is shown in Fig. N-14.

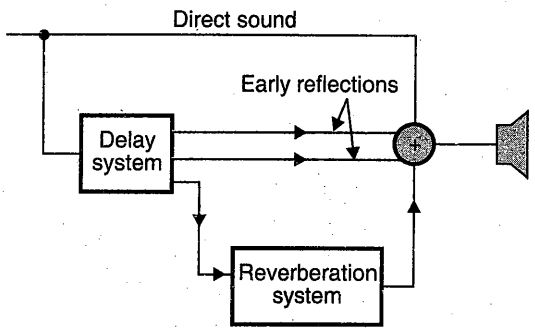


Fig. N-14

Application of electronic delay system :

It is possible to vary the localization of sound source from left speaker to the right speaker by using adjustable delay and gain control as shown in Fig. N-15.

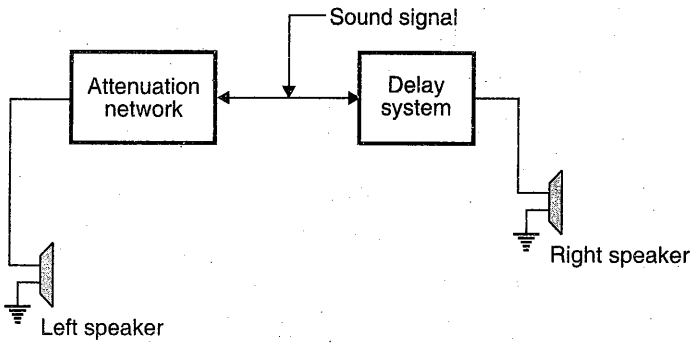


Fig. N-15

As shown in Fig. N-15, by keeping 0 dB loss in the left channel and few milliseconds delay in the right channel gives an impression of a localization of sound source at the left. Now if we lower the left channel signal level by few dB loss then the effect of sound source moving towards the center is observed. The degree of sound broadening can be obtained by phase shifting one channel with respect to other channel by using all pass filter. This scheme is shown in Fig. N-16.

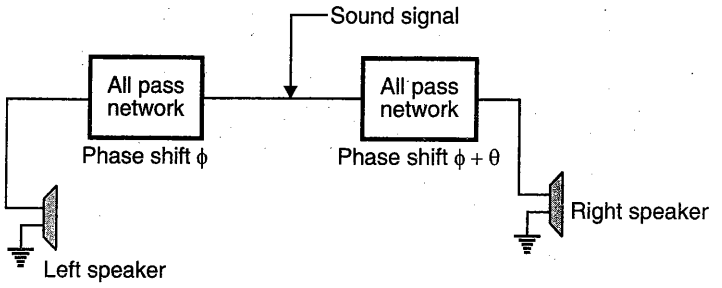


Fig. N-16

1.6 Oversampling A/D Converter :

In DSP, we accept continuous time signal, then passing through sample-hold circuit, we convert it into digital form by means of A/D converter.

If sampling theorem is not followed one cannot recover original signal back due to aliasing. To prevent aliasing, we normally prefer anti-aliasing filter, before sampling. But the problem is anti-aliasing filter, design becomes very critical and of high order. It is usually difficult to implement such a filter in VLSI technology. Moreover, such a filter also introduces undesired phase distortion in its output. Therefore we have to sample analog signal at a rate much higher than the nyquist rate, this will release sharp cut-off requirement of antialiasing filter. Thus filter structure will be simple to built using low-precision analog components.

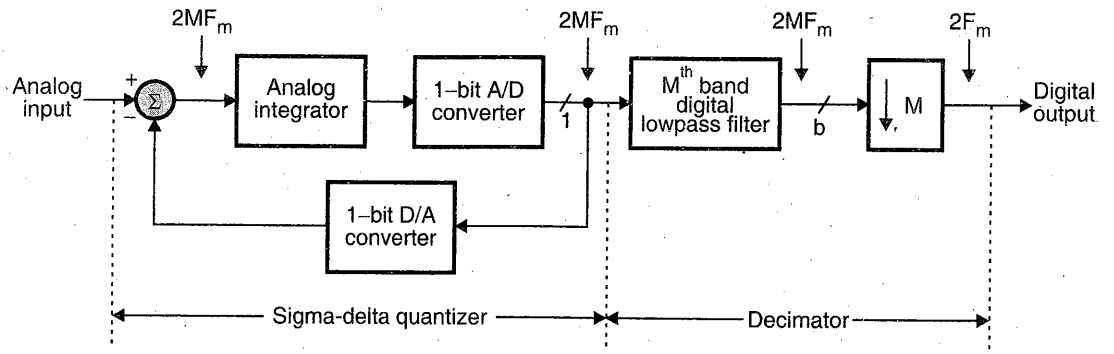


Fig. N-17 : Oversampling Σ Δ A/D converter

Oversampling approach to the input signal is application of multirate digital signal processing.

We know that A/D converter introduces quantization noise. Due to oversampling approach quantization noise power reduces in the signal. One can implement sigma-delta (Σ Δ) quantization scheme to improve noise performance of over sampling A/D converter. Fig. N-17 shows the oversampling approach in Σ Δ A/D converter.

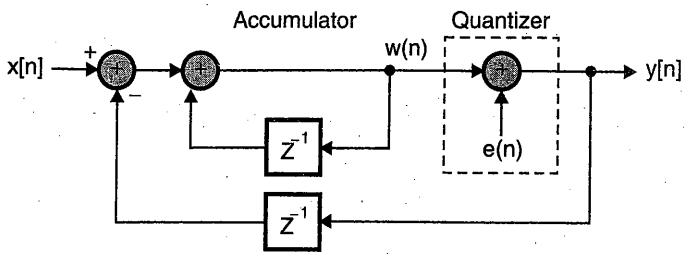


Fig. N-18(a) : Structure of $\Sigma\Delta$ quantizer

As we know, analog signal is sampled at a rate much higher than the Nyquist rate, which results in very closed spaced samples. Therefore the difference between the amplitude of two consecutive samples is very small. This allows us to represent it in digital form using very few bits, usually one bit. The sampling rate can be reduced by passing digital signal through an anti-aliasing low pass into band digital filter to reduce its bandwidth to π / m, and then passing it through a factor of M down sampler. The wordlength of the down sampler output determines the resolution of oversampling A/D converter and it is much higher than that of high rate digital signal due to effect of digital filtering.

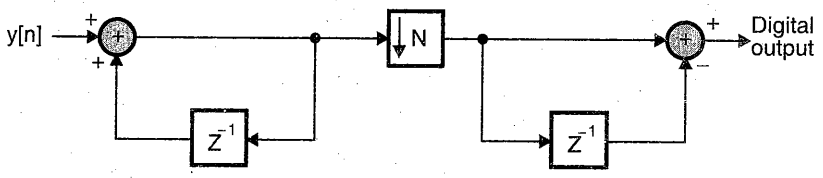


Fig. N-18(b) : Structure of N point decimator

Note that 1 bit output samples of quantizer after decimation becomes b-bit samples. This is just because we get b-bit multiplier coefficients for m^{th} band digital lowpass filter.

Figs. N-18(a) and N-18(b) shows the structure of sigma-delta quantizer and decimator.

1.7 Oversampling D/A Converter :

Following steps will convert digital data to analog signal :

1) Use D/A converter with zero order hold at its output and get staircase continuous-time waveform.

2) Pass this waveform through analog lowpass reconstruction filter.

Again the problem is, if we follow Nyquist rate of sampling we require analog lowpass reconstruction filter with very sharp cutoff in its frequency response. This leads to high precision analog circuit components. To overcome this problem we use over sampling approach. This leads to low precision analog circuit component for analog reconstruction filter. One can still improve the performance of an oversampling D/A converter by employing a digital $\Sigma - \Delta$ 1 bit quantiter at the output of digital interpolator. Fig. N-19 shows the oversampling sigma-delta D/A converter.

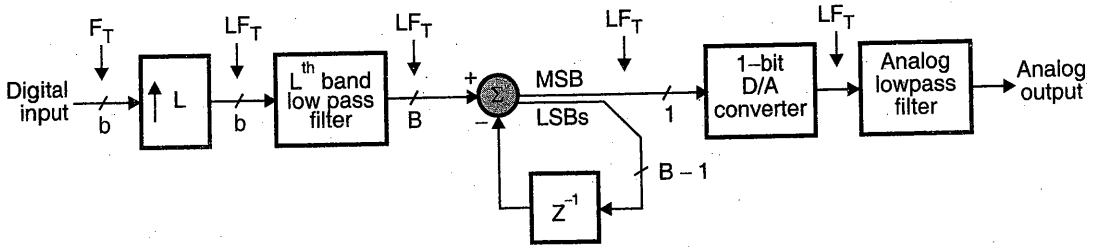


Fig. N-19 : Oversampling sigma-delta D/A converter

As shown in Fig. N-19, Digital input signal is upsampled first. Then, the Upsampled output is passed through L^{th} band LPF. Output of LPF is given to sigma-delta quantiter. Quantizer extracts the MSB from its input. Remaining LSB (i.e. quantization noise) is subtracted from its input.

The MSB output is fed to 1 bit D/A converter and finally passed through analog lowpass reconstruction filter to remove all frequency components beyond signal band.

1.8 Applications of Musical Sound Processing :

Almost all musical programs are produced in basically two stages :

1. Sound from each individual instrument is recorded in an acoustically inert studio on a single track of a multitrack tape recorder.
2. Then, the signals form each track are manipulated by the sound engineer to add special audio effects and are combined in a mix- down system to finally generate the stereo recording on a two- track tape recorder.

The audio effects are artificially generated using various signal processing circuits and devices, and they are being performed using digital signal processing techniques. Some of the methods described below.

1.8.1 Time-Domain Operations :

The sound reaching the listener in a closed space, such as a concert hall, consists of several components viz. direct sound, early reflections, and reverberations. The sound recorded in an inert studio is different from that recorded inside a closed space and, as a result, the former does not sound "natural" to a listener. However, digital filtering can be employed to convert the sound recorded in an inert studio into a natural sounding one by artificially creating the echoes and adding it to the original signal. Echoes are generated by delay units.

The direct sound and a single echo appearing R sampling periods later can be simply generated by the FIR filter, as shown in Fig. N-20.

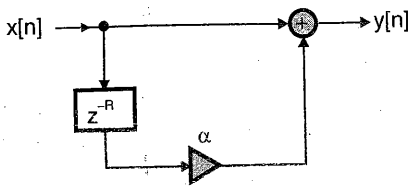


Fig. N-20 : Single echo filter structure

An IIR realization of this filter is shown in Fig. N-21.

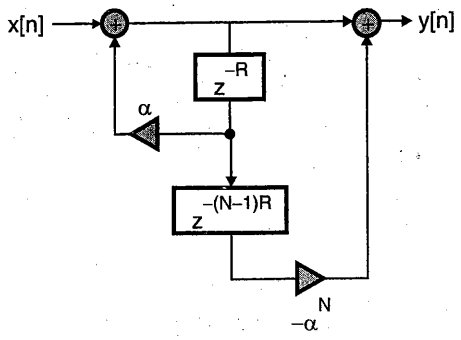


Fig. N-21 : Filter structure generating multiple echoes

It has been observed that approximately 1000 echoes per second are necessary to create a reverberation that sounds free of flutter. To develop a more realistic reverberation, a reverberator with an all pass structure, as shown in Fig. N-22 has been proposed.

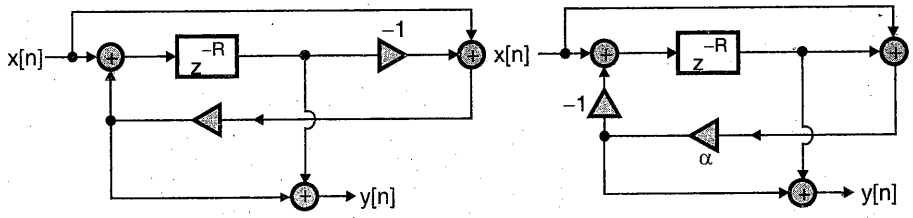


Fig. N-22 : Block diagram representation of all pass reverberator

Let us take an overview of some of the time-domain operations.

(I) Flanging effect :

Flanging is a special sound effect that is used in the mix-down process. By feeding the same musical piece to two tape recorders and then combining their delayed outputs while varying the difference Δt between their delay times gives rise to flanging effect. Δt is varied by slowing down one of the tape recorders by placing the operator's thumb on the flange of the feed reel. The filter structure that creates flanging effect is shown in Fig. N-23. Here, the unit generating the delay of R samples is made a time- varying delay $\beta(n)$. The input-output relation is given by

$$y[n] = x[n] + \alpha x[n - \beta(n)] \qquad ...(1)$$

Periodically varying delay $\beta(n)$ between O and R with a low frequency ω_0 generates a flanging effect on the sound. In flanging the swept notch frequencies are always equally spaced.

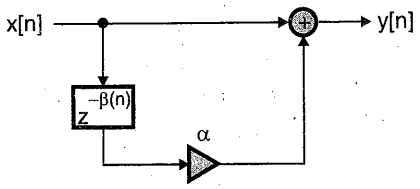


Fig. N-23 : Generation of flanging effect

(II) Chorus effect :

The chorus effect is generated when several musicians are playing the same musical piece at the same time but with small changes in the amplitudes and small timing differences between their sounds.

The chorus generator, shown in Fig. N-24, can crete a chorus of four musicians from the music of a single musician. By randomly varying the delays $\beta_i(n)$, with very slow variations, we can achieve chorus effect.

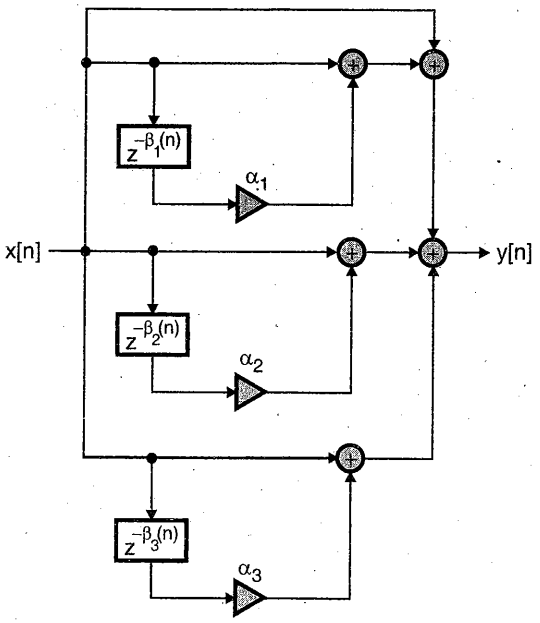


Fig. N-24 : Generation of a chorus effect

(III) Phasing effect :

The phasing effect is produced by processing the signal through a narrowband notch filter with variable notch characteristics and adding a scaled portion of the notch filter output to the original signal. The phase of the signal at the notch filter output can dynamically alter the phase of the combined signal. A phasing effect generator is shown in Fig. N-25. The notch filter in the Fig. N-25 can be replaced by a cascade of tunable notch filters to provide an effect similar to flanging. In phasing the locations of the notch frequencies and their corresponding 3-dB bandwidths are varied independently.

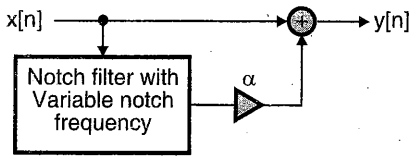


Fig. N-25 : Generation of phasing effect

1.8.2 Frequency-Domain Operations :

The frequency responses of individually recorded instruments or musical sounds of performers are frequently modified by the sound engineer during the mix-down process. These effects are achieved by passing the original signals through an equalizer.

The purpose of the equalizer is to provide "presence" by peaking the midfrequency components in the range of 1.5 to 3 kHz, and to modify the bass-treble relationships by providing "boost" or "cut" to components outside this range. The simple digital filters can be used for implementing these functions.

First-order filters :

The transfer functions of a first-order lowpass filter and a highpass filter with tunable cutoff frequencies are given by equations and respectively,

$$H_{LP}(z) = \frac{1-\alpha}{2} \cdot \frac{1+z^{-1}}{1-\alpha z^{-1}} \qquad ...(1)$$

$$H_{HP}(z) = \frac{1+\alpha}{2} \cdot \frac{1-z^{-1}}{1-\alpha z^{-1}} \qquad ...(2)$$

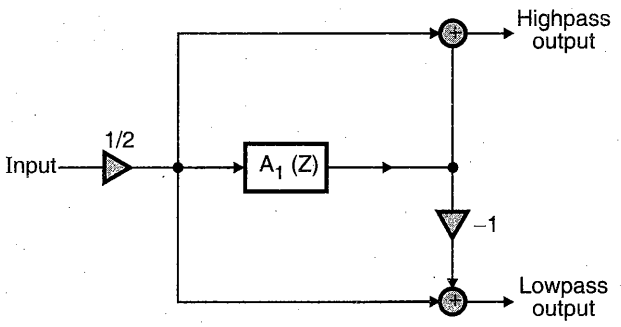


Fig. N-26 : First-order lowpass/highpass filter structure

Combining the outputs of the structure of Fig. N-26 we get a low- frequency shelving filter as shown in Fig. N-27 that is characterized by a transfer function $G_1(z)$ as,

$$G_1(z) = \frac{K}{2}[1 - A_1(z)] + \frac{1}{2}[1 + A_1(z)] \qquad ...(3)$$

where K is a positive constant and $A_1(z)$ is a first-order allpass transfer function given by,

$$A_1(z) = \frac{\alpha - z^{-1}}{1 - \alpha z^{-1}} \qquad ...(4)$$

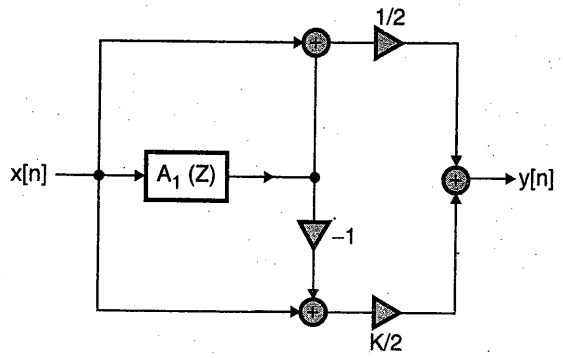


Fig. N-27 : Low-frequency shelving filter

A high-frequency shelving filter is obtained by combining the outputs of the st ucture of Fig. N-26 as shown in Fig. N-28, which is characterized by a transfer function,

$$G_1(z) = \frac{1}{2}[1 - A_1(z)] + \frac{K}{2}[1 + A_1(z)] \qquad ...(5)$$

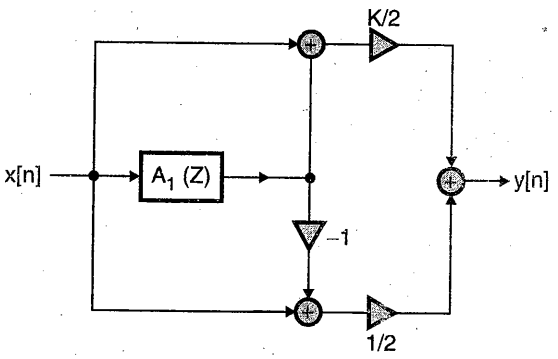


Fig. N-28 : High-frequency shelving filter

Second-order filters and equalizers :

The transfer functions of a second-order bandpass and a bandstop filter with tunable cutoff frequencies and 3-dB bandwidths are given by Equations (1) and (2) respectively.

$$H_{BP}(z) = \frac{1-\alpha}{2} \cdot \frac{1 - z^{-2}}{1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2}}, \qquad ...(1)$$

$$H_{BS}(z) = \frac{1+\alpha}{2} \cdot \frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2}} \qquad ...(2)$$

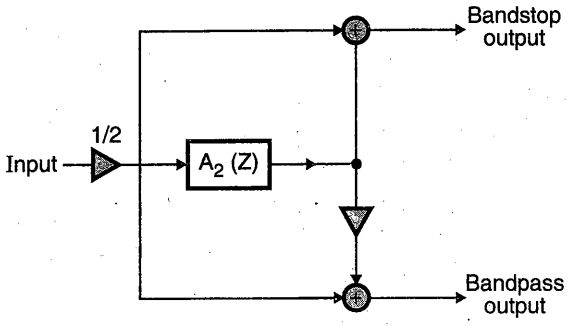


Fig. N-29(a) : Second-order bandpass/bandstop filter structure

A weighted combination of the outputs of Fig. N-28 results in a second-order equalizer as shown in Fig. N-29 with a transfer function $G_2(z)$ given by

$$G_2(z) = \frac{K}{2}[1 - A_2(z)] + \frac{1}{2}[1 + A_2(z)] \quad ...(3)$$

where K is a positive constant and $A_2(z)$ is a second-order transfer function given by,

$$A_2(z) = \frac{\alpha - \beta(1+\alpha)z^{-1} + z^{-2}}{1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2}} \quad ...(4)$$

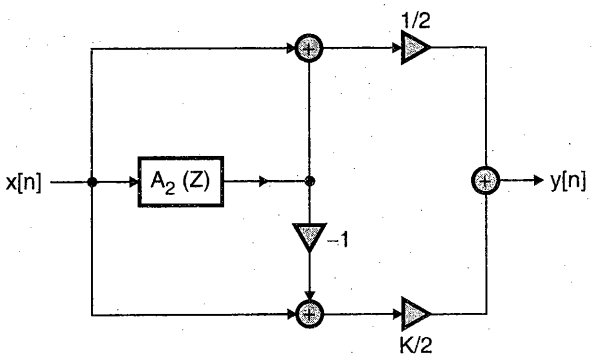


Fig. N-29(b) : Second-order equalizer

Higher-order equalizers :

A graphic equalizer with tunable gain response can be built using a cascade of first-order and second-order equalizer with external control of the maximum gain values of each section in the cascade. Fig. N-30 shows the block diagram of a cascade of one first-order and three second-order equalizer with nominal frequency response parameters.

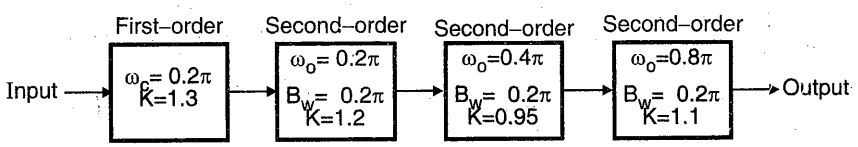


Fig. N-30

1.9 Applications of Multi Rate Signals :

1.9.1 Introduction to Multirate Signal Processing :

The need to process data at more than one sampling rate is increasing in modern digital systems. This has led to the development of a new sub-area in DSP known as multirate processing. The processes of decimation and interpolation are the fundamental operations in multirate signal processing, and they allow the sampling frequency to be decreased or increased without significant undesirable effects of errors such as quantization and aliasing. Decimation reduces sampling rate, effectively compressing the data, and retaining only the desired information. Interpolation increases the sampling rate. The purpose of converting data to a new rate is to make it easier with another system.

Sampling rate reduction : Decimation by integer factor :

A block diagram showing the process of decimating a signal $x(n)$ by an integer factor M is shown in Fig. N-31(a). It consists of a digital anti-aliasing filter, $h(K)$, and a sample rate compressor, shown by a down-arrow and decimation factor, M. The rate compressor reduces the sampling rate from F_s to F_s / M. To prevent aliasing at the lower rate the digital filter is used to bandlimit the input signal to less than $F_s / 2M$ beforehand. Thus the signal $x(n)$ is first bandlimited. Sampling rate reduction is achieved by discarding $(M-1)$ samples for every M samples of the filtered signal, $\omega(n)$. The input-output relationship for the decimation process is,

$$y(m) = \omega(mM) = \sum_{K=-\infty}^{\infty} h(K)\, x(mM-K) \qquad ...(1)$$

where

$$w(n) = \sum_{K=-\infty}^{\infty} h(K)\, x(n-K) \qquad ...(2)$$

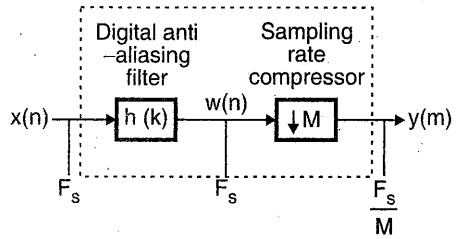


Fig. N-31(a) : Block diagram of decimation by a factor of M

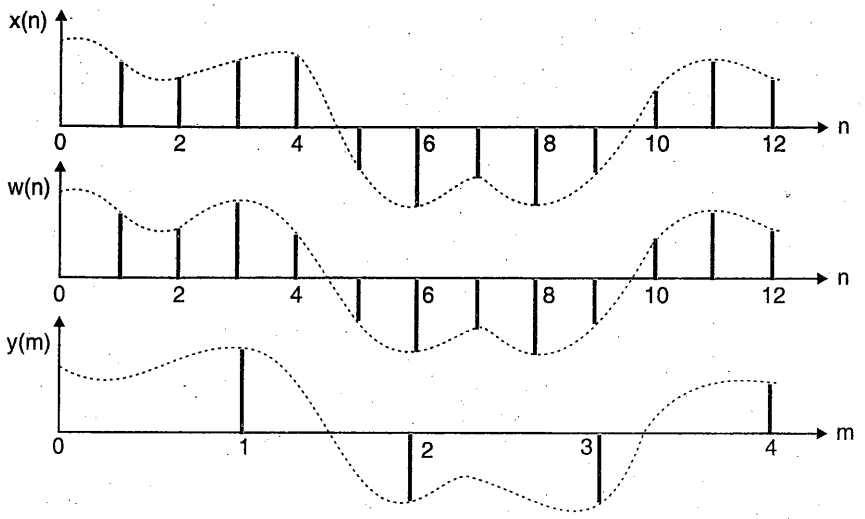


Fig. N-31(b) : An illustration of decimation by a factor of M = 3

Sampling rate increase : Interpolation by integer factor :

In interpolation process, the analog signal is recovered by interpolating the digital samples. Given a signal, x (n), at a sampling frequency F_s, the interpolation process increases the sampling rate by L to LF_s Fig. N-32 shows the interpolator.

It consists of a sample rate expander, shown by an up-arrow and the interpolation factor, L, which indicates the amount by which the rate is increased. For each sample of x (n), the expander inserts (L – 1) zero-valued samples to form the new signal w (m) at a rate of LF_s. This signal is then lowpass filtered to remove image frequencies created by the rate increase to yield y (m). The insertion of (L – 1) zeros spreads the energy of each signal sample over L output samples, effectively attenuating each sample by a factor of L. Thus it is necessary to compensate for this.

The interpolation process is represented by the following input- output relationship,

$$y(m) = \sum_{K=-\infty}^{\infty} h(K)\, w(m-K) \qquad ...(1)$$

where,

$$w(m) = \begin{cases} x(m/L), & m = 0, \pm L, \pm 2L, ... \\ 0 \end{cases}$$

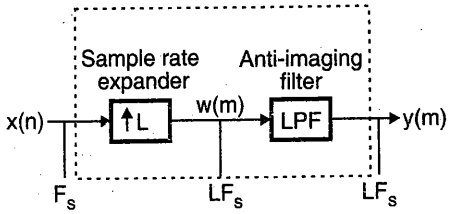


Fig. N-32(a) : Block diagram of interpolation by a factor of L

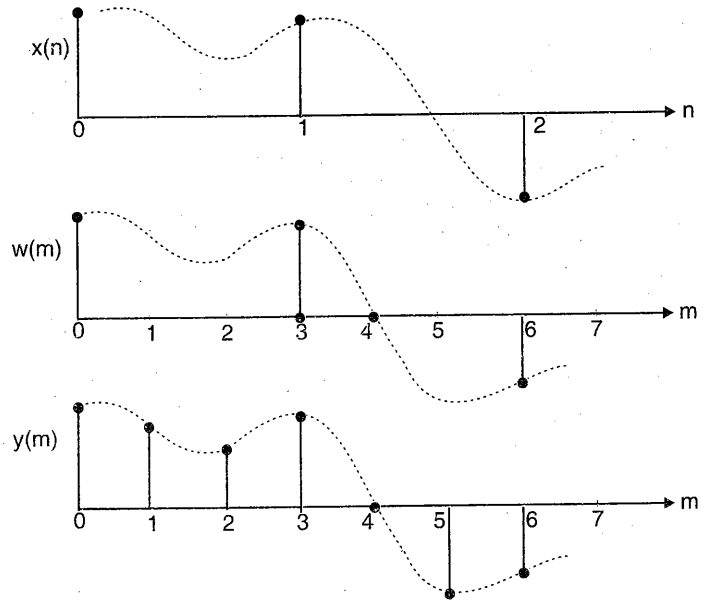


Fig. N-32(b) : An illustration of interpolation by a factor of L = 3

1.9.2 Advantages of Multirate Signal Processing :

1. Multirate technique avoid the use of expensive anti-aliasing analog filters and to handle efficiently signals of different bandwidths which require different sampling frequencies.
2. In speech processing, multirate techniques are used to reduce the storage space or the transmission rate of speech data.
3. The need for inexpensive, high resolution analog-to-digital converters (ADCs) in digital audio had led to the use of oversampling techniques which decreases in-band noise.
4. The use of multirate techniques allows filtering to be performed at a much lower rate, which greatly reduces the filter order.

1.9.3 Applications of Multirate Signal Processing :

1. High quality analog-to-digital conversion for digital audio :

The constant demand in digital audio for better quality, higher resolution and higher speed ADC has led to the development of single bit ADCs using delta-sigma modulation techniques. This

eliminates from conversion process most of the analog circuitry at the front end of a digital audio system, including the analog anti-aliasing filters and sample-and-hold circuits.

A simplified block diagram of a fast single-bit ADC process is shown in Fig. N-33. The analog audio signal is first converted into a single bit stream, using delta-sigma modulation at a 3.072 MHz rate. The single bit stream is then downsampled to 48 kHz, using a multistage decimator, to yield 16- bit PCM words.

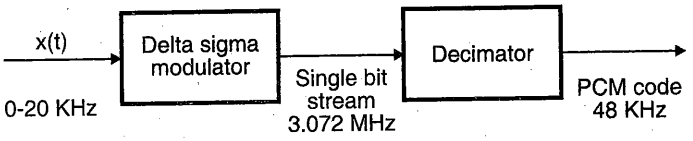


Fig. N-33 : Block diagram of single-bit ADC scheme

2. Compact hi-fi system :

Fig. N-34 shows the process of reconstructing the analog audio signal from the digital signal read from the CD. After decoding and error correction, the digital signals are in 16-bit words, representing the acoustic information at a 44.1 kHz sampling rate.

If these digital codes were converted directly into analog, image frequency bands centered at multiples of sampling frequency of 44.1 kHz would be produced. Although the image frequencies would be inaudiable as they are above the baseband of $0 - 20$ kHz, they could cause overloading if passed on to the player's amplifier and loudspeaker, or they could set up intermodulation distortion. Thus the frequency components above the baseband need to be attenuated by at least 50 dB. Analog filters will have to meet very tight specifications to provide this level of attenuation.

To avoid the problems with analog filters, multirate filtering is employed in CD player. This is achieved by increasing, by interpolation, the sampling frequency of the data by 4 to 176 kHz (4×44.1 kHz $= 176.4$ kHz) before it is applied to DAC. Oversampling the data reduces the noise floor, as the quantization noise is spread over a wider bandwidth.

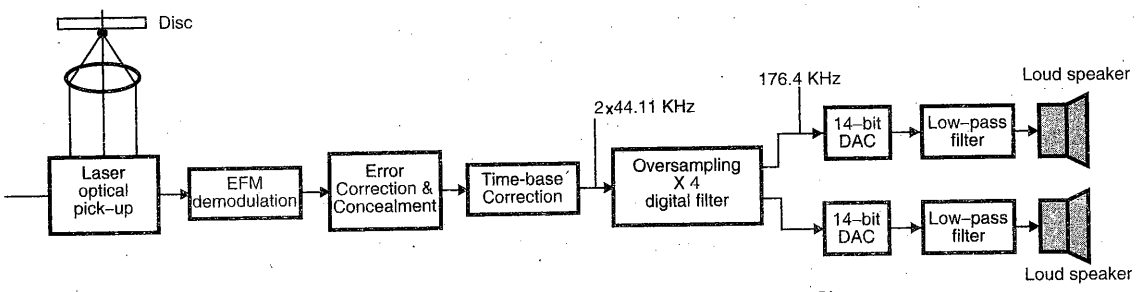


Fig. N-34 : Audio signal reproduction in the compact disc system

3. Application in the acquisition of high quality data :

In the acquisition of data, the need to keep aliasing low often requires the use of relatively complex analog anti-aliasing filter. By using anti-aliasing digital filters, the complex analog filters on each channel can be replaced by much simpler filter leading to cost reduction. Further, the phase matching problems of analog filters is avoided. The difficulty of supporting multiple sampling frequencies is also overcome.

Fig. N-35 shows a block diagram of a multirate data-acquisition system. The desired aliasing level is achieved by the front-end RC filters oversampling the input signal, then downsampling to required frequency using multirate technique.

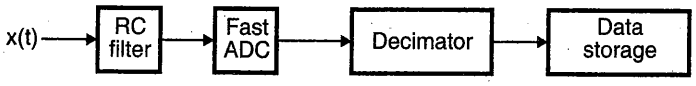


Fig. N-35 : Multirate data acquisition system

1.10 Applications of DSP in Vibration Analysis :

Vibration analysis determines the various defects in the system. Generally, the machine parts such as conveyor belts, shafts etc. vibrate depending upon the speed of their motion. It is a difficult task to determine the fault in such system.

Example : We sense the vibrations of a conveyor belt and perform spectrum analysis on it. It shows that there is a fixed frequency pattern depending upon vibrations. If any fault is created in the conveyor belt, then the frequency spectrum gets changed. The new frequencies that are introduced in the spectrum represents the fault such spectrum analysis is performed by DSP system. If any fault is present then, a message is given indicating the presence of fault.

Besides this, the DSP system also monitors other parameters of the system.

1.11 Applications of DSP in Bio-medical Engineering :

Often, medical data are not well behaved and this gives rise to application of DSP which offers new way of handling the data. Many applications of DSP in bio-medicine involve signal enhancement and extraction of features of clinical interest. Artefacts or signal contamination represents more difficulties. Artefact reduce the clinical usefulness of biomedical signals and make both manual and automatic analysis difficult. Thus, a great deal of care is often required to minimize the distortion of information of clinical interest by signal enhancement process.

In the preceeding section, we will see two applications of DSP in biomedicine that involves signal enhancement and feature extraction.

1.11.1 Fetal ECG Monitoring :

The fetal electrocardiogram (ECG) shows the electrical activity of the baby's heart as measured from the body surface. Cardiotocogram (CTG), is normally used to access the condition of the fetus during labour. But, it leads to unnecessary medical intervention, fetal injury or a failure to intervence when needed. This is overcome by the correct use of combined fetal ECG and CTG analysis, with no adverse effect on neonatal outcome.

The ECG is obtained from a scalp electrode to achieve a good signal-to-noise ratio, band limited to about 0.05 – 100 Hz and digitized to 12 bits accuracy at a rate of 500 samples per second. The fetal scalp ECG is succeptible to low frequency noise and other artefacts which may induce false changes in the waveform. Artefacts hinder feature extraction may lead to inaccurate ECG features and waveform analysis.

A variety of signal processing methods are used to reduce noise and extract key features from the ECG. A fetal ECG signal processing scheme is shown in Fig. N-36. First the R-wave is accurately detected. The baseline shift, muscle noise and power line frequencies are then removed from the raw ECG to obtain a waveform suitable for reliable analysis.

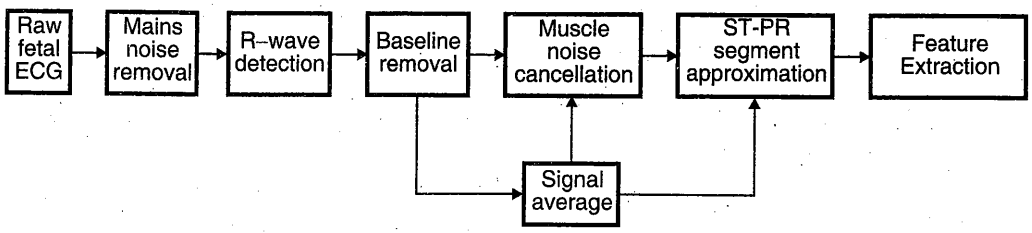


Fig. N-36 : Fetal ECG signal processing scheme

1.11.2 DSP-based Closed Loop Controlled Anaesthesia :

Patients are normally anaesthetized during surgery by injecting anaesthetic drugs, so that they do not feel pain and to create a suitable condition for the surgeon to carry out the operation. To deliver proper amount of drug to induce anaesthesia at required depth as quickly as possible and to maintain the level is most important. Injecting too much drug into a patient leads to side effect while less amount of drug leads to intra-operative awareness which may have long term psychological consequences. Automated drug delivery using closed loop control techniques offers potential benefits to busy anaesthetists and leads to better patient care and reduced costs. Its use reduces the possibility of excessive dosing and enables the anaesthetist to identify and respond. However, automated closed loop controlled drug delivery required a reliable means of monitoring depth of anaesthesia to determine changes to the drug delivery necessary to maintain anaesthesia.

Closed-controlled anaesthesia systems use biological signals to measure depth of anaesthesia and as feedback signals to determine the adjustments to be made to drug delivery. The EEG is the electrical activity of the brain, measured from electrodes placed on the scalp. A simplified block diagram of an EEG-based closed loop controlled anaesthesia system is shown in Fig. N-37.

EEG analyzer is the major component which is connected to the patient through bifrontal electrodes to collect raw EEG signals. A variety of signal processing methods are used in the analyzer to reduce noise, extract features, analyze changes in the features, and to compute a suitable EEG index. These includes wavelet transform, signal averaging, bispectrum analysis and neural networks.

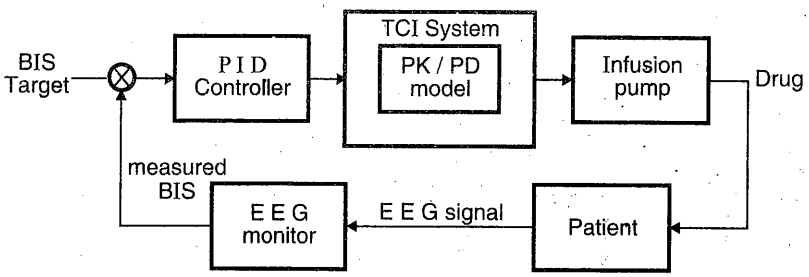


Fig. N-37 : DSP-based closed-controlled anaesthesia system

Theory Questions

Q. 1 Describe a subband speech decoder using neat block diagram.

Q. 2 Explain the role of DSP in speech signal processing briefly.

Q. 3 Write short note on : Subband coding of speech.

Q. 4 Write short notes on following :

i) DSP in speech processing

ii) Echo canceller

iii) Speech analysis

Q. 5 What is speech signal ? Explain mechanism of of Human-speech- reproduction with block diagram.

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Discrete Hilbert Transform

Section No.	Name of the Topic
1.1	Introduction
1.2	Real and Imaginary Part Sufficiency for Causal Sequences
1.3	Minimum Phase Condition
1.4	Design of Hilbert Transforms

1.1 Introduction :

Hilbert transform is an all pass filter. So it passes all the frequencies and produces 90° phase shift. Such transforms are used for the generation of single side band modulated signal. But practically, it is not possible to design ideal all pass digital Hilbert transform.

1.2 Real and Imaginary Part Sufficiency for Causal Sequences :

In the chapter signals and systems; we have studied that any sequence can be expressed in terms of its even and odd components.

$$\therefore \quad x(n) = x_e(n) + x_o(n) \qquad \ldots(1)$$

Here $x_e(n)$ = Even components of $x(n)$

$x_o(n)$ = Odd components of $x(n)$

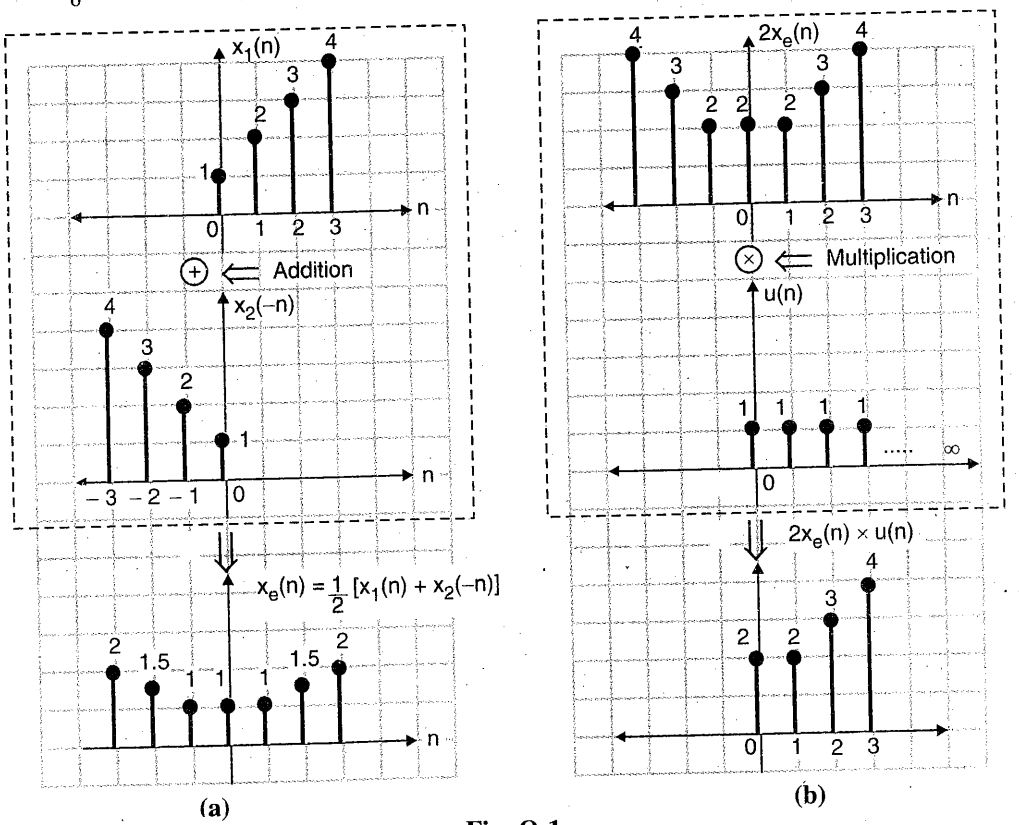


Fig. O-1

We have also studied that even and odd components can be expressed in terms of $x(n)$ as follows :

$$x_e(n) = \frac{1}{2}[x(n) + x(-n)] \qquad \ldots(2)$$

$$\text{and} \quad x_o(n) = \frac{1}{2}[x(n) - x(-n)] \qquad \ldots(3)$$

Now if sequence $x(n)$ is causal then it is possible to recover $x(n)$ from only even components or odd components. It is obtained by using the equation,

$$x(n) = 2x_e(n)u(n) - x_e(0)\delta(n), \quad n \geq 0 \qquad \ldots(4)$$

To verify equation (4) let us consider a sequence

$$x(n) = \{1,2,3,4\}.$$
$$\uparrow$$

Equation (4) is verified as shown in Fig. O-1.

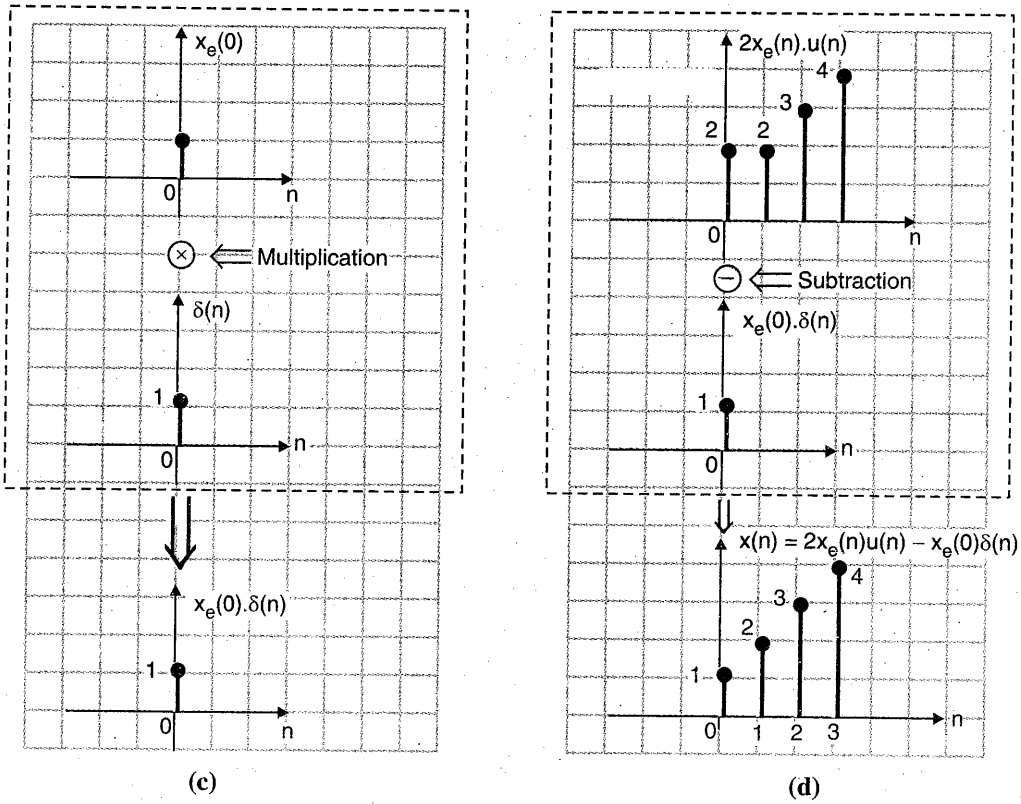


Fig. O-1

Similarly we can prove that, $x(n)$ can be recovered by using only odd components.

$$x(n) = 2x_o(n)u(n) + x(0)\delta(n), \quad n \geq 1 \qquad \ldots(5)$$

Now in terms of fourier transform, sequence $x(n)$ can be expressed as,

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega}) \qquad \ldots(6)$$

When x (n) is causal; we can apply symmetry properties of fourier transform.

$$x_e(n) \xleftrightarrow{F} X_R(e^{j\omega})$$

and $$x_o(n) \xleftrightarrow{F} X_I(e^{j\omega})$$

Thus $X(e^{j\omega})$ can be determined if $X_R(e^{j\omega})$ is known.

1.3 Minimum Phase Condition :

Consider two FIR systems having transfer functions,

$$H_1(Z) = \frac{Z+\frac{1}{2}}{Z} \quad ...(1)$$

and $$H_2(Z) = \frac{\frac{1}{2}Z+1}{Z} \quad ...(2)$$

First system, $H_1(Z)$ has zero at $Z = -\frac{1}{2}$ and the second system, $H_2(Z)$ has zero at $\frac{1}{2}Z+1 = 0 \Rightarrow Z = -2$.. Thus there is reciprocal relation between the zeros of two systems.

Now we will calculate magnitude and phase response of two systems.

Magnitude response :

We know that that magnitude response is obtained by putting $Z = e^{-j\omega}$ and by taking the magnitude of that function.

$$\therefore \quad H_1(\omega) = \frac{e^{-j\omega}+\frac{1}{2}}{e^{-j\omega}} = \frac{\cos\omega - j\sin\omega + \frac{1}{2}}{\cos\omega - j\sin\omega}$$

$$\therefore \quad |H_1(\omega)| = \frac{\sqrt{(\frac{1}{2}+\cos\omega)^2+\sin^2\omega}}{\sqrt{\cos^2\omega+\sin^2\omega}}$$

$$\therefore \quad |H_1(\omega)| = \frac{\sqrt{\frac{1}{4}+\cos\omega+\cos^2\omega+\sin^2\omega}}{1} = \sqrt{\frac{1}{4}+\cos\omega+1}$$

$$\therefore \quad |H_1(\omega)| = \sqrt{\frac{5}{4}+\cos\omega} \quad ...(3)$$

Similarly $$H_2(\omega) = \frac{\frac{1}{2}e^{-j\omega}+1}{e^{-j\omega}} = \frac{\frac{1}{2}\cos\omega - \frac{1}{2}j\sin\omega + 1}{\cos\omega - j\sin\omega}$$

$$\therefore \quad |H_2(\omega)| = \frac{\sqrt{\left(\frac{1}{2}\cos\omega + 1\right)^2 + \left(\frac{1}{2}\sin\omega\right)^2}}{\sqrt{\sin^2\omega + \cos^2\omega}}$$

$$\therefore \quad |H_2(\omega)| = \frac{\sqrt{\frac{1}{4}\cos^2\omega + \cos\omega + 1 + \frac{1}{4}\sin^2\omega}}{1}$$

$$= \sqrt{\frac{1}{4} + 1 + \cos\omega}$$

$$\therefore \quad |H_2(\omega)| = \sqrt{\frac{5}{4} + \cos\omega} \qquad \ldots (4)$$

Thus magnitude response of two systems is same; this is because zeros of $H_1(Z)$ and $H_2(Z)$ are reciprocals.

Phase response :

It is obtained by using the relation $\theta = \tan^{-1}\frac{I_m}{R_e}$

Let phase response of $H_1(Z)$ be $\theta_1(\omega)$.

$$\therefore \quad \theta_1(\omega) = \tan^{-1}\frac{\sin\omega}{\frac{1}{2} + \cos\omega} - \tan^{-1}\frac{\sin\omega}{\cos\omega}$$

$$\therefore \quad \theta_1(\omega) = \tan^{-1}\frac{\sin\omega}{\frac{1}{2} + \cos\omega} - \tan^{-1}(\tan\omega)$$

$$\therefore \quad \theta_1(\omega) = \tan^{-1}\frac{\sin\omega}{\frac{1}{2} + \cos\omega} - \omega \qquad \ldots(5)$$

Similarly for $H_2(Z)$,

$$\theta_2(\omega) = \tan^{-1}\frac{\frac{1}{2}\sin\omega}{\frac{1}{2}\cos\omega + 1} - \tan^{-1}(\tan\omega)$$

$$\therefore \quad \theta_2(\omega) = \tan^{-1}\frac{\sin\omega}{\cos\omega + 2} - \omega \qquad \ldots(6)$$

The plot of phase response for $H_1(Z)$ and $H_2(Z)$ is as shown in Figs. O-2 and O-3 respectively.

For $H_1(Z)$, the zero is at $Z_1 = \frac{1}{2}$ that means zero is inside the unit circle. As shown in Fig. O-2, at $\omega = 0$; $\theta_1(\omega) = 0$ and at $\omega = \pi$, $\theta_1(\omega) = 0$. Thus net phase change For $H_1(Z)$ is zero.

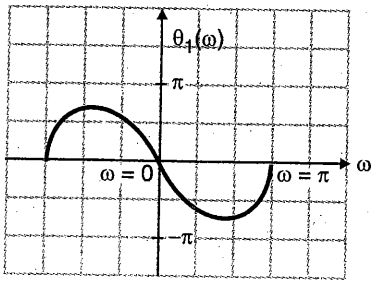


Fig. O-2

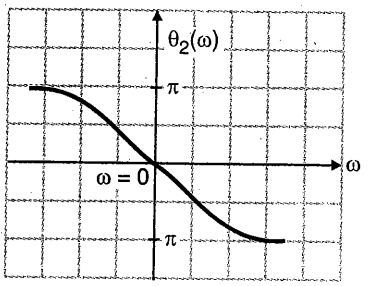


Fig. O-3

As shown in Fig. O-3, at $\omega = 0$, $\theta_2(\omega) = 0$ and at $\omega = \pi$, $\theta_2(\omega) = -\pi$.

Thus net phase change is π. For this system, zero is at, $Z_1 = 2$, that means zero is outside the unit circle. The first system is called as minimum phase system and second system is called as maximum phase system.

Thus for FIR systems :

1) When all zeros are inside the unit circle then it is called as minimum phase system.
2) When all zeros are outside the unit circle then it is called as maximum phase system.
3) When some of the zeros are inside the unit circle and remaining zeros are outside the unit circle then it is called as non-minimum phase system or mixed phase system.

The same principle is applicable for IIR systems. In this case if $H(Z)$ is having minimum phase then its inverse system $H^{-1}(Z)$ is always stable. Alternately if $H(Z)$ is stable then $H^{-1}(Z)$ has minimum phase. But mixed phase and maximum phase systems result in unstable inverse systems.

Prob 1 : FIR filter have impulse responses as given below. Classify them as minimum phase or maximum phase or mixed phase

(i) $h_1(n) = \{1, 0.707, 0.25\}$ (ii) $h_2(n) = \{1, 1.414, 1\}$ (iii) $h_3(n) = \{1, -5, 6\}$

Soln. :

1) Given $h_1(n) = \{1, 0.707, 0.25\}$

$$\text{We have } H_1(Z) = \sum_{n=0}^{2} h_1(n) \cdot Z^{-n}$$

$$\therefore \quad H_1(Z) = 1 + 0.707\,Z^{-1} + 0.25\,Z^{-2}$$

$$\therefore \quad H_1(Z) = 1 + \frac{0.707}{Z} + \frac{0.25}{Z^2}$$

$$\therefore \quad H_1(Z) = \frac{Z^2 + 0.707\,Z + 0.25}{Z^2}$$

To determine the location of zeros; we will obtain roots of numerator terms.

$$Z^2 + 0.707\,Z + 0.25 \Rightarrow -0.707 \pm \frac{\sqrt{(0.707)^2 - 4(0.25)}}{2}$$

$$= \frac{-0.707 \pm \sqrt{0.5-1}}{2} = \frac{-0.707 \pm \sqrt{-0.5}}{2}$$

$$= -0.36 + j\,0.35, -0.36 - j\,0.36$$

Since both zeros are inside unit circle; this is minimum phase system.

(ii) $h_2(n) = \{1, 1.414, 1\}$

$$\text{We have} \quad H_2(Z) = \sum_{n=0}^{2} h_2(n)\, Z^{-n}$$

$$\therefore \quad H_2(Z) = 1 + 1.414\, Z^{-1} + Z^{-2} = 1 + \frac{1.414}{Z} + \frac{1}{Z^2}$$

$$\therefore \quad H_2(Z) = \frac{Z^2 + 1.414\, Z + 1}{Z^2}$$

$$\text{roots of numerator} \Rightarrow \frac{-1.414 \pm \sqrt{2-4}}{2} = \frac{-1.414 \pm \sqrt{-2}}{2}$$

$$= -0.707 + j\,0.707, -0.707 - j\,0.707.$$

Since zeros are inside the unit circle; this is minimum phase system.

(iii) $h_3(n) = \{1, -5, 6\}$

$$\text{We have,} \quad H_3(Z) = \sum_{n=0}^{2} h_3(n)\, Z^{-n}$$

$$\therefore \quad H_3(Z) \quad 1 - 5\, Z^{-1} + 6\, Z^{-2}$$

$$\therefore \quad H_3(Z) = 1 - \frac{5}{Z} + \frac{6}{Z^2} = \frac{Z^2 - 5\, Z + 6}{Z^2}$$

$$\text{roots of numerator} \Rightarrow \frac{5 \pm \sqrt{25-24}}{2} = \frac{5 \pm 1}{2}$$

$$= \frac{5+1}{2}, \frac{5-1}{2} = 3, 2.$$

Since zeros are outside the unit circle; this is maximum phase system.

1.4 Design of Hilbert Transforms :

Ideally Hilbert transform is all pass filter. It produces 90° phase shift of input signal. Thus the frequency response of Hilbert transform is expressed as,

$$H_d(\omega) = \begin{cases} -j & 0 < \omega \le \pi \\ j & -\pi < \omega < 0 \end{cases} \qquad \ldots(1)$$

Hilbert transforms are generally used in communication systems and in digital signal processing.

Unit sample response of Hilbert transform is obtained by taking inverse fourier transform of Equation (1).

$$\therefore \quad h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega$$

$$\therefore \quad h_d(n) = \frac{1}{2\pi} \left[\int_{-\pi}^{0} H_d(\omega) e^{j\omega n} d\omega + \int_{0}^{\pi} H_d(\omega) e^{j\omega n} d\omega \right] \quad ..(2)$$

Putting value of $H_d(\omega)$ from Equation (1) in Equation (2) we get,

$$h_d(n) = \frac{1}{2\pi} \left[\int_{-\pi}^{0} j e^{j\omega n} d\omega - \int_{0}^{\pi} j e^{j\omega n} d\omega \right]$$

$$\therefore \quad h_d(n) = \begin{cases} \frac{2}{\pi} \frac{\sin^2(\pi n/2)}{n}, & n \neq 0 \\ 0, & n = 0 \end{cases} \quad ...(3)$$

Thus $h_d(n)$ is an infinite duration sequence and it is non-causal. Practically it is impossible to design an ideal all pass digital Hilbert transform. But in practical cases its bandwidth is needed to cover the bandwidth of signal to be phase shifted.

The desired frequency response of Hilbert transform is given by,

$$H_{dr}(\omega) = 1, \qquad 2\pi f_1 \leq \omega \leq 2\pi f_2 \quad ...(4)$$

Here f_1 = Lower cut-off frequency

and f_2 = Higher cut-off frequency.

While designing the Hilbert transform using chebyshev approximation method; the filter coefficients are selected in such as manner that the peak approximation error is minimized. Filter coefficients are obtained by using the equation,

$$\delta = \max_{2\pi f_1 \leq \omega \leq 2\pi f_2} [1 - H_r(\omega)] \quad ...(5)$$

Applications :

Hilbert transforms are used for the following applications :

i) Generation of single sideband modulated signal.

ii) Radar signal processing.

iii) Speech signal processing.

Theory Questions

Q. 1 Prove that any D.T. sequence can be recovered using only even components and odd components.

Q. 2 Write a short note on minimum phase condition.

Q. 3 Explain the design of Hilbert transform.



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